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Extreme discrete groups of parabolic type for Jørgensen's inequality

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Abstract

Jørgensen groups of parabolic type parametrized by three real parameters are divided into three types: finite type, countably infinite type and uncoutably infinite type. In the previous papers we found all Jørgensen groups of finite type and of countably infinite type. In this paper we announce that we find all Jørgensen groups of uncoutably infinite type. Consequently, the problem finding all Jørgensen groups of this parabolic type has been completely solved. The proofs will appear elthwhere

1. JØRGENSEN'S INEQUALITY.

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It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup G of the Möbius transformation group is discrete. For this problem there are two important and useful theorems. One is Poincaré's polyhedron theorem, which gives a sufficient condition for G to be discrete. The other is Jørgensen's inequality, which gives a necessary condition for a two-generator Möbius transformation group $G = \langle A, B\rangle$ to be discrete.

In 1976 Jørgensen obtained the following important theorem called *Jørgensen's inequality theorem*, which gives a necessary condition for a non-elementary Möbius transformtion group $G = \langle A, B\rangle$ to be discrete.

THEOREM A (Jørgensen [**1**]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 *is best possible.*

2. JØRGENSENS GROUPS.

2.1. First we will stste some definitions.

DEFINITION 1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ for the ordered pair (A, B) is defined by

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

That is, this number is the left hand side of Jørgensen's inequality.

DEFINITION 2. Let G be a non-elementary two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ *for* G is defined as

$J(G) := \inf\{J(A, B) \mid A$ and B generate $G\}$.

DEFINITION 3. A subgroup G of Möb is called a *Jørgensen group* if G satiafies the following four conditions:

(1) G is a two-generator group.

(2) G is a discrete group.

(3) G is a non-elementary group.

(4) There exist generators A and B of G such that $J(A,B)=1$.

That is, G is a extreme non-elementary discrete group for Jørgensen's inequality.

2.2. Jørgensen and Kiikka showed the following.

THEOREM B (Jørgensen-Kiikka [**2**]). *Let $\langle A,B\rangle$ be a non-elementary discrete group with $J(A,B)=1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A,B\rangle$ is a Jørgensen group such that A is parabolic (resp. elliptic) and $J(A,B)=1$, then we call it a *Jørgensen group of parabolic type* (resp. *of elliptic type*). There are infinite number of Jørgensen groups (Jørgensen-Lascurain-Pignataro [**3**], Sato [**8**]).

Now it gives rise to the following problem.

PROBLEM 1. Find all Jørgensen groups of parabolic type.

2.3. Let $\langle A,B\rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B:=B_{\sigma,\mu}=\begin{pmatrix} \mu\sigma & \mu^2\sigma-1/\sigma \\ \sigma & \mu\sigma \end{pmatrix}$$

where $\sigma\in\mathbf{C}\backslash\{0\}$ and $\mu\in\mathbf{C}$ (see [**4**] for the detail).

We denote by $G_{\sigma,\mu}$ the marked group generated by A and $B_{\sigma,\mu}$: $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$. We say that $(\sigma,\mu) \in \mathbf{C}\backslash\{0\} \times \mathbf{C}$ *is the point representing a marked group* $G_{\sigma,\mu}$ *and that* $G_{\sigma,\mu}$ *is the marked group coresponding to a point* (σ,μ).

2.4. In [8], Sato considered the case of $\mu = ik$ $(k \in \mathbf{R})$. Namely, he considered marked two-generator groups $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix}$$

where $\sigma \in \mathbf{C}\backslash\{0\}$ and $k \in \mathbf{R}$.

Now we have the following conjecture.

CONJECTURE. For any Jørgensen group G of parabolic type there exists a marked group $G_{\sigma,ik}$ $(\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R})$ such that $G_{\sigma,ik}$ is conjugate to G.

If this conjecture is true, then it is suffucient to consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of $\mu = ik$.

2.5. Let C be the following cylinder:

$$C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}.$$

THEOREM C (Sato [8]). *If a marked two-generator group* $G_{\sigma,ik}$ $(\sigma \in \mathbf{C}\backslash\{0\}, k \in \mathbf{R})$ *is a Jørgensen group, then the point* (σ, ik) *representing* $G_{\sigma,ik}$ *lies on the cylinder* C.

If (σ, ik) is a point on the cylinder C, then we set $\sigma = -ie^{i\theta}$ $(0 \le \theta \le 2\pi)$. For simplicity we write $B_{\theta,k}$ and $G_{\theta,k}$ for $B_{\sigma,ik}$ and $G_{\sigma,ik}$ with $\sigma = -ie^{i\theta}$, respectively. If $G_{\theta,k}$ is a Jørgensen group, then we call the group a *Jørgensen group of parabolic type* (θ, k).

Now it gives rise to the following problem.

PROBLEM 2. Find all Jørgensen groups of parabolic type (θ, k).

2.6. We devide Jørgensen groups of this type into three parts as follows:

Part 1. $|k| \leq \sqrt{3}/2,\ 0 \leq \theta \leq 2\pi$ (finite case).

Part 2. $\sqrt{3}/2 < |k| \leq 1,\ 0 \leq \theta \leq 2\pi$ (countably infinte case).

Part 3. $1 < |k|,\ 0 \leq \theta \leq 2\pi$ (uncountably infinte case).

By some lemmas in [**8**], it suffices to consider the case of $0 \leq \theta \leq \pi/2$ and $k \geq 0$ for solvoing Problem 2.

In the previous papers [**4, 5**] we found all Jørgensen groups of finite case and of countably infinite case. In this paper we will announce that we found all Jørgensen groups of parabolic type (θ, k) (Li - Oichi - Sato [**4,5,6**]. That is, Problem 2 is completely solved. The proofs will appear elthwhere.

3. TEOREMS.

THEOREM 1 (Li-Oichi-Sato [4]) (Finite case).
Let $D = \{(\theta, k) \mid 0 \leq \theta \leq \pi/2,\ 0 \leq k \leq \sqrt{3}/2\}$ *and* $\Omega = \{G_{\theta,k} \mid (\theta, k) \in D\}$. *Then*

(1) *There are 16 Jørgensen groups* $G_{\theta,k}$ *in the region* Ω. *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

(2) *The other* $G_{\theta,k}$ *in* Ω *are not Kleinian groups.*

The following table describes 16 Jørgensen groups:

#	$G_{\theta,k}$	θ	k	Kleinian groups	facts
1	$G_{0,0}$	0	0	2-nd kind	(a)
2	$G_{0,1/2}$		$1/2$	2-nd kind	
3	$G_{0,\sqrt{2}/2}$		$\sqrt{2}/2$	2-nd kind	
4	$G_{0,(1+\sqrt{5})/4}$		$(1+\sqrt{5})/4$	2-nd kind	
5	$G_{0,\sqrt{3}/2}$		$\sqrt{3}/2$	2-nd kind	
6	$G_{\pi/6,0}$	$\pi/6$	0	1-st kind	(b)
7	$G_{\pi/6,\sqrt{3}/2}$		$\sqrt{3}/2$	1-st kind	
8	$G_{\pi/4,0}$	$\pi/4$	0	2-nd kind	(c)
9	$G_{\pi/4,1/2}$		$1/2$	1-st kind	
10	$G_{\pi/3,0}$	$\pi/3$	0	1-st kind	(d)
11	$G_{\pi/3,\sqrt{3}/2}$		$\sqrt{3}/2$	1-st kind	
12	$G_{\pi/2,0}$	$\pi/2$	0	2-nd kind	(e)
13	$G_{\pi/2,1/2}$		$1/2$	1-st kind	
14	$G_{\pi/2,\sqrt{2}/2}$		$\sqrt{2}/2$	1-st kind	
15	$G_{\pi/2,(1+\sqrt{5})/4}$		$(1+\sqrt{5})/4$	1-st kind	
16	$G_{\pi/2,\sqrt{3}/2}$		$\sqrt{3}/2$	1-st kind	

THEOREM 2 (Li-Oichi-Sato [5]) (Countably infinite case).

The group $G_{\theta,k}$ *with* $0 \leq \theta \leq \pi/2$ *and* $\sqrt{3}/2 < k \leq 1$ *is a Jørgensen group (discrete) if and only if one of the following conditions holds.*

(a) $\theta = 0$ *and* $k = 1$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_{\theta,k})/G_{\theta,k}$ *is a union of two Riemann surfaces with signature* $(0; 2, 3, \infty)$.

(b) $\theta = 0$ *and* $k = \cos(\pi/n)$ $(n = 7, 8, \ldots)$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_k)/G_k$ *is a union of two Riemann surfaces with signatures* $(0; 2, 3, n)$ *and* $(0; 2, 3, \infty)$.

(c) $\theta = \pi/4$ *and* $k = 1$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/4,1})$ *of the* 3*-orbifold for* $G_{\pi/4,1}$ *is*

$$V(G_{\pi/4,1}) = 8(2L(\pi/4) - L(\pi/12) - L(5\pi/12)).$$

(d) **(Sato-Yamada [11])** $\theta = \pi/2$ *and* $k = 1$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 3, 3, \infty)$.

(e) **(Sato-Yamada [11])** $\theta = \pi/2$ *and* $k = \cos(\pi/n)$ $(n = 7, 8, \ldots)$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 3, 3, n)$.

COROLLARY. *There are countably infinite Jørgensen groups on the region* $\{(\theta, k) \mid 0 \leq \theta \leq \pi/2, \sqrt{3}/2 < k \leq 1\}$.

THEOREM 3 (Uncountably infinite case).

The group $G_{\theta,k}$ *with* $0 \leq \theta \leq \pi/2$ *and* $1 < k$ *is a Jørgensen group if and only if one of the following conditions holds.*

(a) $\theta = 0$ *and* $k > 1$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_{\theta,k})/G_{\theta,k}$ *is a union of two Riemann surfaces with signatures* $(0; 2, 3, \infty)$ *and* $(0; 2, 2, 2, 3)$.

(b) (1) $\theta = \pi/6$ *and* $k = \sqrt{3}n/2$ $(n = 2, 4, 6, \ldots)$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/6,\sqrt{3}n/2})$ *of the* 3*-orbifold for* $G_{\pi/6,\sqrt{3}n/2}$ *is*

$$V(G_{\pi/6,\sqrt{3}n/2}) = 3L(\pi/3).$$

(2) $\theta = \pi/6$ *and* $k = \sqrt{3}n/2$ $(n = 3, 5, 7, \ldots)$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/6,\sqrt{3}n/2})$ *of the* 3*-orbifold for* $G_{\pi/6,\sqrt{3}n/2}$ *is*

$$V(G_{\pi/6,\sqrt{3}n/2}) = 6L(\pi/3).$$

(c) (1) $\theta = \pi/4$ *and* $k = 3/2$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/4,3/2})$ *of the* 3*-orbifold for* $G_{\pi/4,3/2}$ *is*

$$\begin{aligned} &V(G_{\pi/4,3/2}) \\ &\quad = 3\{7L(\pi/3)/2 - L(\varphi_0 + \pi/6) + L(\varphi_0 - \pi/6)\} \\ &\quad = 3V(G_{\pi/2,1/2}), \end{aligned}$$

where $\varphi_0 = \sin^{-1}(1/2\sqrt{3})$.

(2) $\theta = \pi/4$ *and* $k = 1 + \sqrt{2}/2$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/4,1+\sqrt{2}/2})$ *of the* 3*-orbifold for* $G_{\pi/4,1+\sqrt{2}/2}$ *is*

$$\begin{aligned} &V(G_{\pi/4,1+\sqrt{2}/2}) \\ &\quad = 2(2L(\pi/4) - L(5\pi/12) - L(\pi/12)) \\ &\quad +4V(G_{\pi/2,1/2}) \\ &\quad = V(G_{\sqrt{2}\pi/2,1/2}) + 4V(G_{\pi/2,1/2}). \end{aligned}$$

(3) $\theta = \pi/4$ *and* $k = (5+\sqrt{5})/4$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/4,(5+\sqrt{5})/4})$ *of the* 3*-orbifold for* $G_{\pi/4,(5+\sqrt{5})/4}$ *is*

$$\begin{aligned} &V(G_{\pi/4,1+(1+\sqrt{5})/4}) \\ &\quad = 2L(\pi/10) + 2L(2\pi/5) + 4V(G_{\pi/2,1/2}) \\ &\quad = V(G_{\pi/4,(1+\sqrt{5})/4}) + 4V(G_{\pi/2,1/2}). \end{aligned}$$

(4) $\theta = \pi/4$ *and* $k = 1+\sqrt{3}/2$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/4,1+\sqrt{3}/2})$ *of the* 3*-orbifold for* $G_{\pi/4,1+\sqrt{3}/2}$ *is*

$$\begin{aligned} &V(G_{\pi/4,1+\sqrt{3}/2}) \\ &\quad = 5L(\pi/3) + 4V(G_{\pi/2,1/2}) \\ &\quad = V(G_{\pi/4,\sqrt{3})/2}) + 4V(G_{\pi/2,1/2}). \end{aligned}$$

(5) $\theta = \pi/4$ *and* $k = 1 + \cos(\pi/n)$ $(n = 7, 8, \ldots)$. *In this case,* G_k *are Kleinian groups of the second kind, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 3, 3, n)$.

(6) $\theta = \pi/4$ *and* $k = 2$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 3, 3, \infty)$.

(7) $\theta = \pi/4$ *and* $k > 2$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 2, 3, 3)$.

(d) $\theta = \pi/3$ *and* $k = \sqrt{3}n/2$ $(n = 2, 3, \ldots)$. *In this case,* G_k *is a Kleinian group of the first kind, and the volume* $V(G_{\pi/3,\sqrt{3}n/2})$ *of the* 3*-orbifold for* $G_{\pi/3,\sqrt{3}n/2}$ *is*

$$V(G_{\pi/3,\sqrt{3}n/2}) = 3L(\pi/3).$$

(e) **(Sato-Yamada [11])** $\theta = \pi/2$ *and* $k > 1$. *In this case,* G_k *is a Kleinian group of the second kind, and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 2, 3, 3)$.

COROLLARY. *There are uncountably infinite Jørgensen groups on the region* $\{(\theta, k) \mid 0 \leq \theta \leq \pi/2, 1 < k\}$.

4. OPEN PROBLEMS.

1. Find all Jørgensen groups of elliptic type.

2. Let r be a real number with $r \geq 1$. Then is there a discrete group whose Jørgensen number is equal to r ? (cf. Sato [**10**])

3. The group $G_{\pi/6, \sqrt{3}/2}$ is conjugate to the figure-eight knot group (cf. Sato [**8**]). The group $G_{\pi/2,0}$ is conjugate to the modular group. The group $G_{\pi/2, 1/2}$ is conjugate to the Picard group (cf. Sato [**9**]). Are Jørgensen groups $G_{\pi/2, \sqrt{2}/2}$, $G_{\pi/2,(1+\sqrt{5})/4}$, and $G_{\pi/2,\sqrt{3}/2}$ conjugate to familiar groups, respectively?

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