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NOTES ON DISCRETE SUBGROUPS OF PU(1,2;C) WITH PARABOLIC ELEMENTS

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1 Introduction

In the study of discrete groups it is important to find out conditions for a group to be discrete. Shimizu's lemma gives a necessary condition for a subgroup of PSL(2;C) containing a parabolic element to be discrete. In this paper we give analogues of Shimizu's lemma for a subgroup of isometries of complex hyperbolic 2-space.
This is a joint work with John. R. Parker (University of Durham).

2 Shimizu's lemma

Let $B(z) = (az+b)/(cz+d)$ with $a, b, c, d \in \mathbf{C}$ and $ad - bc = 1$. If B does not fix ∞, the isometric circle $I(B)$ of B is defined as a circle centered at $B^{-1}(\infty)$ with radius $1/|c|$, that is,

$$I(B) = \left\{ z \in \hat{\mathbf{C}} | \ |z - B^{-1}(\infty)| = \frac{1}{|c|} \right\}.$$

We denote the radius of $I(B)$ by r_B.

Theorem 2.1 ([11], [13]). *Let G be a discrete subgroup of* PSL(2;C) *containing a parabolic element A with $A(z) = z + t$ $(t > 0)$. Then for any element B of G with $B(\infty) \neq \infty$, $r_B \leq t$.*

This theorem is known as Shimizu's lemma. In general, we say that a set Y is precisely invariant under the subgroup H in G, if
(1) H is the stabilizer of Y in G, and

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(2) $B(Y) \cap Y = \emptyset$ for all $B \in G \setminus H$.
Where there is no danger of confusion, we will simply say that Y is precisely invariant under H. As a corollary to Theorem 2.1, we have

Corollary 2.2. *Let Γ be a Fuchsian group acting on the upper half plane $\mathbf{H}^1_{\mathbf{C}} = \{z \in \mathbf{C} | \ Im(z) > 0\}$, and let $A \in \Gamma$ with $A(z) = z + t \ (t > 0)$. If the stabilizer Γ_∞ of ∞ is generated by A, then*

$$U = \{z \in \mathbf{H}^1_{\mathbf{C}} | \ Im(z) > t\}$$

is precisely invariant under Γ_∞.

Remark 2.3. This corollary shows that the action of Γ on U is the same as that of the cyclic subgroup generated by A on U, whenever A generates the stabilizer of ∞.

3 Preliminaries

We give some definitions and fix notation. Let $\mathbf{C}^{2,1}$ be a complex vector space of dimension 3, equipped with the Hermitian form of signature (2,1) given by

$$\langle z^*, w^* \rangle = z_1^* \overline{w_3^*} + z_2^* \overline{w_2^*} + z_3^* \overline{w_1^*}$$

for $z^* = (z_1^*, z_2^*, z_3^*)$, $w^* = (w_1^*, w_2^*, w_3^*) \in \mathbf{C}^{2,1}$. An automorphism A of $\mathbf{C}^{2,1}$, that is a linear bijection such that $\langle A(z^*), A(w^*) \rangle = \langle z^*, w^* \rangle$ for any $z^*, w^* \in \mathbf{C}^{2,1}$, is called a unitary transformation. We denote the group of all unitary transformations by U(1,2; C). Let V_0 be the set of points z^* in $\mathbf{C}^{2,1}$ such that $\langle z^*, z^* \rangle = 0$ and let V_- be the set of points z^* in $\mathbf{C}^{2,1}$ satisfying $\langle z^*, z^* \rangle < 0$. It is clear that both V_0 and V_- are invariant under U(1,2; C). Let π be the canonical projection map from $\mathbf{C}^{2,1} - \{0\}$ to $\pi(\mathbf{C}^{2,1} - \{0\})$ defined by $\pi(z_1^*, z_2^*, z_3^*) = (z_1, z_2)$, where $z_i = z_i^*/z_3^*$ for $i = 1, 2$. We write ∞ for $\pi(1, 0, 0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$\mathbf{H}^2_{\mathbf{C}} = \{(z_1, z_2) \in \mathbf{C}^2 | \ 2Re(z_1) + |z_2|^2 < 0\}.$$

Set PU(1,2;C)=U(1,2;C)/(center). We can introduce the Bergman metric in $\mathbf{H}^2_{\mathbf{C}}$. With respect to this metric, an element of PU(1,2;C) acts on $\mathbf{H}^2_{\mathbf{C}}$ as an isometry. We see that PU(1,2;C) is the group of all biholomorphic isometries of $\mathbf{H}^2_{\mathbf{C}}$. Now we define H-coordinate system in $\overline{\mathbf{H}^2_{\mathbf{C}}} - \{\infty\}$, where $\overline{\mathbf{H}^2_{\mathbf{C}}} = \mathbf{H}^2_{\mathbf{C}} \cup \partial \mathbf{H}^2_{\mathbf{C}}$. The H-coordinates of a point (z_1, z_2) in $\overline{\mathbf{H}^2_{\mathbf{C}}} - \{\infty\}$ are defined by $(\zeta, v, k)_H$ in $\mathbf{C} \times \mathbf{R} \times \mathbf{R}_+$, where $z_1 = -|\zeta|^2 - k + iv$ and $z_2 = \sqrt{2}\zeta$.

We introduce the Cygan metric ρ, which is appropriate to our situation. The Cygan metric $\rho(p, q)$ for $p = (\zeta_1, v_1, k_1)_H$, $q = (\zeta_2, v_2, k_2)_H$ is defined by

$$\rho(p,q) = ||\zeta_1 - \zeta_2|^2 + |k_1 - k_2| + i(v_1 - v_2 + 2Im(\zeta_1\overline{\zeta_2}))|^{\frac{1}{2}}.$$

We note that this Cygan metric is invariant under Heisenberg translations. Now we define the isometric sphere $I(B)$ of an element B of PU(1,2;**C**) with $B(\infty) \neq \infty$. If B is of the form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix},$$

then the isometric sphere $I(B)$ of B is defined as ρ-sphere centered at $B^{-1}(\infty)$ with radius $\sqrt{1/|g|}$, that is,

$$I(B) = \left\{ z \in \overline{\mathbf{H}^2_{\mathbf{C}}} \mid \rho(z, B^{-1}(\infty)) = \sqrt{\frac{1}{|g|}} \right\}.$$

We denote the radius of $I(B)$ by R_B.

4 Discrete subgroups of PU(1,2; C) with parabolic elements

We show complex hyperbolic versions of Shimizu's lemma.

Theorem 4.1 ([6], [7]). *Let G be a discrete subgroup of* PU(1,2; **C**), *which contains a vertical translation A with the form*

$$\begin{pmatrix} 1 & 0 & it \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $t > 0$. Then for any element B of G with $B(\infty) \neq \infty$,

$$R_B^2 \leq t.$$

Theorem 4.2 ([12]). *Let G be a discrete subgroup of* PU(1,2; **C**). *Let $A \in G$ be a Heisenberg translation with the form*

$$\begin{pmatrix} 1 & -\sqrt{2}\overline{\tau} & s+it \\ 0 & 1 & \sqrt{2}\tau \\ 0 & 0 & 1 \end{pmatrix},$$

where $s = -|\tau|^2$. *If* B *is an element of* G *with* $B(\infty) \neq \infty$, *then*

$$R_B^2 \leq \rho(AB(\infty), B(\infty))\rho(AB^{-1}(\infty), B^{-1}(\infty)) + 4|\tau|^2.$$

Remark 4.3. If A is a vertical translation, then $\rho(AB(\infty), B(\infty))\rho(AB^{-1}(\infty), B^{-1}(\infty)) = t$ and $\tau = 0$. Therefore we have $R_B^2 \leq t$. Thus Theorem 4.2 is a generalization of Theorem 4.1.

By using Theorems 4.1 and 4.2, we can construct precisely invariant regions.

Theorem 4.4 ([7], [12]). *Let* G *be a discrete subgroup of* PU(1,2; **C**). *Assume that the stabilizer* G_∞ *of* ∞ *consists of Heisenberg translations.*

(1) If G_∞ *contains a vertical translation* A, *then*

$$U_A = \{(\zeta, v, k)_H | \ k > \rho(A(z), z)^2 = t\}$$

is precisely invariant under G_∞.

(2) If G_∞ *does not contain a vertical translation* A, *then*

$$U_A = \{(\zeta, v, k)_H | \ k > \rho(A(z), z)^2 + 8|\tau|^2\}$$

is precisely invariant under G_∞.

Next we discuss a discrete group with a screw parabolic element.

Theorem 4.5. *Let* G *be a discrete subgroup of* PU(1,2; **C**) *containing a screw parabolic element* A *with the form*

$$\begin{pmatrix} 1 & 0 & it \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $u = e^{i\theta}$, $|u-1| < \frac{1}{4}$ *and* $t \sin\theta > 0$. *If* B *is an element of* G *with* $B(\infty) \neq \infty$, *then*

$$R_B^2 \leq \frac{\rho(AB(\infty), B(\infty))\rho(AB^{-1}(\infty), B^{-1}(\infty))}{K^2},$$

where $K = \frac{1+\sqrt{1-4|u-1|}}{2}$.

Remark 4.6. If $|u-1| = 0$, then A is a vertical translation and $R_B^2 \leq t$.

We construct a precisely invariant region in the case where a discrete group G contains a screw parabolic element.

Theorem 4.7. *Let G be a discrete subgroup of* PU(1,2; **C**). *Let A be a screw parabolic element of G with the form*

$$\begin{pmatrix} 1 & 0 & it \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $u = e^{i\theta}$, $|u-1| < \frac{2}{9}$ and $t \sin\theta > 0$. Assume that the stabilizer of ∞ is generated by A. Then the sub-horospherical regin U defined by

$$U = \left\{ (\zeta, v, k)_H \middle| \ k > \frac{2|2|\zeta|^2(u-1)+it|}{1-6|u-1|+\sqrt{1-4|u-1|}} \right\}.$$

is precisely invariant under G_∞ in G.

Remark 4.8. If $u = 1$, then A is a vertical translation and $U = \{(\zeta, v, k)_H | \ k > t\}$, which coincides with U_A in (1) of Theorem 4.4.

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