数理解析研究所講究録 1387 巻 2004 年 59-69

End invariants and Jorgensen's angle invariants of punctured torus groups

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1 Introduction

Let T be the once-punctured torus, $\mathcal{QF}$ its quasifuchsian space, and $\overline{\mathcal{QF}}$ be the closure of $\mathcal{QF}$ in the representation space. Then the end invariant $\lambda = (\lambda^-, \lambda^+)\colon \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ is defined by using conformal structures on the ideal boundary of hyperbolic 3-manifolds obtained from elements of $\overline{\mathcal{QF}}$. On the other hand, the angle invariant $\nu = (\nu^-, \nu^+)\colon \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ is also defined by using the combinatorial structure of the Ford domains of Kleinian groups obtained from elements of $\overline{\mathcal{QF}}$. Let us denote by $m_\rho^\pm(\gamma)$ the extremal length of a loop γ on T with respect to the conformal structure on the ideal boundary. The main theorem in this paper is the following.

Theorem 1.1. *The following holds.*

(1) *There is a positive number ϵ_0 with the following property. If $\rho \in \overline{\mathcal{QF}}$ satisfies the conditions that $\nu^s(\rho)$ is contained in a Farey triangle $\langle s_0, s_1, s_2\rangle$ and that $\pi/2 - \theta^s_{s_0}(\rho) < \epsilon_0$, then the extremal length $m^s_\rho(\gamma)$ is the minimum among those lengths of essential simple loops, where γ is a loop corresponding to s_0.*

(2) *For any $\epsilon > 0$, there is $\delta > 0$ with the following property. If $\rho \in \overline{\mathcal{QF}}$ satisfies the conditions that $\nu^s(\rho)$ is contained in a Farey triangle $\langle s_0, s_1, s_2\rangle$ and that $\pi/2 - \theta^s_{s_j}(\rho) \geq \epsilon$ for all $j \in \{0,1,2\}$, then the hyperbolic distance between $\nu^s(\rho)$ and $\lambda^s(\rho)$ is at most δ.*

The angle invariant[1] is defined in an unfinished paper [Jor] by Jorgensen. In the paper, he claims that the restriction of the angle invariant to $\mathcal{QF}$,

[1]This is called the "side parameter" in the Jorgensen's paper.

$\nu|_{\mathcal{QF}}\colon \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$, is a bijective map. In the joint work of Sakuma, Wada, Yamashita and the author, they try to give the proof to the claim. However, they have the proof at present only for the surjectivity and the continuity of $\nu|_{\mathcal{QF}}$.

Question 1.2. Is $\nu|_{\mathcal{QF}}\colon \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ injective?

One way to answer the question is to show that $\nu|_{\mathcal{QF}}\colon \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is both proper and locally injective. As a corollary to Theorem 1.1, we obtain the following.

Corollary 1.3. *The map $\nu|_{\mathcal{QF}}\colon \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is proper.*

2 Punctured torus

Let T be the once-punctured torus, and fix a marking $\{\alpha, \beta\} \subset H_1(T)$ of T. The Teichmüller space, $\mathcal{T}$, of T can be identified with the hyperbolic plane $\mathbb{H}^2$ as follows. For any ω in the upper half plane, which is identified with $\mathbb{H}^2$, one can construct a marked conformal structure, λ_ω, on T from the parallelogram $\langle 0, 1, 1+\omega, \omega\rangle$ in $\mathbb{C}$ by identifying the opposite edges by parallel translations, removing a point, and identifying the loop α (resp. β) with that obtained from the interval $[0,1]$ (resp. $[0,\omega]$). Then it can be seen that the map $\mathbb{H}^2 \ni \omega \mapsto \lambda_\omega \in \mathcal{T}$ gives a global parametrization.

For any loop γ on T, the *extremal length*, $m_\lambda(\gamma)$, of γ with respect to a conformal structure λ is equal to $l_\mu(\gamma^*)^2/A(\mu)$, where $l_\mu(\gamma^*)$ is the length of the geodesic γ^* freely homotopic to γ with respect to the Euclidean metric μ in the conformal class λ defined as above, and $A(\mu)$ is the area of T with respect to μ (see [Ahl]).

3 Quasifuchsian space and end invariants

In this section, we summarize definitions and results concerning the quasifuchsian space of T and the end invariants of marked punctured torus groups. The study on this subject has a long history, which has been proceeded by Ahlfors, Bers, Kra, Marden, Maskit, Thurston, Bonahon, Minsky and many others.

Fix an arbitrary complete hyperbolic structure on T of finite volume, and let $\rho_0\colon \pi_1(T) \to PSL(2,\mathbb{R})$ be the holonomy representation for it. By composing with the natural embedding $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$, ρ_0 can be regarded as an element of $\mathrm{Hom}(\pi_1(T), PSL(2,\mathbb{C}))$. Let $\mathcal{R}$ be the representation space of $\pi_1(T)$ to $PSL(2,\mathbb{C})$, i.e., $\mathcal{R}$ is the quotient space of

$\mathrm{Hom}(\pi_1(T), PSL(2,\mathbb{C}))$ by conjugations. We define the *quasifuchsian space*, $\mathcal{QF}$, of T to be the subspace of $\mathcal{R}$ consisting of the conjugacy classes $[\rho]$ of $\rho \in \mathrm{Hom}(\pi_1(T), PSL(2,\mathbb{C}))$ such that $\rho(g) = w \circ \rho_0(g) \circ w^{-1}$ $(\forall g \in \pi_1(T))$ for some quasiconformal homeomorphism $w\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. We denote the closure of $\mathcal{QF}$ in $\mathcal{R}$ by $\overline{\mathcal{QF}}$. From now on, we abuse the notations and write the conjugacy classes in the representation space as representations. Then it can be seen that every element ρ of $\overline{\mathcal{QF}}$ is a faithful, discrete, type-preserving representation.

For an arbitrary element $\rho \in \overline{\mathcal{QF}}$, let M_ρ be the quotient manifold $\mathbb{H}^3/\rho(\pi_1(T))$. Then M_ρ is homeomorphic to $T \times (-1,1)$. The domain of discontinuity Ω for the Kleinian group $\rho(\pi_1(T))$ is partitioned into two $\rho(\pi_1(T))$-invariant subsets $\Omega_\pm$, each with the property that the points in the set can be reached from the subset of M_ρ corresponding to $T \times (-1,0]$ or $T \times [0,1)$ according to the signs $\pm$. Each Ω_s $(s \in \{-,+\})$ satisfies one of the following three conditions:

1. Ω_s is a topological disk, and $\Omega_s/\rho(\pi_1(T))$ is homeomorphic to T.

2. Ω_s is an infinite union of round disks, and $\Omega_s/\rho(\pi_1(T))$ is homeomorphic to the thrice-punctured sphere.

3. Ω_s is empty.

If the condition 3 in the above is satisfied, then there is a sequence $\{\gamma_n\}$ of essential simple loops in T whose geodesic representative $\{\gamma_n^*\}$ exits the corresponding "end" of M_ρ. It can be seen that the sequence $\{\gamma_n\}$ converges in the projective lamination space, $\mathcal{PML}$, of T. Moreover the limit does not depend on the choice of such sequence $\{\gamma_n\}$.

Recall that the Teichmüller space, $\mathcal{T}$, of T is naturally identified with $\mathbb{H}^2$. Then the pair $(\mathcal{T}, \mathcal{PML})$ is identified with $(\mathbb{H}^2, \partial\mathbb{H}^2)$ via Thurston compactification.

Definition 3.1. The end invariant

$$\lambda = (\lambda^-, \lambda^+)\colon \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$$

is defined as

1. $\lambda^s(\rho) \in \mathcal{T}$ is the marked conformal structure of $\Omega_s/\rho(\pi_1(T))$ when Ω_s is a topological disk;

2. $\lambda^s(\rho) \in \mathcal{PML}$ is the marked conformal structure of $\Omega_s/\rho(\pi_1(T))$ when Ω_s is an infinite union of round disks;

3. $\lambda^s(\rho) \in \mathcal{PML}$ is the limit of a sequence $\{\gamma_n\}$ such that its geodesic representative exits the corresponding "end" of M_ρ when Ω_s is empty.

The following theorem is proved by Minsky [Min].

Theorem 3.2. *The following holds.*

1. *The map* $\lambda\colon \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$ *is bijective. Moreover, the map* λ^{-1} *is continuous.*

2. *Every marked punctured torus group, i.e., the image of* $\rho \in \mathcal{R}$ *which is faithful and sends peripheral elements to parabolics, is contained in* $\overline{\mathcal{QF}}$.

3. *Every Bers slice is a closed disk, and every Maskit slice is a closed disk with one boundary point removed.*

4 Generators of $\pi_1(T)$

Convention 4.1. In the rest of the paper, we fix a peripheral element K of $\pi_1(T)$.

Definition 4.2. We call a pair of elements, (A, B), of $\pi_1(T)$ a *generator pair* if A and B generates $\pi_1(T)$ and satisfies $ABA^{-1}B^{-1} = K$. For such a pair, A (resp. B) is called a *left* (resp. *right*) *generator*, or simply a *generator*.

Remark 4.3. The situation may be more clear if we introduce the notion of *elliptic generator triple*, for which we need to extend the group $\pi_1(T)$ to the fundamental group of the orbifold obtained as the quotient space of T by the hyperelliptic involution (cf. [ASWY1]).

One can see that every generator in the above sense has a simple closed curve in T as a representative. Recall that we have fixed a framing $\{\alpha, \beta\} \subset H_1(T)$ in Section 2.

Definition 4.4. For each generator X which represents an element $p\alpha+q\beta \in \mathbb{H}_1(T)$, the *slope*, $s(X)$, of X is defined by $p/q \in \widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.

Definition 4.5. The Farey triangulation of $\mathbb{H}^2$ is an ideal triangulation consisting of the ideal triangles $\{\gamma\sigma_0 \mid \gamma \in PSL(2, \mathbb{Z})\}$, where σ_0 is the ideal triangle with vertices $\infty, 0, 1 \in \partial\mathbb{H}^2$.

Lemma 4.6. *The following holds.*

1. *For any generator pair (A, B), the slopes of A, AB and B span an ideal triangle in the Farey triangulation.*

2. *For any ideal edge (resp. ideal triangle) σ in the Farey triangulation, there is a generator pair (A, B) such that the slopes of A and B (resp. A, AB and AB) span σ.*

5 Angle invariants

The combinatorial structure of the Ford domain of a Kleinian group $\rho(\pi_1(T))$ for $\rho \in \overline{\mathcal{QF}}$ is completely understood via Jorgensen's geometric continuity argument ([Jor], cf. [ASWY1, ASWY2]).

In what follows we use the upper half space model of $\mathbb{H}^3$. So, $\mathbb{H}^3$ and $\partial\mathbb{H}^3$ are identified with the upper half space of $\mathbb{R}^3$ and the Riemann sphere $\widehat{\mathbb{C}}$ respectively.

Definition 5.1. For an element $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{C})$ with $c \neq 0$, the *isometric circle*, $I(\gamma)$, of γ is defined to be the circle in $\mathbb{C}$ with center $-d/c$ and radius $1/|c|$, and the *isometric hemisphere*, $Ih(\gamma)$, of γ is defined to be the totally geodesic plane in $\mathbb{H}^3$ bounded by $I(\gamma)$. We denote the exterior of $I(\gamma)$ in $\mathbb{C}$ (resp. the exterior of $Ih(\gamma)$ in the upper half space) by $E(\gamma)$ (resp. $Eh(\gamma)$).

Definition 5.2. For a Kleinian group Γ, let Γ_∞ be the subgroup of Γ consisting of the elements which stabilizes ∞. The *(extended) Ford domain*, $Eh(\Gamma)$, of Γ is defined by

$$Eh(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} Eh(\gamma).$$

Similarly, we define the *(extended) Ford polygon*, $E(\Gamma)$, of Γ by

$$E(\Gamma) = \bigcap_{\gamma \in \Gamma - \Gamma_\infty} E(\gamma).$$

Remark 5.3. In general, the extended Ford domain of a Kleinian group Γ is not a fundamental domain of the action of Γ on $\mathbb{H}^3$. One needs to find a fundamental domain of the stabilizer subgroup of ∞ and then take the intersection with $Eh(\Gamma)$ to get a desired one.

In order to define the angle invariant, we need a normalization of the punctured torus groups.

Convention 5.4. In what follows, we fix a generator pair (A_0, B_0), and normalize $\rho \in \overline{\mathcal{QF}}$ so that $\rho(A_0B_0A_0^{-1}B_0^{-1}) = \rho(K)$ is the parallel translation by 2 and that the center of $I(\rho(A_0B_0))$ is the origin of $\mathbb{C}$.

The following proposition will be one of the most important results of the geometric continuity method in [Jor]. Now one can see a moving picture of Ford domains by using the computer program OPTi [Wad] by Wada.

Proposition 5.5. *Let ρ be an element of $\mathcal{QF}$. Then $E(\rho(\pi_1(T)))$ consists of precisely two connected components. Each of the components has the following property:*

1. *It is simply connected.*

2. *The boundary consists of circular arcs (edges) contained in the isometric circles of some generators. Moreover, the edges are partitioned into three (or two) families so that (i) the edges in the same family are contained in the isometric circles of generators with a common slope, (ii) the slopes of the families constitute the vertex set of a Farey triangle (or a Farey edge), (iii) the union of edges in the same family is invariant by the parallel translation by 1, and (iv) the edges in the same family are translated to one another by parallel translations $z \mapsto z+n$ $(n \in \mathbb{Z})$.*

3. *If the edges are partitioned into three families, then for any three consecutive edges, there is a generator pair (A, B) such that the three edges are contained in the isometric circles of $\rho(A)$, $\rho(AB)$ and $\rho(B)$ in this order. If the edges are partitioned into two families, then for any two consecutive edges, there is a generator pair (A, B) such that the three edges are contained in the isometric circles of $\rho(A)$ and $\rho(B)$ in this order. (See Figure 1.)*

4. *One of the components contains a region $\{z \in \mathbb{C} \mid \Im(z) \leq -L\}$ and the other contains $\{z \in \mathbb{C} \mid \Im(z) \geq L\}$ for a sufficiently large $L > 0$.*

Definition 5.6. Let ρ be an element of $\mathcal{QF}$. We denote by $E^-(\rho)$ (resp. $E^+(\rho)$) the component of $E(\rho(\pi_1(T)))$ containing $\{z \in \mathbb{C} \mid \Im(z) \leq -L\}$ (resp. $\{z \in \mathbb{C} \mid \Im(z) \geq L\}$) for a sufficiently large $L > 0$. Let $\sigma^s = \langle s_0, s_1, s_2 \rangle$ $(s \in \{-, +\})$ be the Farey triangle corresponding to $E^s(\rho)$. We denote by $\theta^s_{s_j}(\rho)$ the half of the angle of the boundary edge of $E^s(\rho)$ in $I(\rho(X))$ with $s(X) = s_j$, and let $\theta^s(\rho) = (\theta^s_{s_0}(\rho), \theta^s_{s_1}(\rho), \theta^s_{s_2}(\rho))$.

Proposition 5.7. *For any $\rho \in \mathcal{QF}$ and for each sign s, the sum of the components of $\theta^s(\rho)$ is equal to $\pi/2$.*

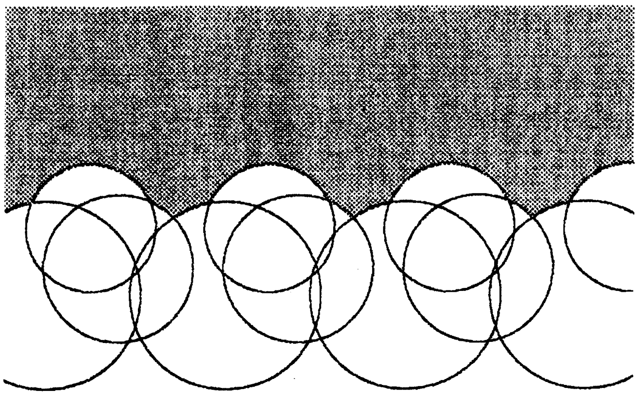

Figure 1: A component of $E(\rho(\pi_1(T)))$ with three families of edges: greyed region is the component. If the edges of one of the three families shrink, then the component changes into the one with two families of edges.

Definition 5.8. We define a map $\nu = (\nu^-, \nu^+)\colon \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ as follows. We call ν the *angle invariant.*

1. For each $s \in \{-, +\}$, let $\nu^s(\rho)$ lie in the Farey triangle, $\sigma^s = \langle s_0, s_1, s_2 \rangle$, corresponding to the component $E^s(\rho)$.
2. The triple $\theta^s(\rho)$ determines a point, which we define to be $\nu^s(\rho)$, in σ^s via the barycentric coordinates (cf. Proposition 5.7).

The following is the characterization of the combinatorial structures of the Ford domains of quasifuchsian punctured torus groups by Jorgensen (see [Jor], cf. [ASWY1, ASWY2]).

Theorem 5.9. *The angle invariant*

$$\nu = (\nu^-, \nu^+)\colon \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$$

satisfies the following conditions.

(1) *The combinatorial structure of the Ford domain of $\rho(\pi_1(T))$ is determined from the sequence of Farey triangles whose interiors intersect the geodesic segment connecting $\nu^-(\rho)$ and $\nu^+(\rho)$.*

(2) *The map ν is continuous with respect to the algebraic topology on $\mathcal{QF}$ and the product topology on $\mathbb{H}^2 \times \mathbb{H}^2$.*

(3) *The image $\nu(\mathcal{QF})$ is equal to $\mathbb{H}^2 \times \mathbb{H}^2$.*

One can apply Jorgensen's method to geometrically infinite boundary groups to obtain the following theorem ([Aki]).

Theorem 5.10. *The angle invariant extends to*

$$\nu = (\nu^-, \nu^+)\colon \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$$

so that the following conditions are satisfied.

(1) *The combinatorial structure of the Ford domain of $\rho(\pi_1(T))$ is determined from the sequence of Farey triangles whose interiors intersect the geodesic segment connecting $\nu^-(\rho)$ and $\nu^+(\rho)$.*

(2) *The map ν is continuous with respect to the strong topology on $\overline{\mathcal{QF}}$ and the product topology on $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2} - \mathrm{diag}(\partial\mathbb{H}^2)$, i.e., if a sequence $\{\rho_n\} \subset \overline{\mathcal{QF}}$ converges strongly to some $\rho_\infty \in \overline{\mathcal{QF}}$, then the sequence $\{\nu(\rho_n)\}$ converges to $\nu(\rho_\infty)$ in the product topology.*

(3) *For an element $\rho \in \overline{\mathcal{QF}}$ and a sign $s \in \{-,+\}$, $\nu^s(\rho) \in \partial\mathbb{H}^2$ if and only if $\lambda^s(\rho) \in \partial\mathbb{H}^2$. Moreover, if $\nu^s(\rho) \in \partial\mathbb{H}^2$, then $\nu^s(\rho) = \lambda^s(\rho)$.*

6 Idea of the proof of Theorem 1.1

Lemma 6.1. *For $\rho \in \overline{\mathcal{QF}}$, suppose that X is a generator such that $s(X)$ is a vertex of the Farey triangle containing $\nu^s(\rho)$. Take a point from an edge of $\partial E^s(\rho(\pi_1(T)))$ contained in $I(\rho(X))$. Then $I(\rho(X^{-1}))$ is the next isometric circle in the family of isometric circles of generators with slope $s(X)$. Moreover, the arc in $E^s(\rho(\pi_1(T)))$ connecting the point and the equivalent point in the edge contained in $I(\rho(X^{-1}))$ projects onto a loop in T contained in the free homotopy class of X. (See Figure 2.)*

Idea of the proof of Theorem 1.1(1). If $\pi/2-\theta^s_{s_0}(\rho)$ becomes smaller and smaller, then the figure of $E^s(\rho(\pi_1(T)))$ is more and more close to Figure 3. Thus one can find an arc in $E^s(\rho(\pi_1(T)))$ which projects onto a loop, γ, in T with slope s_0 and is contained in a wide Euclidean annulus, Q, in the conformal class (figure 4). Then it follows

$$m^s_\rho(\gamma) = m_{\lambda^s(\rho)}(\gamma) = l_{\mu^s(\rho)}(\gamma^*)^2/A(\mu^s(\rho)) \le l_{\mu^s(\rho)}(\gamma^*)^2/A(Q) \to 0,$$

where $\mu^s(\rho)$ is a Euclidean metric in the conformal class $\lambda^s(\rho)$, γ^* is a geodesic loop freely homotopic to γ with respect to $\mu^s(\rho)$, and $A(Q)$ is the area of a wide Euclidean annulus Q contained in the Euclidean torus $(T, \mu^s(\rho))$. Notice

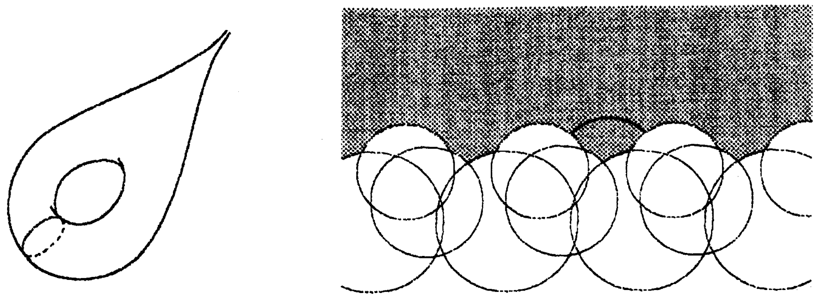

Figure 2: One can find a simple loop in the ideal boundary corresponding to generators which support $\partial E(\rho(\pi_1(T)))$ like this: the arc in the right figure projects onto a simple loop in T.

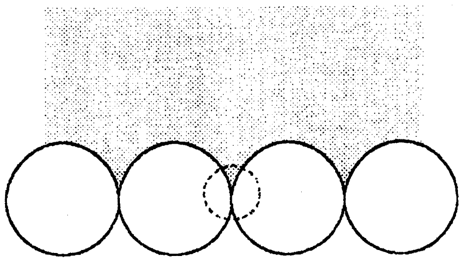

Figure 3: In the limit, $\partial E^s(\rho(\pi_1(T)))$ consists of only one family of edges that have the common slope s_0. The isometric circles of those generators are tangent to one another.

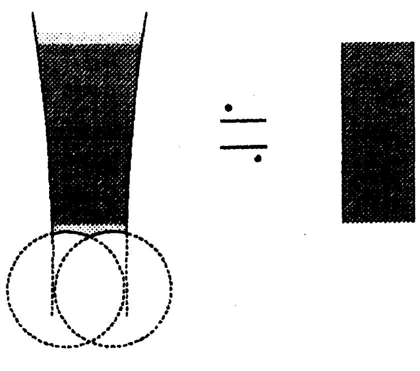

Figure 4: The left figure is obtained if one zooms up the region surrounded by the dotted circle in Figure 3. It is almost conformal to a wide Euclidean annulus depicted in the right figure.

the following two facts: (i) There is a relation $\pi m/2 \leq l \leq \pi m$ between the extremal length m and the hyperbolic length l in the conformal class ([Mas]). (ii) There is a constant $L_0 > 0$ such that if a hyperbolic length of a loop in T is less than L_0, then it is shortest. Then we can conclude that if $\pi/2 - \theta^s_{s_0}(\rho)$ is sufficiently small, then the extremal length $m^s_\rho(\gamma)$ is shortest. □

Idea of the proof of Theorem 1.1(2). By studying the proofs of Theorems 5.9 and 5.10 carefully, we can show the following key lemma.

Lemma 6.2. *For any $\epsilon > 0$, there is $\delta' > 0$ with the following property. If $\rho \in \overline{\mathcal{QF}}$ satisfies the conditions that $\nu^s(\rho)$ is contained in a Farey triangle $\langle s_0, s_1, s_2 \rangle$ and that $\pi/2 - \theta^s_{s_j}(\rho) \geq \epsilon$ for all $j \in \{0,1,2\}$, then the Euclidean distance in $\mathbb{C}$ between $E^s(\rho(\pi_1(T)))$ and the limit set of $\rho(\pi_1(T))$ is at least δ'.*

The key idea for the proof of Lemma 6.2 is to find a collar neighborhood of $\partial E^s(\rho(\pi_1(T)))$ in Ω_s by using the fact that $E^s(\rho(\pi_1(T)))$ is a fundamental domain for the action of $\rho(\pi_1(T))$ on Ω_s. We omit the detail here.

Once we obtain Lemma 6.2, the proof of Theorem 1.1(2) is immediate. By Koebe's theorem, there is $D > 0$ such that for any $x \in E^s(\rho(\pi_1(T)))$, the inequality $\|\mu_{hyp}|x\| \leq D\|\mu\ |x\|$ between the hyperbolic metric μ_{hyp} on Ω_s and the Euclidean metric μ on $\mathbb{C}$ holds. Then there exists $L > 0$ depending only on ϵ with the following property. If $\max\{l_{hyp}(\gamma_0), l_{hyp}(\gamma_1), l_{hyp}(\gamma_2)\} \leq L$, which is equivalent to the condition that $\max\{m^s_\rho(\gamma_0), m^s_\rho(\gamma_1), m^s_\rho(\gamma_2)\} \leq \pi L$ by [Mas], then the hyperbolic distance in $\mathbb{H}^2$ between the end invariant $\lambda^s(\rho)$ and the center of the Farey triangle $\langle s_0, s_1, s_2 \rangle$ is bounded. On the other hand, by the assumption, the hyperbolic distance in $\mathbb{H}^2$ between the angle parameter $\nu^s(\rho)$ and the center of the Farey triangle $\langle s_0, s_1, s_2 \rangle$ is also bounded. Thus the hyperbolic distance in $\mathbb{H}^2$ between $\lambda^s(\rho)$ and $\nu^s(\rho)$ is bounded. □

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