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Some Factorization Theorem for Hardy Spaces and Commutators on Morrey Spaces

(Joint Work with Yasuo Komori in Tokai University)

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Abstract of the Talk

We show a factorization theorem on Hardy space $H^1(R^n)$ in terms of the fractional integral operator and both functions in classical Morrey space and functions generated by blocks. Consequently, we show that the commutator $[M_b, I_\alpha]$ of the multiplication operatorM_b by b and the fractional integral operator I_α is bounded from the Morrey space $L^{p,\lambda}(R^n)$ to the Morrey space $L^{q,\lambda}(R^n)$ where $1<p<\infty, 0<\alpha<n, 0<\lambda<n-\alpha p$ and $1/q=1/p-\alpha/(n-\lambda)$ if and only if b belongs to $BMO(R^n)$.

1 Introduction

Let I_α, $0<\alpha<n$, be the fractional integral operator defined by

$$I_\alpha f(x)=\int_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

We consider the commutator

$$[M_b, I_\alpha]f(x)=b(x)I_\alpha f(x)-I_\alpha(bf)(x), \qquad b\in L^1_{\mathrm{loc}}(R^n).$$

Chanillo [1] and Komori [7] obtained the necessary and sufficient condition for which the commutator $[M_b, I_\alpha]$ is bounded on $L^p(R^n)$. Di Fazio and Ragusa [4] obtained the necessary and sufficient condition for which the commutator $[M_b, I_\alpha]$ is bounded on Morrey spaces for some α.

In this paper we refine their results in [4] by using the duality argument. Our proof is different from the one in [4].

Definition 1. (Morrey Spaces) Let $1\leq p<\infty, \lambda\geq 0$. We define the classical Morrey space by

$$L^{p,\lambda}(R^n)=\{f\in L^p_{\mathrm{loc}}(R^n)\,;\|f\|_{L^{p,\lambda}}<\infty\}$$

where

$$\|f\|_{L^{p,\lambda}} = \sup_{\substack{x\in R^n \\ t>0}} \left(\frac{1}{t^\lambda}\int_{B(x,t)} |f(y)|^p dy\right)^{1/p}.$$

Remark 1.(Some properties of Morrey space $L^{p,\lambda}(R^n)$) For the classical Morrey space $L^{p,\lambda}(R^n)$, the next results are well-known. If $1 \le p < \infty$, then $L^{p,0}(R^n) = L^p(R^n)$ and $L^{p,n}(R^n) = L^\infty(R^n)$ (isometrically), and if $n < \lambda$, then $L^{p,\lambda}(R^n) = \{0\}$. So we consider the case $0 \le \lambda \le n$.

Definition 2.(((q,r)-blocks, Taibleson and Weiss [13], Long [10]. See also Lu, Taibleson and Weiss [11])

Let $1 \le q < r \le \infty$. Then a function $b(x)$ is called a (q,r)-block, if there exists a ball $B(x_0,t)$ such that

$$\text{supp } b \subset B(x_0,t), \quad \|b\|_{L^r} \le t^{n(1/r-1/q)}.$$

Definition 3.(Function spaces generated by blocks, Long [10])

Let $1 \le q < r \le \infty$. We define the space generated by blocks by

$$h_{q,r}(R^n) = \{f = \sum_{j=1}^{\infty} m_j b_j \ ; b_j \text{ are } (q,r)-\text{blocks}, \ \|f\|_{h_{q,r}} < \infty\}$$

where

$$\|f\|_{h_{q,r}} = \inf \sum_{j=1}^{\infty} |m_j|.$$

where the infimum extends over all representations $f = \sum_{j=1}^{\infty} m_j b_j$.

Remark 2. Each (q,r)−block b_j belongs to $L^q(R^n)$ and $\|b_j\|_q \le 1$.

So the series of blocks $\sum_j m_j b_j$ converges in $L^q(R^n)$ and absolutely almost everywhere provided $\sum_j |m_j| < \infty$.

Hence each space $h_{q,r}(R^n)$ is a function space and a Banach space (see Long [10], p.17).

Definition 4. (Hardy space and John-Nirenberg space)

$H^1(R^n)$ is the Hardy space in the sense of Fefferman and Stein [5].

$BMO(R^n)$ is the John-Nirenberg space ([6]), that is, $BMO(R^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_{\substack{x\in R^n \\ t>0}} \frac{1}{|B(x,t)|}\int_{B(x,t)} |b(y) - b_B| dy$$

where

$$b_B = \frac{1}{|B(x,t)|}\int_{B(x,t)} b(y)dy.$$

Remark 3. Latter [9] obtained a decomposition theorem of Hardy space $H^1(R^n)$ in terms of atoms. Fefferman and Stein [5] showed that the Banach space dual of $H^1(R^n)$ is isomorphic to $BMO(R^n)$, that is,

$$\|b\|_* \approx \sup_{\|f\|_{H^1}\le 1}\left|\int b(x)f(x)dx\right|.$$

2. Known Results

The L^p theory about the commutator $[M_b, I_\alpha]$ is as follows ;

Theorem A.(Chanillo [1] and Komori [7])
Let $1/q = 1/p - \alpha/n, 1 < p < n/\alpha$ and $0 < \alpha < n$.
The commutator $[M_b, I_\alpha]$ is a bounded operator from $L^p(R^n)$ to $L^q(R^n)$ if and only if $b \in BMO(R^n)$.

Remark 4. Theorem A says about the results for the particular Morrey spaces $L^{p,0}(R^n)$ and $L^{q,0}(R^n)$.

Recently, Di Fazio and Ragusa [4] obtained the next results corresponding to index $\lambda, 0 < \lambda < n$.

Theorem B.(Di Fazio and Ragusa [4])
Let $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n-\lambda)$ and $1/q + 1/q' = 1$.
If $b \in BMO(R^n)$, then $[M_b, I_\alpha]$ is a bounded operator from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$.
Conversely, if $n - \alpha$ is an even integer and $[M_b, I_\alpha]$ is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$ for some p, q, λ as above, then $b \in BMO(R^n)$.

Remark 5. As we can see easily, the conditions for the converse part of Theorem B are very strong. In fact, when $n = 1, 2$ there does not exist α satisfying the conditions. When $n = 3$, the assumptions are satisfied only for $\alpha = 1$. When $n = 4$, the assumptions are satisfied for $\alpha = 1, 2$.

The aim of this note is to remove this restriction in the converse part of Theorem B.

3. Results of this note

Our result is the following.

Theorem 1. (Komori and Mizuhara [8])
Let $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n - \lambda)$. If the commutator $[M_b, I_\alpha]$ is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$ for some p, q, λ as above, then $b \in BMO(R^n)$ and $\|b\|_*$ is bounded by $C_n\|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}}$ where C_n is a positive constant depending only on n.

Theorem 1 is a consequence of Theorem 2 below.

Theorem 2.(Komori and Mizuhara [8]) If $1 < p < \infty$, $0 < \alpha < n$, $0 < \lambda < n - \alpha p$, $1/q = 1/p - \alpha/(n - \lambda)$, $1/q + 1/q' = 1$ and $f \in H^1(R^n)$, then there exist $\{\varphi_j\}_{j=1}^\infty \subset L^{p,\lambda}(R^n)$ and $\{\psi_j\}_{j=1}^\infty \subset h_{nq/(nq-n+\lambda),q'}(R^n)$ such that

$$f = \sum_{j=1}^{\infty}(\varphi_j \cdot I_\alpha\psi_j - \psi_j \cdot I_\alpha\varphi_j),$$

$$\sum_{j=1}^{\infty} \|\varphi_j\|_{L^{p,\lambda}}\|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}} \le C_n\|f\|_{H^1}.$$

Remark 6. Uchiyama [15] showed the factorization theorem on $H^p(X)$ when X is the space of homogeneous type, in the sense of Coifman-Weiss [3]. His result is corresponding to the case $\lambda = 0$ for Morrey spaces $L^{p,\lambda}(R^n)$. Also he applied his result to the boundedness problem of the commutators of the Calderón-Zygmund singular integral operator T.

Applying Uchiyama's method, Komori [7] showed the boundedness of the commutators of the fractional integral operator I_α when $X = R^n$ and $\lambda = 0$.

4. Some Lemmas

We need four lemmas in order to prove our theorems. The first lemma is proved easily from the definitions.

Lemma 1. Let $1 \le p < \infty, 0 \le \lambda \le n$, $1 \le q < r \le \infty$. Then we have

$$\|\chi_{B(x_0,t)}\|_{L^{p,\lambda}} \le C_n t^{\frac{n-\lambda}{p}}, \quad \|\chi_{B(x_0,t)}\|_{h_{q,r}} \le C_n t^{\frac{n}{q}}$$

where C_n is a positive constant depending only on n.

The folowing two lemmas are proved by Long [10].

Lemma 2. (Long [10])

Let X be the whole space R^n or the unit cube Q^n in R^n. If $1 \leq q < p' < \infty,\ q = \frac{np}{np-n+\lambda}$ and $1/p + 1/p' = 1$, then we have

$$\|\phi\|_{L^{p,\lambda}(X)} = \sup_{b:(q,p')-blocks} \left| \int_X \phi(x)b(x)dx \right|,$$

where the spaces $L^{p,\lambda}(Q^n)$ and $h_{q,p'}(Q^n)$ are defined by slightly modifying Definitions 1, 2 and 3.

Lemma 3. (Duality of $h_{q,p'}$ and $L^{p,\lambda}$) Let $1 \leq q < p' < \infty, q = \frac{np}{np-n+\lambda}$ and $1/p + 1/p' = 1$, then the Banach space dual of $h_{q,p'}(R^n)$ is isomorphic to $L^{p,\lambda}(R^n)$.

The last lemma is obtained from the elementary properties of $H^1(R^n)$.

Lemma 4.

If $\int f(x)dx = 0$ and $|f(x)| \leq (\chi_{B(x_0,1)} + \chi_{B(y_0,1)})$ where $N > 10$ and $|x_0 - y_0| = N$, then we have $\|f\|_{H^1} \leq C_n \log N$.

5. Proofs of Theorems 1 and 2

First we prove Theorem 2.

Proof of Theorem 2.

Using the atomic decomposition of H^1 (see Latter [9] or Torchinsky[14], p.347), we may consider for an atom a such that

$$\text{supp } a \subset B(x_0, t),\ \|a\|_{L^\infty} \leq t^{-n} \text{ and } \int a(x)dx = 0.$$

We apply the method due to Komori [7]. Let N be a large integer and take $y_0 \in R^n$ such that $|x_0 - y_0| = Nt$ and set

$$\begin{aligned} \varphi(x) &= N^{n-\alpha}\chi_{B(y_0,t)}(x), \\ \psi(x) &= -a(x)/I_\alpha\varphi(x_0). \end{aligned}$$

By Lemma 1,

$$\|\varphi\|_{L^{p,\lambda}} \leq C_n N^{n-\alpha} t^{\frac{n-\lambda}{p}},$$
$$\|\psi\|_{h_{nq/(nq-n+\lambda),q'}} \leq C_n t^{-n-\alpha} t^{\frac{nq-n+\lambda}{q}},$$

and

$$\|\varphi\|_{L^{p,\lambda}} \|\psi\|_{h_{nq/(nq-n+\lambda),q'}} \leq C_n N^{n-\alpha}. \tag{1}$$

We write

$$a - (\varphi \cdot I_\alpha \psi - \psi \cdot I_\alpha \varphi) = \frac{a \cdot (I_\alpha \varphi(x_0) - I_\alpha \varphi)}{I_\alpha \varphi(x_0)} - \varphi \cdot I_\alpha \psi,$$

then we have

$$\int \{a - (\varphi \cdot I_\alpha \psi - \psi \cdot I_\alpha \varphi)\} dx = 0,$$
$$|a - (\varphi \cdot I_\alpha \psi - \psi \cdot I_\alpha \varphi)| \leq C_n N^{-1} t^{-n} (\chi_{B(x_0,t)} + \chi_{B(y_0,t)}).$$

By Lemma 4,

$$\|a - (\varphi \cdot I_\alpha \psi - \psi \cdot I_\alpha \varphi)\|_{H^1} \leq C_n N^{-1} \log N. \tag{2}$$

Next for any $f \in H^1$ such that $\|f\|_{H^1} \leq 1$, we can write $f = \sum_j m_j a_j$ where $\{a_j\}$ are atoms and $\sum_j |m_j| \leq C_n$ by the atomic decomposition.

Then there exist

$$\{\varphi_j\}_{j=1}^\infty \subset L^{p,\lambda} \quad \text{and} \quad \{\psi_j\}_{j=1}^\infty \subset h_{nq/(nq-n+\lambda),q'}$$

such that

$$\|\varphi_j\|_{L^{p,\lambda}} \|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}} \leq C_n N^{n-\alpha},$$

$$\|a_j - (\varphi_j I_\alpha \psi_j - \psi_j I_\alpha \varphi_j)\|_{H^1} \leq C_n N^{-1} \log N$$

by (1) and (2). So we have

$$\|f - \sum_j \{(m_j \varphi_j) I_\alpha \psi_j - \psi_j I_\alpha (m_j \varphi_j)\}\|_{H^1} \leq C_n N^{-1} \log N \sum_j |m_j| \leq 1/2$$

if N is sufficiently large and

$$\sum_j \|m_j\varphi_j\|_{L^{p,\lambda}}\|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}} \le C_n N^{n-\alpha}\sum_j |m_j| \le C_{n,N}$$

Repeating this process, we get the desired result. □

Next, applying Theorem 2, we prove Theorem 1.

Proof of Theorem 1.

We assume that the commutator $[M_b, I_\alpha]$ is bounded from $L^{p,\lambda}(R^n)$ to $L^{q,\lambda}(R^n)$ for some p, q, λ in Theorem 1.

Let $f \in H^1(R^n)$. Then, by Theorem 2 and Lemma 3, we have

$$\begin{aligned}\left|\int_{R^n} b(x)f(x)dx\right| &\le \sum_j \left|\int_{R^n} b(x)[\varphi_j(x)I_\alpha\psi_j(x) - \psi_j(x)I_\alpha\varphi_j(x)]dx\right| \\ &= \sum_j \left|\int_{R^n} \psi_j(x)[b(x)I_\alpha\varphi_j(x) - I_\alpha(b\varphi_j)(x)]dx\right| \\ &\le C_n\sum_j \|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}}\|[M_b, I_\alpha]\varphi_j\|_{L^{q,\lambda}}.\end{aligned}$$

From the assumption and Theorem 2 again, this is bounded by

$$\begin{aligned}&C_n\sum_j \|\psi_j\|_{h_{nq/(nq-n+\lambda),q'}}\|\varphi_j\|_{L^{p,\lambda}}\|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}} \\ &\qquad \le C_n\|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}}\|f\|_{H^1}.\end{aligned}$$

By the duality for $H^1(R^n)$ and $BMO(R^n)$, we have that $b \in BMO(R^n)$ and $\|b\|_*$ is bounded by $C_n\|[M_b, I_\alpha]\|_{L^{p,\lambda}\to L^{q,\lambda}}$. This completes the proof. □

6. Some Problems

Some problems are open.

Problem 1. Can we get the boundedness or the compactness of the commutators $[M_b, I_\alpha]$ from $L^{p,\lambda}(R^n)$ to $L^{q,\mu}(R^n)$ for some p, q, λ, μ ?.

Problem 2. Can we get the $H^p(R^n)$ $(0 < p < 1)$ version of Theorem 2 ?.

Problem 3. In the setting of spaces of homogenous type, can we get any results corresponding to Theorems 1 and 2 ?.

7. References

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