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L_p - L_q ESTIMATES OF THE OSEEN SEMIGROUP IN EXTERIOR DOMAINS

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We consider the following Oseen equation:

$$(1) \qquad \begin{cases} u_t - \Delta u + (u_\infty \cdot \nabla) u + \nabla \pi = f & \text{in } (0,\infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0,\infty) \times \Omega, \\ u|_{\partial\Omega} = 0, \ u|_{t=0} = a, \end{cases}$$

where Ω is an exterior domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ ($n \geqslant 3$). When $u_\infty = 0$, the equation is the Stokes one. Our treatment below is including the Stokes equation.

First of all, we introduce the notation throughout the paper. For two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y. We put

$$\begin{aligned} B_b^n &= \{x \in \mathbb{R}^n \mid |x| < b\} \qquad (b > 0), \\ \Omega_b &= \Omega \cap B_b^n, \\ C_{0,\sigma}^\infty(\Omega)^n &= \{u \in C_0^\infty(\Omega)^n \mid \nabla \cdot u = 0\}, \\ J_p(\Omega) &= \overline{C_{0,\sigma}^\infty(\Omega)^n}^{\|\cdot\|_{L_p}}, \\ J_{p,b}(\Omega) &= \{u \in J_p(\Omega) \mid u(x) = 0 \text{ for } |x| > b\}, \\ G_p(\Omega) &= \{\nabla \pi \in L_p(\Omega)^n \mid \pi \in L_{p,loc}(\Omega)\} \end{aligned}$$

and $\varphi_b(x)$ is a function in $C^\infty(\mathbb{R}^n)$ such that $\varphi_b(x) = 0$ for $|x| \leqslant b-1$ and $\varphi_b(x) = 1$ for $|x| \geqslant b$.

The Banach space $L_p(\Omega)^n$ admits the Helmholtz decomposition:

$$L_p(\Omega)^n = J_p(\Omega) \oplus G_p(\Omega).$$

Let $\mathbb{P}$ be a continuous projection from $L_p(\Omega)^n$ onto $J_p(\Omega)$. Applying $\mathbb{P}$ to the Oseen equation, we have

$$\begin{cases} u_t + \mathbb{P}(-\Delta + (u_\infty \cdot \nabla))u = \mathbb{P}f, \\ u|_{\partial\Omega} = 0, \ u|_{t=0} = a. \end{cases}$$

Let us define the operator $\mathbb{O}_{u_\infty}$ by $\mathbb{O}_{u_\infty} = \mathbb{P}(-\Delta + (u_\infty \cdot \nabla))$ with the domain:

$$\mathcal{D}_p(\mathbb{O}_{u_\infty}) = \{u \in J_p(\Omega) \cap W_p^2(\Omega) \mid u|_{\partial\Omega} = 0\}$$

By Miyakawa [3], we know that $\mathbb{O}_{u_\infty}$ generates an analytic semigroup $\{T_{u_\infty}(t)\}_{t \geqslant 0}$.

Our main theorem is the followiong one.

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Theorem 1. *Let σ_0 be a positive number and $1 \leqslant p \leqslant q \leqslant \infty$. Assume that $|u_\infty| \leqslant \sigma_0$. For any $t > 0$,*

$$\|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant C_{p,q,\sigma_0} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|a\|_{L_p(\Omega)} \quad (p,q) \neq (1,1), (\infty,\infty), \tag{2}$$

$$\|\nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant C_{p,q,\sigma_0} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|a\|_{L_p(\Omega)} \quad 1 \leqslant p \leqslant q \leqslant n,\ (p,q) \neq (1,1). \tag{3}$$

Moreover, if $|u_\infty| \neq 0$ and $t > 1$ then we have

$$\|\partial_t T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant C_{p,q,\sigma_0} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \|a\|_{L_p(\Omega)} \quad (p,q) \neq (1,1), (\infty,\infty), \tag{4}$$

(5)

$$\|\partial_t \nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant C_{p,q,\sigma_0} t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-1} \|a\|_{L_p(\Omega)} \quad 1 \leqslant p \leqslant q \leqslant n,\ (p,q) \neq (1,1).$$

A crucial step of our approach is to show the following local energy decay of the Oseen semigroup.

Theorem 2. *Let $1 < p < \infty$ and $\sigma_0 > 0$. Assume that $|u_\infty| \leqslant \sigma_0$. Then, for any $b > b_0$ and any nonnegative integer k, there exists a positive constant $C_{k,p,b,\sigma_0,n}$ such that*

$$\|\partial_t^k T_{u_\infty}(t)a\|_{W_p^2(\Omega_b)} \leqslant C_{k,p,b,\sigma_0,n}\, t^{-\frac{n+k}{2}} \|a\|_{L_p(\Omega)} \quad \text{for } {}^\forall t \geqslant 1,\ {}^\forall a \in J_{p,b}(\Omega),$$

where b_0 is a fixed positive number such that $\Omega^c \subset B^n_{b_0-3}$.

To use a cut-off technique later on under the Helmholtz decomposition, we use the following Bogovskii lemma.

Lemma 3 ([1], [2]). *Let $1 < p < \infty$ and let m be a nonnegative integer. Then, there exists a bounded linear operator $\mathbb{B} : \dot{W}^m_{p,a}(D) \longrightarrow \dot{W}^{m+1}_p(D)$ such that*

$$\nabla \cdot \mathbb{B}[f] = f \text{ in } D \quad \text{and} \quad \|\mathbb{B}[f]\|_{W_p^{m+1}(D)} \leqslant C\|f\|_{W_p^m(D)}$$

where D is a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, $\dot{W}^m_p(D) = \overline{C_0^\infty(D)}^{\|\cdot\|_{W_p^m}}$ and $\dot{W}^m_{p,a}(D) = \{u \in \dot{W}^m_p(D) \mid \int_D u dx = 0\}$.

Sketch of proof of Theorem 1. We define a solution operator in $\mathbb{R}^n$. Let $c(x)$ be a function in $L_p(\mathbb{R}^n)^n$ satisfying $\nabla \cdot c = 0$ in $\mathbb{R}^n$. We define $S_{u_\infty}(t)c(x)$ by the formula:

$$S_{u_\infty}(t)c(x) = \left(\frac{1}{4\pi t}\right)^n \int_{\mathbb{R}^n} e^{-\frac{|x-y-tu_\infty|^2}{4t}} c(y)dy.$$

Put $v(t,x) = S_{u_\infty}(t)c(x)$, then v satisfies the equation:

$$\begin{cases} v_t - \Delta v + (u_\infty \cdot \nabla)v = 0 & \text{in } (0,\infty) \times \mathbb{R}^n, \\ \nabla \cdot v = 0 & \text{in } (0,\infty) \times \mathbb{R}^n, \\ v|_{t=0} = c. \end{cases}$$

Moreover, when $1 \leqslant p \leqslant q \leqslant \infty$, by the Young inequality we can show that

$$\|\partial_t^j \partial_x^\alpha v(t)\|_{L_q(\mathbb{R}^n)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j+|\alpha|}{2}} \|\|_{L_p(\Omega)} \qquad t \geqslant 1, \tag{6}$$

$$\|\partial_t^j \partial_x^\alpha v(t)\|_{L_q(\mathbb{R}^n)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\left(j+\frac{|\alpha|}{2}\right)} \|c\|_{L_p(\Omega)} \qquad 0 < t \leqslant 1. \tag{7}$$

For $t \geqslant 1$, we will prove the following L_p - L_q estimates:

$$\|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|a\|_{L_p(\Omega)} \qquad 1 < p \leqslant q \leqslant \infty, \tag{8}$$

$$\|\nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|a\|_{L_p(\Omega)} \qquad 1 < p \leqslant q \leqslant n. \tag{9}$$

To do this put $\tilde{a}(x) = T_{u_\infty}(1)a(x)$ and $u(t,x) = T_{u_\infty}(t)\tilde{a}(x)$. By the analytic semigroup theory, for any nonnegative integer N

$$\tilde{a} \in \mathcal{D}_p(\mathbb{O}^N_{u_\infty}) \quad \text{and} \quad \|\tilde{a}\|_{W^{2N}_p(\Omega)} \leqslant C\|a\|_{L_p(\Omega)}.$$

1st step.

Let m be a nonnegative integer. For any $t \geqslant 0$, we shall prove the following estimates:

$$\|u(t)\|_{W^{2m}_p(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)}, \tag{10}$$

$$\|u_t(t)\|_{W^{2m}_p(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|a\|_{L_p(\Omega)}, \tag{11}$$

$$\|u(t)\|_{W^{2m}_\infty(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)}, \tag{12}$$

$$\|u_t(t)\|_{W^{2m}_\infty(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|a\|_{L_p(\Omega)}, \tag{13}$$

where $1 < p < \infty$.

Let N be a natural number such that $N \geqslant \frac{1}{2}\left(\frac{n}{p} + 2m + 6\right)$ and let $1 < p < \infty$. There exists a $c(x) \in W^{2N}_p(\mathbb{R}^n)$ such that $c(x) = \tilde{a}(x)$ on Ω and $\nabla \cdot c = 0$ in $\mathbb{R}^n$. Moreover,

$$\|c\|_{W^{2N}_p(\mathbb{R}^n)} \leqslant C\|\tilde{a}\|_{W^{2N}_p(\Omega)} \leqslant C\|a\|_{L_p(\Omega)}. \tag{14}$$

By (6), (7) and (14), for any $t \geqslant 0$ we put $v(t,x) = S_{u_\infty}(t)c(x)$ then we have

$$\|\partial_t^j v(t)\|_{W^{2m+1}_\infty(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2p}-\frac{j}{2}}\|a\|_{L_p(\Omega)},$$

where $j = 0,\ 1,\ 2$. Let us define w by the following formula:

$$w = u - \varphi_{b+1}v - \mathbb{B}[(\nabla\varphi_{b+1}) \cdot v].$$

Then, $w = u$ in Ω_b and w satisfies the equation:

$$\begin{cases} w_t - \Delta w + (u_\infty \cdot \nabla)w + \nabla\pi = g & \text{in } (0,\infty)\times\Omega, \\ \nabla \cdot w = 0 & \text{in } (0,\infty)\times\Omega, \\ w|_{\partial\Omega} = 0, \ w|_{t=0} = d, \end{cases}$$

where

$$\begin{aligned} g &= -2(\nabla\varphi_{b+1})(\nabla v) - (\Delta\varphi_{b+1})v + [(u_\infty\cdot\nabla)\varphi_{b+1}]v \\ &\quad - (\partial_t - \Delta + (u_\infty \cdot \nabla))\,\mathbb{B}[(\nabla\varphi_{b+1})\cdot v], \\ d &= \varphi_{b+1}c - \mathbb{B}[(\nabla\varphi_{b+1})\cdot c]. \end{aligned}$$

It is easy to show that g and d satisfy the properties:

$$\begin{aligned} \partial_t^j g(t) \in \mathcal{D}_p(\mathbb{O}^m_{u_\infty}) \cap J_{p,b+1}(\Omega), &\quad \|\partial_t^j g(t)\|_{W^{2m}_p(\Omega)} \leqslant C(1+t)^{-\frac{n}{2p}-\frac{j}{2}}\|a\|_{L_p(\Omega)}, \\ d \in \mathcal{D}_p(\mathbb{O}^N_{u_\infty}) \cap J_{p,b+1}(\Omega), &\quad \|d\|_{W^{2N}_p(\Omega)} \leqslant C\|a\|_{L_p(\Omega)}. \end{aligned}$$

By Duhamel's principle, w is represented by

$$w(t,x) = T_{u_\infty}(t)d(x) + \int_0^t T_{u_\infty}(t-s)g(s)ds.$$

Since g and d have compact supports, we can use the local energy decay theorem. Then, by the local energy decay estimate, we have

$$\|w(t)\|_{W_p^{2m}(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)}.$$

Moreover, we have

$$\|w_t(t)\|_{W_p^{2m}(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}-\frac{1}{2}}\|a\|_{L_p(\Omega)}.$$

Therefore we obtain (10) and (11). Since m is arbitrary, by Sobolev's embedding theorem, we obtain (12) and (13).

2nd step.

We estimate the pressure π, that is, for $t \geqslant 0$ we prove

$$\|\pi(t)\|_{W_p^{2m}(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)}, \tag{15}$$

$$\|\pi(t)\|_{W_\infty^{2m}(\Omega_b)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)}, \tag{16}$$

where $1 < p < \infty$.

We may assume without loss of generality that $\int_{\Omega_b}\pi(x)dx = 0$. By Poincaré's inequality, we have

$$\begin{aligned}\|\pi(t)\|_{W_p^{2m}(\Omega_b)} &\leqslant C\|\nabla\pi(t)\|_{W_p^{2m-1}(\Omega_b)} \\ &\leqslant C\|u_t - \Delta u + (u_\infty\cdot\nabla)u\|_{W_p^{2m-1}(\Omega_b)},\end{aligned}$$

which implies that (15) holds. Using the Sobolev's embedding theorem again, we obtain (16).

3rd step.

We shall prove the following L_p - L_q estimates:

$$\|u(t)\|_{L_q(\Omega)} \leqslant C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|a\|_{L_p(\Omega)} \qquad 1 < p \leqslant q \leqslant \infty, \tag{17}$$

$$\|\nabla u(t)\|_{L_q(\Omega)} \leqslant C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|a\|_{L_p(\Omega)} \qquad 1 < p \leqslant q \leqslant n. \tag{18}$$

Since we have already had the estimate of u in Ω_b, in order to obtain (17) and (18), it is sufficient to estimate u outside of Ω_b. Let us define z by the formula:

$$z = (1-\varphi_b)u + \mathbb{B}[(\nabla\varphi_b)\cdot u].$$

Then, $z = u$ in Ω_b^c and z satisfies the equation:

$$\begin{cases} z_t - \Delta z + (u_\infty\cdot\nabla)z + \nabla[(1-\varphi_b)\pi] = h & \text{in } (0,\infty)\times\mathbb{R}^n, \\ \nabla\cdot z = 0 & \text{in } (0,\infty)\times\mathbb{R}^n, \\ z|_{t=0} = e, \end{cases}$$

where

$$h = 2(\nabla\varphi_b)(\nabla u) + (\Delta\varphi_b)u + [(u_\infty \cdot \nabla)\varphi_b]u - (\nabla\varphi_b)\pi$$
$$+ (\partial_t - \Delta + (u_\infty \cdot \nabla))\,\mathbb{B}[(\nabla\varphi_b)\cdot u],$$
$$e = (1-\varphi_b)\tilde{a} + \mathbb{B}[(\nabla\varphi_b)\cdot\tilde{a}].$$

It is easy to show that h and e satisfy the inequalities:

$$\|h(t)\|_{W_p^{2m-1}(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)},$$
$$\|e\|_{W_p^{2m}(\mathbb{R}^n)} \leqslant C\|a\|_{L_p(\Omega)},$$

where $1 < p < \infty$. By Duhamel's principle, z is represented by

$$z(t,x) = S_{u_\infty}(t)e(x) + \int_0^t S_{u_\infty}(t-s)\mathbb{P}h(s)ds.$$

By L_p - L_q estimate, we have

$$\|S_{u_\infty}(t)e\|_{L_q(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|a\|_{L_p(\Omega)},$$
$$\|\nabla S_{u_\infty}(t)e\|_{L_q(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|a\|_{L_p(\Omega)},$$

where $1 \leqslant p \leqslant q \leqslant \infty$. Let ρ be a number such that $1 < \rho < \min\left(\frac{n}{2}, p\right)$. Since

$$\|\mathbb{P}h(t)\|_{L_\rho(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)},$$

we have

$$\|\int_0^t S_{u_\infty}(t-s)\mathbb{P}h(s)ds\|_{L_q(\mathbb{R}^n)} \leqslant CI_\rho(t)\|a\|_{L_p(\Omega)},$$
$$\|\nabla\int_0^t S_{u_\infty}(t-s)\mathbb{P}h(s)ds\|_{L_q(\mathbb{R}^n)} \leqslant CJ_\rho(t)\|a\|_{L_p(\Omega)},$$

where

$$I_\rho(t) = \int_0^t (1+t-s)^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{q}\right)}(1+s)^{-\frac{n}{2p}}ds,$$
$$J_\rho(t) = \int_0^t (1+t-s)^{-\frac{n}{2}\left(\frac{1}{\rho}-\frac{1}{q}\right)-\frac{1}{2}}(1+s)^{-\frac{n}{2p}}ds$$

and $1 < p \leqslant q \leqslant \infty$. Therefore, we obtain

$$\|z(t)\|_{L_q(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|a\|_{L_p(\Omega)} \qquad 1 < p \leqslant q \leqslant \infty,$$
$$\|\nabla z(t)\|_{L_q(\mathbb{R}^n)} \leqslant C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|a\|_{L_p(\Omega)} \qquad 1 < p \leqslant q \leqslant n.$$

Now, for $0 < t \leqslant 1$, we shall prove the following L_p - L_q estimates:

$$\|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|a\|_{L_p(\Omega)}, \tag{19}$$
$$\|\nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|a\|_{L_p(\Omega)}, \tag{20}$$

where $1 < p \leqslant q < \infty$. In the similar manner, we have

$$\|\partial_t T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|a\|_{L_p(\Omega)}, \tag{21}$$

$$\|\partial_t \nabla T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1}\|a\|_{L_p(\Omega)}. \tag{22}$$

If u together with some π satisfies the equation:

$$\begin{cases} \lambda u - \Delta u + (u_\infty \cdot \nabla)u + \nabla\pi = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then there exists an $R > 0$ such that for $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbb{C} \mid |u_\infty|^2 \mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|^2 > 0\}$ with $|\lambda| \geqslant R$,

$$|\lambda|\|u\|_{L_p(\Omega)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{L_p(\Omega)} + \|\nabla^2 u\|_{L_p(\Omega)} \leqslant C\|f\|_{L_p(\Omega)}, \tag{23}$$

where $1 < p < \infty$. The analytic semigroup $T_{u_\infty}(t)a$ is represented by

$$T_{u_\infty}(t)a = \int_\Gamma e^{\lambda t}(\lambda + \mathbb{O}_{u_\infty})^{-1}a \; d\lambda$$

with suitable contour Γ in some sector. By the resolvent estimate (23), for $0 < t \leqslant 1$

$$\|T_{u_\infty}(t)a\|_{L_p(\Omega)} + t^{\frac{1}{2}}\|\nabla T_{u_\infty}(t)a\|_{L_p(\Omega)} + t\|\nabla^2 T_{u_\infty}(t)a\|_{L_p(\Omega)} \leqslant C\|f\|_{L_p(\Omega)}. \tag{24}$$

In view of the complex interpolation: $W_p^{n(\frac{1}{p}-\frac{1}{q})} = [L_p,\ W_p^2]_{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$, interpolating (24) and Sobolev's embedding theorem, we obtain the L_p - L_q estimates (19) and (20).

Next, for $0 < t \leqslant 1$ we shall prove

$$\|T_{u_\infty}(t)a\|_{L_\infty(\Omega)} \leqslant Ct^{-\frac{n}{2p}}\|a\|_{L_p(\Omega)} \qquad 1 < p < \infty. \tag{25}$$

A Besov space $B_{p,1}^{\frac{n}{p}}$ is continuously included in L_∞ and it is obtained by the real interpolation: $B_{p,1}^{\frac{n}{p}} = [L_p,\ W_p^2]_{\frac{n}{2p},1}$. Interpolating two formulas: $\|T_{u_\infty}(t)a\|_{L_p(\Omega)} \leqslant C\|a\|_{L_p(\Omega)}$ and $\|T_{u_\infty}(t)a\|_{W_p^2(\Omega)} \leqslant Ct^{-1}\|a\|_{L_p(\Omega)}$, we have (25).

Finally, for $t > 0$ we shall prove

$$\|T_{u_\infty}(t)a\|_{L_q(\Omega)} \leqslant Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|a\|_{L_1(\Omega)} \qquad 1 < q \leqslant \infty. \tag{26}$$

For $a \in J_1(\Omega)$, we define $T_{u_\infty}(t)a$ by the duality

$$(T_{u_\infty}(t)a, b) = (a, T_{-u_\infty}(t)b) \qquad \text{for } {}^\forall b \in C_{0,\sigma}^\infty(\Omega).$$

Then, we have

$$\begin{aligned} |(T_{u_\infty}(t)a, b)| &\leqslant C\|a\|_{L_1(\Omega)}\|T_{-u_\infty}(t)b\|_{L_\infty(\Omega)} \\ &\leqslant C\|a\|_{L_1(\Omega)}\; t^{-\frac{n}{2q'}}\|b\|_{L_{q'}(\Omega)}, \end{aligned}$$

where $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$ which implies that (26) holds. This completes the proof of Theorem 1. □

Now, we shall prove our local energy decay theorem. Before going to a sketch of the proof of Theorem 2, we introduce the following definition concerning some regularity of the resolvent operator.

Definition 4. Let B be a Banach space and $\|\cdot\|_B$ its norm. Let $T_{u_\infty}(t)$ be a function in $C^\infty(\mathbb{R}^n\backslash\{0\})$ with its value in B, which depends on $u_\infty \in \mathbb{R}^n$. Let σ_0 be a positive number. Assume that $\|T_{u_\infty}(t)\|_{L_\infty(\mathbb{R},B)} \leqslant C$ when $|u_\infty| \leqslant \sigma_0$ for some constant C independent of σ_0. We say that $T_{u_\infty}(t)$ is *uniformly n-regular* in B if whenever $|u_\infty| \leqslant \sigma_0$, $T_{u_\infty}(t)$ satisfies the following properties :
When n is even, for any nonnegative integers m, M and N with $N \geqslant m$ there hold the following seven inequalities :

$$\|\Delta_h^2[s^N\partial_s^{\frac{n}{2}-1+m}T_{u_\infty}(s)]\|_{L_1(\mathbb{R},B)} \leqslant C|h|;$$
$$\|\Delta_h[s^N\partial_s^{\frac{n}{2}-1+m}T_{u_\infty}(s)]\|_{L_q(\mathbb{R},B)} \leqslant C|h|^{\frac{1}{2}} \quad 1 \leqslant {}^\forall q < 2;$$
$$\|\Delta_h[s^{N+1}\partial_s^{\frac{n}{2}+m}T_{u_\infty}(s)]\|_{L_1(\mathbb{R},B)} \leqslant C|h|^{\frac{1}{2}};$$
$$\|s^N\partial_s^{\frac{n}{2}+m}T_{u_\infty}(s)\|_{L_q(\mathbb{R},B)} \leqslant C \quad 1 \leqslant {}^\forall q < \infty;$$
$$\|s^{N+1}\partial_s^{\frac{n}{2}+m}T_{u_\infty}(s)\|_{L_q(\mathbb{R},B)} \leqslant C \quad 1 \leqslant {}^\forall q < 2;$$
$$\|\Delta_h[s^M\partial_s^{m}T_{u_\infty}(s)]\|_{L_q(\mathbb{R},B)} \leqslant C|h|^r$$
$$1 \leqslant {}^\forall q < \infty,\ 0 \leqslant m \leqslant \frac{n}{2}-2,\ r=1 \text{ and } \frac{1}{2};$$
$$\|s^M\partial_s^{m}T_{u_\infty}(s)\|_{L_\infty(\mathbb{R},B)} \leqslant C \quad 1 \leqslant m \leqslant \frac{n}{2}-2.$$

When n is odd, for any nonnegative integer m, M and N with $N \geqslant 2m$ there hold the following seven inequalities :

$$\|\Delta_h^2[s^{N+1}\partial_s^{[\frac{n}{2}]+m}T_{u_\infty}(s)]\|_{L_1(\mathbb{R},B)} \leqslant C|h|;$$
$$\|\Delta_h[s^N\partial_s^{[\frac{n}{2}]+m}T_{u_\infty}(s)]\|_{L_1(\mathbb{R},B)} \leqslant C|h|^{\frac{1}{2}};$$
$$\|\Delta_h[s^{N+1}\partial_s^{[\frac{n}{2}]+m}T_{u_\infty}(s)]\|_{L_\infty(\mathbb{R},B)} \leqslant C;$$
$$\|s^N\partial_s^{[\frac{n}{2}]+m}T_{u_\infty}(s)\|_{L_q(\mathbb{R},B)} \leqslant C \quad 1 \leqslant {}^\forall q < 2;$$
$$\|\Delta[s^M\partial_s^{m}T_{u_\infty}(s)]\|_{L_q(\mathbb{R},B)} \leqslant C|h| \quad 1 \leqslant {}^\forall q < 2,\ 0 \leqslant m \leqslant \left[\frac{n}{2}\right]-1;$$
$$\|\Delta_h[s^M\partial_s^{m}T_{u_\infty}(s)]\|_{L_q(\mathbb{R},B)} \leqslant C|h|^{\frac{1}{2}} \quad 1 \leqslant {}^\forall q < \infty,\ 0 \leqslant m \leqslant \left[\frac{n}{2}\right]-1;$$
$$\|s^M\partial_s^{m}T_{u_\infty}(s)\|_{L_\infty(\mathbb{R},B)} \leqslant C \quad 1 \leqslant m \leqslant \left[\frac{n}{2}\right]-1,$$

where $\left[\frac{n}{2}\right] = \frac{n-1}{2}$. Here, the constant C depends on n, m, r, M, N and σ_0, but is independent of h and u_∞; and for any B-valued function $g(s)$ and $h \in \mathbb{R}\backslash\{0\}$ we have put

$$\begin{aligned} \|g\|_{L_q(\mathbb{R},B)} &= \left\{\int_{-\infty}^{\infty} \|g(s)\|_B^q ds\right\}^{\frac{1}{q}} \quad 1 \leqslant q < \infty; \\ \|g\|_{L_\infty(\mathbb{R},B)} &= \operatorname*{ess\,sup}_{s\in\mathbb{R}\backslash\{0\}} \|g(s)\|_B; \\ \Delta_h^2 g(s) &= g(s+h) - 2g(s) + g(s-h); \\ \Delta_h g(s) &= g(s+h) - g(s). \end{aligned}$$

Theorem 5. *Let $X = \mathcal{L}\left(L_{p,b}(\mathbb{R}^n), W_p^2(B_b^n)\right)$, and $\sigma_0 > 0$. Assume $|u_\infty| \leqslant \sigma_0$. If we put*

$$U_{u_\infty}(s) = \left(\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \frac{e^{ix\cdot\xi}}{|\xi|^2 + is + i(u_\infty\cdot\xi)} \frac{\xi_j\xi_k}{|\xi|^2} d\xi\right)$$

and $E_{u_\infty}(s)f = U_{u_\infty}(s) * f$, then $E_{u_\infty}(s)$ is uniformly n-regular in X.

To prove Theorem 2, we construct a parametrix. For $f(x) \in L_{p,b}(\Omega)$, we put $f_0(x) = f(x)$ for $x \in \Omega$ and $f_0(x) = 0$ for $x \notin \Omega$. Let us put

$$\begin{aligned} \Phi_{u_\infty}(\lambda)f &= \varphi_{b-1}E_{u_\infty}(\lambda)f_0 + (1-\varphi_{b-1})F_{u_\infty}(\lambda)f + G_{u_\infty}(\lambda)f, \\ P_{u_\infty}(\lambda)f &= \varphi_{b-1}\Pi f_0 + (1-\varphi_{b-1})\Pi_{u_\infty}(\lambda)f, \end{aligned}$$

where $G_{u_\infty}(\lambda)f = \mathbb{B}[(\nabla\varphi_{b-1})\cdot(E_{u_\infty}(\lambda)f_0 - F_{u_\infty}(\lambda)f)]$ and $(v,\pi) = (F_{u_\infty}(\lambda)f, \Pi_{u_\infty}(\lambda)f)$ is a solution to the Oseen equation:

$$\begin{cases} (\lambda - \Delta + (u_\infty\cdot\nabla))\,v + \nabla\pi = f & \text{in } \Omega_b, \\ \nabla\cdot v = 0 & \text{in } \Omega_b, \\ v = 0 & \text{on } \partial\Omega_b. \end{cases}$$

Then, $\Phi_{u_\infty}(\lambda)f$ and $P_{u_\infty}(\lambda)f$ satisfy the equation:

$$\begin{cases} (\lambda - \Delta + (u_\infty\cdot\nabla))\,\Phi_{u_\infty}(\lambda)f + \nabla P_{u_\infty}(\lambda)f = (I + \Psi_{u_\infty}(\lambda))f & \text{in } \Omega, \\ \nabla\cdot\Phi_{u_\infty}(\lambda)f = 0 & \text{in } \Omega, \\ \Phi_{u_\infty}(\lambda)f = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $\Phi_{u_\infty}(\lambda)f$ is uniformly n-regular in $C^\infty\left(\mathbb{R}\backslash\{0\}; \mathcal{L}(L_{p,b}(\Omega), W_p^2(\Omega_b))\right)$. For $I + \Psi_{u_\infty}(\lambda)$, we obtain the following lemma.

Lemma 6. *Let $1 < p < \infty$ and $\lambda \in \Sigma_{u_\infty} \cup \{0\}$. Then, $I + \Psi_{u_\infty}(\lambda) : L_{p,b}(\Omega) \longrightarrow L_{p,b}(\Omega)$ has the bounded inverse $(I + \Psi_{u_\infty}(\lambda))^{-1}$. Moerover, $(I + \Psi_{u_\infty}(\lambda))^{-1}$ is uniformly n-regular in $\mathcal{L}\left(L_{p,b}(\Omega), L_{p,b}(\Omega)\right)$.*

Note that, the resolvent operator of the Oseen equation is represented by

$$(\lambda + \mathbb{O}_{u_\infty})^{-1}f = \Phi_{u_\infty}(\lambda)\left(I + \Psi_{u_\infty}(\lambda)\right)^{-1} f \qquad \text{for } {}^\forall f \in J_{p,b(\Omega)}.$$

Sketch of proof of Theorem 2. Let $X = \mathcal{L}\left(J_{p,b}(\Omega), W_p^2(\Omega_b)\right)$. Using a cut off function $\varphi_R(s)$, we have

$$\begin{aligned} T_{u_\infty}(t) &= \int_{-\infty}^{\infty} e^{-its}\varphi_R(s)\Phi_{u_\infty}(is)\left(I + \Psi_{u_\infty}(is)\right)^{-1} ds \\ &\quad + \int_{-\infty}^{\infty} e^{-its}(1-\varphi_R(s))(is + \mathbb{O}_{u_\infty})^{-1} ds \\ &= I_1(t) + I_2(t) \ \in \ X. \end{aligned}$$

In order to estimate $I_2(t)$, we use the following theorems about the resolvent.

Theorem 7. *Let $1 < p < \infty$. Then, $\rho(\mathbb{O}_{u_\infty}) \supset -\Sigma_{u_\infty}$. Moreover, for any $\sigma_0 > 0$ and $\lambda_0 > 0$ there exists a $C_{p,\sigma_0,\lambda_0} > 0$ such that*

$$\|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{W_p^2(\Omega)} + |\lambda| \|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{L_p(\Omega)} \leqslant C_{p,\sigma_0,\lambda_0}\|f\|_{L_p(\Omega)}, \qquad {}^\forall f \in \mathbb{J}_p(\Omega),$$

provided that $\operatorname{Re}\lambda \geqslant 0$, $|\lambda| \geqslant \lambda_0$ *and* $|u_\infty| \leqslant \sigma_0$.

Theorem 8. *Let $1 < p < \infty$, $\sigma_0 > 0$ and $|u_\infty| \leqslant \sigma_0$. Then there exist $0 < \delta_0 < \dfrac{\pi}{2}$ and $R_0 = R_0(p, \sigma_0) > 0$ indepedent of u_∞ such that*

$$|\lambda| \|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{L_p(\Omega)} + \|(\lambda + \mathbb{O}_{u_\infty})^{-1} f\|_{W_p^2(\Omega)} \leqslant C_p \|f\|_{L_p(\Omega)} \quad {}^\forall f \in \mathbb{J}_p(\Omega)$$

provided that $|\lambda| \geqslant R_0$ *and* $|\arg \lambda| \leqslant \pi - \delta_0$.

By Theorems 7 and 8, we have

$$\|\partial_t^k I_2(t)\|_X \leqslant C_{k,l}\, t^{-l} \qquad {}^\forall k,\ {}^\forall l \in \mathbb{N}.$$

Next, we estimate $I_1(t)$. Observe that

$$\partial_t^k I_1(t) = \int_{-\infty}^{\infty} (-is)^k e^{-its}\varphi_R(s)\Phi_{u_\infty}(is)\left(I + \Psi_{u_\infty}(is)\right)^{-1} ds.$$

To estimate $I_1(t)$, we introduce the following space.

Definition 9 ([4]). Let X be a Banach space with norm $|\cdot|_X$. Let N be a positive integer and $\alpha = N + \sigma$ with $0 < \sigma \leqslant 1$. Put

$$C^\alpha(\mathbb{R}; X) = \left\{ f \in C^{N-1}(\mathbb{R}; X) \cap C^\infty(\mathbb{R}\backslash\{0\}; X) \mid \langle\!\langle f \rangle\!\rangle_{\alpha,X} < \infty \right\},$$

where

$$\langle\!\langle f \rangle\!\rangle_{\alpha,X} = \sum_{j=0}^{N} \int_{-\infty}^{\infty} \left| \left(\frac{d}{d\tau}\right)^j f(\tau) \right|_X d\tau + \sup_{h \neq 0} |h|^{-\sigma} \int_{-\infty}^{\infty} \left| \Delta_h \left(\frac{d}{d\tau}\right)^N f(\tau) \right|_X d\tau \qquad \text{if } 0 < \sigma < 1,$$

$$\langle\!\langle f \rangle\!\rangle_{\alpha,X} = \sum_{j=0}^{N} \int_{-\infty}^{\infty} \left| \left(\frac{d}{d\tau}\right)^j f(\tau) \right|_X d\tau + \sup_{h \neq 0} |h|^{-1} \int_{-\infty}^{\infty} \left| \Delta_h^2 \left(\frac{d}{d\tau}\right)^N f(\tau) \right|_X d\tau \qquad \text{if } \sigma = 1.$$

Theorem 10. *If* $f \in C^{\alpha}(\mathbb{R}; X)$ *then*

$$\|\hat{f}(\tau)\|_X \leqslant C(1+|\tau|)^{-\alpha}\langle\!\langle f \rangle\!\rangle_{\alpha,X},$$

where

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} e^{-i\tau t} f(t)dt.$$

Since $(-is)^k \varphi_R(s)\Phi_{u_\infty}(is)\,(I+\Psi_{u_\infty}(is))^{-1} \in C^{\frac{n+k}{2}}(\mathbb{R}, X)$, by Theorem 10 we have

$$\|\partial_t^k I_1(t)\|_X \leqslant C_{n,k}(1+t)^{-\frac{n+k}{2}} \qquad {}^\forall k \in \mathbb{N}.$$

This completes the proof of Theorem 2. □

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