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FATOU AND LITTLEWOOD THEOREMS FOR POISSON INTEGRALS WITH RESPECT TO NON-INTEGRABLE KERNELS

相川 弘明 (HIROAKI AIKAWA)
島根大学 総合理工学部

1. Fatou Theorem and Littlewood Theorem

In 1906 Fatou [5] proved the following:

Theorem (Fatou Theorem). *Let f be a bounded analytic function on the unit disk $U = \{|z| < 1\}$ in $\mathbb{C}$. Then f has non-tangential limit at a.e. $e^{i\theta} \in \partial U$.*

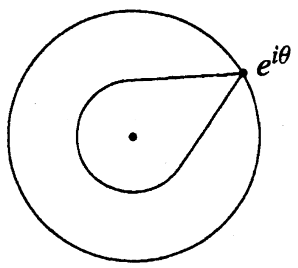


Figure 1. Fatou Theorem.

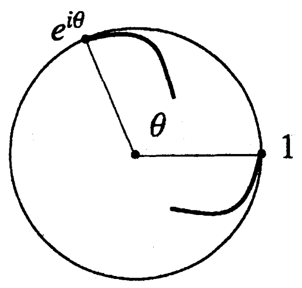


Figure 2. Littlewood Theorem.

In 1927 Littlewood [9, 10] proved the sharpness of non-tangential approach.

Theorem (Littlewood Theorem). *Let $\gamma \subset U$ be a tangential curve at 1 and let γ_θ be the rotation. Then there exists a bounded analytic function f on U such that the limit of f along γ_θ does not exists for a.e. $e^{i\theta} \in \partial U$.*

There are many generalizations of Fatou theorem as follows:

- Hardy space H^p
- Harmonic functions
- Local Fatou theorem
- Invariant harmonic functions. Korányi (1969) [8]

- Square root of the Poisson kernel. Sjögren (1983) [18, 19, 20]
- Non non-tangential convergence. Nagel-Stein (1984) [13]
- Harmonic functions on trees
- Symmetric spaces

On the other hand, there are rather few works for Littlewood theorem:

- Zygmund (1949) [21]. *(Blaschke product/Real Analysis)*
- Lohwater-Piranian (1957) [11]. *(Blaschke product. Everywhere divergence)*
- Hakim-Sibony (1983) [6]. *(Invariant harmonic functions)*
- Aikawa (1990) [1, 2]. *(Everywhere divergence)*
- Salvatori-Vignati (1997) [17]. *(Homogeneous tree).*
- Di Biase (1998) [4]. *(General tree)*
- Hirata (2003) [7]. *(Invariant harmonic functions in the unit ball of $\mathbb{C}^n$)*

In this note, we would like to observe that Fatou Theorem and Littlewood Theorem should go hand in hand.

2. Fatou and Littlewood Theorems for harmonic functions on $\mathbf{R}^{n+1}_+$

Let $\Psi(x) = (1+|x|^2)^{-(n+1)/2}$ for $x \in \mathbf{R}^n$ and put $\Psi_t(x) = \dfrac{1}{t^n}\Psi(\dfrac{x}{t})$ for $t > 0$. Then $\Psi_t * 1 = c_n$ and

$$\frac{\Psi_t * f(x)}{\Psi_t * 1} = \frac{1}{c_n}\int_{\mathbf{R}^n} \frac{tf(y)dy}{(|x-y|^2+t^2)^{(n+1)/2}}$$

is the Poisson integral $Pf(x,t)$ for the half space $\mathbf{R}^{n+1}_+ = \{(x,t) : x \in \mathbf{R}^n, t > 0\}$. By A we denote a positive constant whose value may change from occurrence to the next. If two positive functions f and g satisfy $f \le Ag$ for some $A \ge 1$, then we write $f \lesssim g$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$. Let $h(t)$ be a positive function for $t > 0$. Define the approach region

$$\mathscr{A}_h(\xi) = \{(x,t) : |x-\xi| < h(t)\} \quad \text{for } \xi \in \mathbf{R}^n.$$

If $h(t) \sim t$, then $\mathscr{A}_h(\xi)$ gives a nontangential approach to ξ. We say that a function u in $\mathbf{R}^{n+1}_+$ has a nontangential limit at ξ if the limit of u along $\mathscr{A}_h(\xi)$ exists for every nontangential approach $\mathscr{A}_h(\xi)$.

Theorem A (Fatou Theorem). *Let $1 \le p \le \infty$. If $f \in L^p(\mathbf{R}^n)$, then $Pf(x,t)$ has nontangential limit $f(\xi)$ at a.e. $\xi \in \mathbf{R}^n$.*

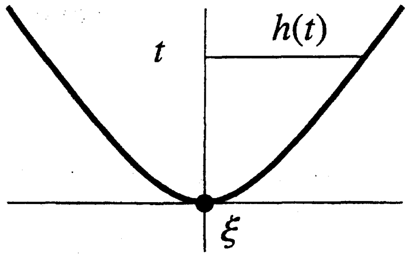


FIGURE 3. Approach region $\mathcal{A}_h(\xi)$.

Theorem B (Littlewood Theorem). *If* $\limsup_{t\to 0} h(t)/t = \infty$, *then there exists* $f \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ *such that*

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathcal{A}_h(\xi)}} Pf(x,t) \text{ fails to exist at every } \xi \in \mathbf{R}^n.$$

If γ *is a tangential curve in* $\mathbf{R}^{n+1}_+$ *ending at* $\partial\mathbf{R}^{n+1}_+$, *then there exists* $f \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ *such that*

$$\lim_{\substack{t\to 0\\ (x,t)\in\gamma+\xi}} Pf(x,t) \text{ fails to exist at every } \xi \in \mathbf{R}^n.$$

The above theorems suggest that the higher integrability of the boundary function f does not improve the admissible tangency.

3. NON-INTEGRABLE KERNEL

Sjögren [18, 19, 20] gave extensions of the Fatou theorem for fractional Poisson integrals. Let

$$P(z,\zeta) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-\zeta|^2}$$

be the Poisson kernel for the unit disk U. Then the classical Poisson integral

$$Pf(z) = \int_{\partial U} P(z, e^{i\theta}) f(e^{i\theta})\,d\theta$$

is, of course, harmonic, i.e., $\Delta Pf = 0$.

Consider the fractional integral, or the λ-Poisson integral

$$u = P_\lambda f(z) = \int_{\partial U} P(z, e^{i\theta})^{\lambda+1/2} f(e^{i\theta})\,d\theta.$$

Then, with the invariant or hyperbolic Laplacian

$$\widetilde{\Delta} = \frac{1}{4}(1 - |z|^2)^2\Delta,$$

u enjoys $\widetilde{\Delta} u = (\lambda^2 - \frac{1}{4})u$. Sjögren studied the boundary behavior of the normalization

$$\mathcal{P}_\lambda f(z) = \frac{P_\lambda f(z)}{P_\lambda 1(z)}.$$

If $\lambda > 0$, then the Fatou theorem holds for $\mathcal{P}_\lambda f$ almost verbatim.

Theorem C. *If $f \in L^1(\partial U)$, then $\mathcal{P}_\lambda f(z)$ has nontangential limit $f(e^{i\theta})$ at a.e. $e^{i\theta} \in \partial U$.*

If $\lambda = 0$, then suddenly tangential limits appear (Sjögren [18, 19, 20] and Rönning [14, 15, 16]).

Theorem D. *Suppose $f \in L^p(\partial U)$ with $1 \le p \le \infty$. Then $\mathcal{P}_0 f(z)$ has limit $f(e^{i\theta})$ along $\mathcal{A}_h(e^{i\theta})$ at a.e. $e^{i\theta} \in \partial U$, where*

$$h(t) \lesssim \begin{cases} t(\log 1/t)^p & \text{if } 1 \le p < \infty, \\ t^{1-\varepsilon} \quad \text{for all } \varepsilon > 0 & \text{if } p = \infty. \end{cases}$$

How should we understand the tangential nature? It seems that the tangential nature is caused by the *non-integrability of the kernel.*

$$P(z, \zeta)^{1/2} = \sqrt{\frac{1}{2\pi}\frac{1 - |z|^2}{|z - \zeta|^2}} \sim \frac{1}{|z - \zeta|}.$$

Let us observe this phenomenon with the half space version due to Brundin [3] and Mizuta-Shimomura [12]. Define $(P_0 f)(x, t)$ by

$$\int_{\mathbf{R}^n} \left[\frac{t}{c_n(|x - y|^2 + t^2)^{(n+1)/2}}\right]^{n/(n+1)} f(y)dy.$$

Then $(P_0 1)(x, t) \equiv \infty$ (non-integrable). Fix a bounded open set $\Omega \subset \mathbf{R}^n$ and regard $(P_0 \chi_\Omega)(x, t)$ as a substitute of $(P_0 1)(x, t)$. Let us study the normalization $(P_0 f)(x, t)/(P_0 \chi_\Omega)(x, t)$.

Theorem E. *Let $1 \le p \le \infty$. Suppose, for small $t > 0$,*

(3.1) $\qquad h(t) \lesssim t(\log 1/t)^{p/n} \qquad$ *if $1 \le p < \infty$,*

(3.2) $\qquad h(t) \lesssim t^{1-\varepsilon}$ *for all $\varepsilon > 0$* $\quad$ *if $p = \infty$.*

If $f \in L^p(\mathbf{R}^n)$, then

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} \frac{(P_0 f)(x,t)}{(P_0\chi_\Omega)(x,t)} = f(\xi) \quad \textit{for a.e. } \xi \in \Omega.$$

Observe that

- For the critical power $n/(n+1)$, certain tangential limits exist.
- Possible tangency depends on the Lebesgue exponent p for which $f \in L^p(\mathbf{R}^n)$.

The tangential nature in Theorem E is caused by the *non-integrability of the kernel.* Let $\Phi(x) = \Psi(x)^{n/(n+1)} = (1 + |x|^2)^{-n/2}$. Then

$$\frac{(P_0 f)(x,t)}{(P_0\chi_\Omega)(x,t)} = \frac{\Phi_t * f(x)}{\Phi_t * \chi_\Omega(x)}.$$

Observe that $\Phi \notin L^1(\mathbf{R}^n)$; $\Phi \in L^p(\mathbf{R}^n)$ for $1 < p \le \infty$; and $\Phi_t * \chi_\Omega(x) \sim \log 1/t$ as $t \to 0$ for $x \in \Omega$. This is a sharp contrast between Ψ and Φ.

From now on we need not the explicit form $(1 + |x|^2)^{-n/2}$. Instead we suppose

- $\Phi(x) > 0$ is a doubling function of $|x|$.
- $\Phi \notin L^1(\mathbf{R}^n)$, $\Phi \in L^p(\mathbf{R}^n)$ for $1 < p \le \infty$.

Let

$$\varphi(r) = \int_{|x|<r} \Phi(x)dx.$$

Then $\varphi(r) \uparrow \infty$ is doubling. Assume

$$\lim_{r\to\infty} \frac{\varphi(2r)}{\varphi(r)} = 1. \tag{3.3}$$

This condition looks technical; but it turns out to be crucial as observed in Proposition 1 below. Fix a bounded open set $\Omega \subset \mathbf{R}^n$. Study the boundary behavior of the normalization

$$(\mathscr{P}_0 f)(x,t) = \frac{\Phi_t * f(x)}{\Phi_t * \chi_\Omega(x)}.$$

Proposition 1. *Condition* (3.3) *holds if and only if*

$$\lim_{t\to 0}(\mathscr{P}_0 f)(x,t) = f(x) \quad \textit{for } x \in \Omega$$

for all $f \in C_0(\mathbf{R}^n)$.

With (3.3) we obtain the following Fatou theorem for $(\mathscr{P}_0 f)(x,t)$.

Theorem 1. *Let $1 \le p \le \infty$. Suppose, for small $t > 0$,*

$$h(t) \lesssim t\varphi(1/t)^{p/n} \quad \text{if } 1 \le p < \infty, \tag{3.4}$$

$$\lim_{t\to 0} \frac{\varphi(h(t)/t)}{\varphi(1/t)} = 0 \quad \text{if } p = \infty. \tag{3.5}$$

If $f \in L^p(\mathbf{R}^n)$, then

$$\lim_{\substack{t\to 0\\ (x,t)\in \mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) = f(\xi) \quad \text{for a.e. } \xi \in \Omega.$$

Remark 1. Theorem 1 extends Theorem E.

- (3.4) $\Longrightarrow$ (3.5).
- If $\Phi(x) = (1 + |x|^2)^{-n/2}$, then
  (i) $\varphi(r) \sim \log r$ for large $r > 0$;
  (ii) (3.1) $\Longleftrightarrow$ (3.4), (3.2) $\Longleftrightarrow$ (3.5).

What is a Littlewood type theorem? The cases $1 \le p < \infty$ and $p = \infty$ are different.

Theorem 2. *Let $1 \le p < \infty$. If (3.4) does not hold, i.e.,*

$$\limsup_{t\to 0} \frac{h(t)}{t\varphi(1/t)^{p/n}} = \infty. \tag{3.6}$$

then there exists $f \in L^p(\Omega)$ such that for all $\xi \in \Omega$,

$$-\infty = \liminf_{\substack{t\to 0\\ (x,t)\in \mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in \mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) = \infty.$$

Theorem 3. *If (3.5) does not hold, i.e.,*

$$\limsup_{t\to 0} \frac{\varphi(h(t)/t)}{\varphi(1/t)} > 0. \tag{3.7}$$

then there exists $f \in L^\infty(\Omega)$ such that for all $\xi \in \Omega$,

$$\liminf_{\substack{t\to 0\\ (x,t)\in \mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in \mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t).$$

Let us close this section with the proof of Proposition 1. Let $B(x,r)$ be the open ball with center at x, radius r and $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. By $\operatorname{diam}\Omega$ we denote the diameter of Ω.

Proof of Proposition 1. For simplicity we assume that Ω is a bounded Lipschitz domain. For all $x \in \Omega$, there exists a cone $\Gamma(x) \subset \Omega$ with vertex at x and fixed aperture α and radius r_0. Change of variable gives

$$A\varphi\left(\frac{r_0}{t}\right) \le \Phi_t * \chi_\Omega(x) \le \varphi\left(\frac{\operatorname{diam}\Omega}{t}\right),$$

where $A > 0$ depends only on the aperture α. Since φ is doubling, it follows that

$$\Phi_t * \chi_\Omega(x) \sim \varphi\left(\frac{1}{t}\right) \quad \text{for } x \in \Omega. \tag{3.8}$$

Let $x \in \Omega$ and let $0 < \varepsilon < \delta_\Omega(x)$. Then (3.8) and the doubling of φ gives

$$\frac{\varphi(\delta_\Omega(x)/t) - \varphi(\varepsilon/t)}{\varphi(\varepsilon/t)} \lesssim (\mathscr{P}_0\chi_{\Omega\setminus B(x,\varepsilon)})(x,t) \lesssim \frac{\varphi(\operatorname{diam}\Omega/t) - \varphi(\varepsilon/t)}{\varphi(\varepsilon/t)}.$$

Hence $\lim_{t\to 0}(\mathscr{P}_0\chi_{\Omega\setminus B(x,\varepsilon)})(x,t) = 0$ if and only if (3.3) holds. Proposition 1 follows from this. □

4. Ingredients of Proof of Theorem 1

We state some estimates needed for the proof of Theorem 1. The complete proof will be given elsewhere. First we estimate the influence of the local part of f. If $p = \infty$, this is stated as follows.

Lemma 1. *Suppose h satisfies* (3.5). *Then*

$$\lim_{(x,t)\to(\xi,0)} (\mathscr{P}_0\chi_{B(x,4h(t))})(x,t) = 0 \quad \textit{for } \xi \in \Omega.$$

If $1 \le p < \infty$, then the Lebesgue point argument gives an estimate at almost every boundary point.

Lemma 2. *Let $1 \le p < \infty$ and $f \in L^p(\mathbf{R}^n)$. Suppose h satisfies* (3.4). *Then for a.e. $\xi \in \Omega$,*

$$\lim_{\substack{t\to 0\\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0[\chi_{B(x,4h(t))}f])(x,t) = 0.$$

On the other hand the influence of the global part is controlled by maximal functions. Define the truncated maximal function by

$$M_t f(x) = \sup_{r>t} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy$$

with $t \geq 0$. $Mf(x) = M_0 f(x)$ is the classical Hardy-Littlewood maximal function. Define another maximal function $\mathscr{M}_h f(\xi)$ by

$$\sup_{(x,t)\in\mathscr{A}_h(\xi)} \left| \frac{1}{\Phi_t * \chi_\Omega(x)} \int_{|x-y|\geq 4h(t)} \Phi_t(x-y)f(y)dy \right|$$

associated with the approach region $\mathscr{A}_h(\xi)$.

Lemma 3. *There is A such that*

$$\mathscr{M}_h f(\xi) \leq AMf(\xi) \quad \textit{for } \xi \in \Omega$$

for arbitrary $h(t) > 0$.

Lemma 4. *Let* $f \in L^p(\Omega)$ *with* $1 \leq p < \infty$. *Then*

$$\lim_{t\to 0} \|(\mathscr{P}_0 f)(\cdot, t) - f\|_p = 0.$$

As a result, for a.e. $x \in \Omega$, *some subsequence* $\{(\mathscr{P}_0 f)(x, t_j)\}_j$ *converges to* $f(x)$.

5. Outline of Proof of Theorem 2

Let us prove Theorem 2 with the aid of the following two lemmas, whose proof will be given elsewhere.

Lemma 5 (Lower Estimate). *We find* $0 < {}^{\exists}A_0 < 1$ *such that*

$$(\mathscr{P}_0 \chi_{B(x,r)})(x,t) \geq A_0 \frac{\varphi(r/t)}{\varphi(1/t)}$$

for $x \in \Omega$, $t > 0$, $r > 0$ *small.*

Lemma 6 (Upper Estimate). *If* $f \in L^1(\Omega)$, *then*

$$|(\mathscr{P}_0 f)(x,t)| \lesssim M_t f(x) \quad \textit{for } x \in \Omega.$$

Proof of Theorem 2. By (3.6) we find $t_j \downarrow 0$ such that

$$\frac{t_j \varphi(1/t_j)^{p/n}}{h(t_j)} \to 0.$$

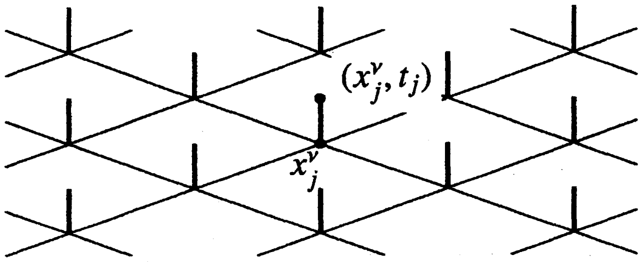
Let $\{x_j^\nu\}_\nu$ be lattice points $(h(t_j)/\sqrt{n})\mathbf{Z}^n$. Observe x_j^ν are vertices of cubes of side length $h(t_j)/\sqrt{n}$. Hence we have $x_j^\nu \in B(\xi, h(t_j))$.

If $\xi \in \Omega$, then

$$(x_j^\nu, t_j) \in \mathscr{A}_h(\xi) \quad \text{with } x_j^\nu \in \Omega,$$

provided j is sufficiently large.

Put vertical line segments connecting $(x_j^\nu, 0)$ and (x_j^ν, t_j). We obtain a bed of thorns. We observe that $\mathscr{A}_h(\xi)$ cannot touch Ω without being pierced by



some thorn. Now we construct f_j such that $(\mathscr{P}_0 f_j)(x,t)$ is large on each "thorn". Put

$$f_j = \varphi(\frac{1}{t_j})\chi_{D_j} \quad \text{with } D_j = \bigcup_\nu B(x_j^\nu, t_j) \cap \Omega.$$

Extract subsequence, find $c_j \uparrow \infty$ and let

$$f = \sum_{j=1}^{\infty} (-1)^j c_j f_j \in L^p(\mathbf{R}^n).$$

If j is even and $j \to \infty$, then

$$(\mathscr{P}_0 f)(x_j^\nu, t_j) \to \infty;$$

if j is odd and $j \to \infty$, then

$$(\mathscr{P}_0 f)(x_j^\nu, t_j) \to -\infty.$$

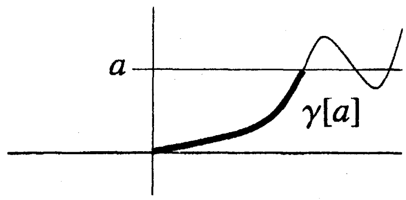
Since $\mathscr{A}_h(\xi)$ cannot touch Ω without being pierced by some thorn, we obtain

$$-\infty = \liminf_{\substack{t\to 0 \\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0 \\ (x,t)\in\mathscr{A}_h(\xi)}} (\mathscr{P}_0 f)(x,t) = \infty.$$

□

6. Oscillating limits along curves

If $p = \infty$, then a result stronger than Theorem 3 can be obtained. Let γ be a curve in $\mathbf{R}^{n+1}_+$ ending at the boundary. Let $\gamma[a]$ be the connected component of $\gamma \cap \{(x,t) : 0 \le t \le a\}$ containing the end point of γ.



Theorem 4. *Assume $\varphi(2r)/\varphi(r)$ is nonincreasing of r. Suppose γ is more tangential than* (3.5), *i.e.,*

$$\limsup_{t\to 0} \frac{\varphi(\operatorname{diam}(\gamma[t])/t)}{\varphi(1/t)} > 0. \tag{6.1}$$

Then there exists $f \in L^\infty(\Omega)$ such that for every $\xi \in \Omega$,

$$\liminf_{\substack{t\to 0\\ (x,t)\in\gamma+\xi}} (\mathscr{P}_0 f)(x,t) < \limsup_{\substack{t\to 0\\ (x,t)\in\gamma+\xi}} (\mathscr{P}_0 f)(x,t).$$

The proof of this theorem will be given elsewhere.

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Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: haikawa@math.shimane-u.ac.jp