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On 2-D Euler equations with initial vorticity in bmo

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INTRODUCTION.

In this paper we consider a two-dimensional ideal incompressible fluid described by the Euler equations:

$$\text{(E)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + u\cdot\nabla u + \nabla \mathbf{p} = 0, \quad \operatorname{div} u = 0 \quad \text{in } x \in \mathbb{R}^2,\ t\in(0,T), \\ u|_{t=0} = u_0 \end{cases}$$

where $u = (u^1(x,t), u^2(x,t))$ and $\mathbf{p} = \mathbf{p}(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t)\in\mathbb{R}^2\times(0,T)$, respectively, while $a = (a^1(x), a^2(x))$ is the given initial velocity vector. In this paper we consider a *nondecaying* initial data $u_0 \in L^\infty$ with initial vorticity $\omega_0 = \operatorname{rot} u_0 \in \mathbf{bmo}$. (Here $\mathbf{bmo} = \mathbf{BMO} \cap L^1_{\text{unif.loc}}$).

Many researchers have investigated the 2 dimensional Euler equations when the initial data has the decay property: $|u(x)| \to 0$ as $|x|\to\infty$ and $|\omega_0(x)|\to 0$ as $|x|\to\infty$ in some sense. For example, Di Perna-Majda[15] showed that if $\omega_0 = \operatorname{rot} u_0 \in L^1\cap L^p$ for $1<p<\infty$, then there exist a weak solution u on $[0,\infty)$ with

$$u\in L^\infty(0,\infty; W^{1,p}_{loc}(\mathbb{R}^2)), \quad \omega = \operatorname{rot} u \in L^\infty(0,\infty; L^p(\mathbb{R}^2)).$$

It is notable that Giga-Miyakawa-Osada[19] proved the similar result to [15] without the assumption $\omega_0\in L^1$ by using a different method. Chae[7] proved that if $\omega_0 \in L\log L$ $(\subset L^1)$, then there exist a weak solution u on $[0,\infty)$ with

$$u\in L^\infty(0,\infty; L^2(\mathbb{R}^2)).$$

Concerning the uniqueness theorem, Yudovich[38] showed that a solution u satisfying

$$u\in L^\infty(0,T;L^2), \quad \omega = \operatorname{rot} u\in L^\infty(0,T;L^\infty)$$

is uniquely determined by the initial data u_0. Moreover, in [39], he proved the uniqueness theorem for unbounded vorticity rot u. He showed that, for the Euler equations in a bounded domain Ω in $\mathbb{R}^n$, a solution u satisfying

$$\omega = u\in L^\infty(0,T;L^2(\Omega)), \quad \operatorname{rot} u\in L^\infty(0,T;V^\Theta)$$

is uniquely determined by the initial data u_0. Here V^Θ was introduced by Yudovich, is wider than $L^\infty(\Omega)$ and includes $\log^+\log^+(1/|x|)$. For the detail see [39]. Recently, Vishik showed the new uniqueness theorem for the solutions to (E) in the n-dimensional whole space $\mathbb{R}^n$. He proved that the uniqueness holds in the class

$$\text{(0.1)} \qquad \omega\in L^\infty(0,T;L^p(\mathbb{R}^n)\cap B_\Gamma(\mathbb{R}^n)) \quad \text{for some } 1<p<n,$$

where B_Γ is a space of Besov type and wider than $B^0_{\infty,\infty}$ and $\mathbf{bmo}$. Moreover, in the case $n=2$, he also proved that global existence of solutions to (E) in the class (0.1). However, for his global existence theorem, he imposed the slightly strong assumption on the initial vorticity ω_0: $\omega_0\in B_{\Gamma'}(\mathbb{R}^2)$, where $B_{\Gamma'}$ is strictly smaller than B_Γ and can not include **bmo**. He also imposed the integrability condition on ω_0: $\omega_0 \in L^p(\mathbb{R}^2)$ for some $1 < p < 2$. That is, he assumed that the initial vorticity ω_0 decays at infinity in some sense.

On the other hand, flows having *nondecaying* velocity at infinity are not only physically but also mathematically interesting. In this case, it is known that there exists a solution to the Euler equations which blows up in finite time. See e.g. Constantin[12]. Concerning bounded initial data with bounded vorticity, Serfati[30] proved the unique global existence of solution to (E) in $\mathbb{R}^2$ with initial data $(u_0, \omega_0) \in L^\infty \times L^\infty$ without any integrability condition. (In [29] he had proved it for the initial data $u_0 \in C^{1+\alpha}$.) In this paper, we improve his global existence theorem. We show that there exists a *global* solution to (E) in $\mathbb{R}^2$ with initial data $(u_0, \omega_0) \in L^\infty \times \mathbf{bmo}$. In [30], the well-known a-priori estimate $\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}$ plays important role. However, it seems to be difficult to establish the corresponding estimate in **bmo**. To overcome this difficulty, we introduce the uniformly localized version of Yudovich's space which is wider than **bmo** and we establish L^p_{ul}-estimate for solutions to the 2-D vorticity equation.

It is notable that, with respect to the Navier-Stokes equations, Cannon-Knightly[5], Cannone[6] and Giga-Inui-Matsui[17] proved the local existence of solutions to the Navier-Stokes equations with initial velocity $u_0 \in L^\infty$. Recently Giga-Matsui-Sawada[18] proved the global existence of solutions to the 2-dimensional Navier-Stokes equations with $u_0 \in L^\infty(\mathbb{R}^2)$.

1. Preliminaries and Main Results

Before presenting our results, we give some definitions. Let $B(x, r)$ denote the ball centered at x of radius r and let

$$\|f\|_{p;\Omega} \equiv \left(\int_{y\in\Omega} |f(y)|^p dy\right)^{1/p},$$

$$|\!|\!| f |\!|\!|_{p,\lambda} \equiv \sup_{x\in\mathbb{R}^2} \left(\|f\|_{p;B(x,\lambda)}\right) = \sup_x \left(\int_{|x-y|<\lambda} |f(y)|^p dy\right)^{1/p},$$

$$L^p_{ul} \equiv L^p_{\text{unif,loc}} = \{f \in L^1_{loc};\ |\!|\!| f |\!|\!|_{p,1} < \infty\},$$

$$\|f\|_{L^p_{ul}} \equiv |\!|\!| f |\!|\!|_{p,1} = \sup_x \left(\int_{|x-y|<1} |f(y)|^p dy\right)^{1/p}.$$

For $m = 0, 1, 2, \cdots$, and $0 < \alpha < 1$, let $C^{m+\alpha}$ denote the uniform Hölder space: $\left\{f\ ;\ \sum_{|\beta|\le m} \|\partial^\beta f\|_\infty + \sum_{|\beta|=m} \sup_{x\ne y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} < \infty\right\}$.

We first introduce the definition of weak solution to (E) as follows:

<u>DEFINITION 1. (WEAK SOLUTION)</u>

u is called weak solution to (E) on $[0, T]$ with initial data u_0, if u satisfies the following conditions:

$$u \in L^2_{loc}(\mathbb{R}^2 \times [0,T]), \quad \nabla \cdot u = 0 \text{ in } \mathcal{D}',$$
$$\int_0^T \int_{\mathbb{R}^2} \{-u \cdot \frac{\partial}{\partial t}\varphi - u^k u^l \frac{\partial}{\partial x_l}\varphi^k\} dx dt = \int_{\mathbb{R}^2} u_0 \cdot \varphi(0) dx$$
$$\text{for all } \varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^2) \text{ with } \nabla \cdot \varphi = 0.$$

Now we recall the Littlewood-Paley decomposition $\psi, \phi_j \in \mathcal{S}$, $j = 0, 1, \cdots$, such that

$$\text{supp } \hat{\psi} \subset \{|\xi| < 1\}, \quad \text{supp } \hat{\phi} \subset \{1/2 < |\xi| < 2\}, \quad \phi_j(x) = 2^{2j}\phi(2^j x)$$

$$(1.2) \qquad 1 = \hat{\psi}(\xi) + \sum_{j=0}^{\infty} \hat{\phi}_j(\xi) \quad (\xi \in \mathbb{R}^2) \quad \text{and} \quad 1 = \sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) \quad (\xi \neq 0),$$

where $\hat{f}$ denotes the Fourier transform of f.

We state the Besov spaces.

<u>Definition 2</u> (Besov Space cf. [2]). The inhomogeneous and homogeneous Besov spaces $B^s_{p,\rho}$ and $\dot{B}^s_{p,\rho}$ are defined as follows.

$$B^s_{p,\rho} \equiv \{f \in \mathcal{S}'; \|f\|_{B^s_{p,\rho}} < \infty\}, \quad \dot{B}^s_{p,\rho} \equiv \{f \in \mathcal{S}'; \|f\|_{\dot{B}^s_{p,\rho}} < \infty\},$$

where

$$\|f\|_{B^s_{p,\rho}} = \|\psi * f\|_p + (\sum_{j=0}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho}, \quad \|f\|_{\dot{B}^s_{p,\rho}} = (\sum_{j=-\infty}^{\infty} \|2^{js}\phi_j * f\|_p^\rho)^{1/\rho},$$

for $s \in \mathbb{R}$, $1 \le p, \rho \le \infty$. While $B^s_{p,\rho}$ is a Banach space, $\dot{B}^s_{p,q}$ is a semi-normed space, since

$$\|f\|_{\dot{B}^s_{p,q}} = 0 \text{ if and only if } f \text{ is a polynomial.}$$

It is notable that there holds

$$(1.3) \qquad \{f \in \mathcal{S}'; \ \|f\|_{\dot{B}^s_{p,\rho}} < \infty, \ f = \sum_{j=-\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}'\} \cong \dot{B}^s_{p,\rho}/\mathcal{P},$$

if

$$(1.4) \qquad s < n/p, \quad \text{or} \quad s = n/p \text{ and } \rho = 1,$$

for the detail see [23]. Here $\mathcal{P}$ denotes the set of all polynomials. Hence when s, p, ρ satisfy (1.4), we may modify the definition of Besov space as

$$(1.5) \qquad \dot{B}^s_{p,\rho} \equiv \{f \in \mathcal{S}'; \ \|f\|_{\dot{B}^s_{p,\rho}} < \infty, \ f = \sum_{j=-\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}'\}.$$

From now on we use (1.5) as the definition of $\dot{B}^s_{p,\rho}$ when s,p,ρ satisfy (1.4). Then if s,p,ρ satisfy (1.4), $\dot{B}^s_{p,\rho}$ is a Banach space and

$$\|f\|_{\dot{B}^s_{p,q}} = 0 \text{ if and only if } f = 0 \text{ in } \mathcal{S}'.$$

Next we state the Riesz operator $R_k = \partial_k(-\Delta)^{-1/2}$ $(k = 1,2)$ on Besov spaces. Let s,p,ρ satisfy (1.4) and let $f \in \dot{B}^s_{p,\rho}$. Then $R_k f$ can be defined by

$$R_k f \equiv \sum_{j=-\infty}^{\infty} (R_k\tilde{\phi}_j) * \phi_j * f \text{ in } \mathcal{S}' \tag{1.6}$$

where $\tilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}$. We note that $\hat{\tilde{\phi}}_j\hat{\phi}_j = \hat{\phi}_j$. Using this definition, we see that R_k is a bounded operator in $\dot{B}^s_{p\rho}$ as a subspace of $\mathcal{S}'$, if s,p,ρ satisfy (1.4). In particular, R_k is bounded in $\dot{B}^0_{\infty,1}$.

We introduced the space of **bmo**. For the detail, see e.g. [34]

<u>DEFINITION 3.</u> $\mathbf{bmo}(\mathbb{R}^n)$ is the space defined as a set for an $L^1_{loc}(\mathbb{R}^n)$ function f such that

$$\begin{aligned}\|f\|_{\mathbf{bmo}} \equiv & \sup_{0<r<1, x\in\mathbb{R}^n} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - \bar{f}_{B(x,r)}|dy \\ & + \sup_{x\in\mathbb{R}^n} \frac{1}{|B(x,1)|}\int_{B(x,1)} |f(y)|dy < \infty,\end{aligned} \tag{1.7}$$

where $\bar{f}_B$ stands for the average of f over B: $|B|^{-1}\displaystyle\int_B f(y)dy$.

Yudovich introduced some function space to prove the uniqueness theorem for the Euler equations. Here we introduce a slightly modified version of his space.

<u>DEFINITION 4.(YUDOVICH)</u>

Let $\Theta(p) \geq 1$ be a nondecaying function on $[1,\infty)$.

$Y^\Theta(\Omega) \equiv \{f \in \bigcap_{1\leq p<\infty} L^p(\Omega); \|f\|_{Y^\Theta(\Omega)} < \infty\}$, where

$$\|f\|_{Y^\Theta(\Omega)} \equiv \sup_{p\geq 1} \frac{\|f\|_{L^p(\Omega)}}{\Theta(p)}.$$

We note that $\log^+(1/|x|) \in Y^\Theta(\Omega)$, when $\Theta(p) = p$. Moreover, if Ω is a bounded domain, then $L^\infty(\Omega) \subset Y^\Theta(\Omega)$, since $\Theta(p) \geq 1$.

However, when $\Omega = \mathbb{R}^n$, $L^\infty(\mathbb{R}^n)$ and **bmo** can not be included in $Y^\Theta(\mathbb{R}^n)$. We want to consider wider spaces than **bmo** and L^∞. So we introduce a uniformly localized version of Y^Θ as follows.

<u>DEFINITION 5.</u>

Let $\Theta(p) \geq 1$ be a nondecaying function on $[1,\infty)$.

$Y_{ul}^{\Theta}(\mathbb{R}^n) \equiv \{f \in \bigcap_{1\le p<\infty} L_{ul}^p(\mathbb{R}^n); \|f\|_{Y_{ul}^{\Theta}(\mathbb{R}^n)} < \infty\}$, where

$$\|f\|_{Y_{ul}^{\Theta}(\mathbb{R}^n)} \equiv \sup_{p\ge 1} \frac{\|f\|_{L_{ul}^p(\mathbb{R}^n)}}{\Theta(p)},$$

$$\|f\|_{L_{ul}^p(\mathbb{R}^n)} \equiv \sup_{x\in\mathbb{R}^n} \left(\int_{|y-x|\le 1} |f(y)|^p dy\right)^{1/p}.$$

Then obviously there holds

$$L^\infty(\mathbb{R}^n) \subset Y_{ul}^{\Theta}(\mathbb{R}^n).$$

Moreover, we observe that

$$\mathbf{bmo}(\mathbb{R}^n) \subset Y_{ul}^{\Theta}(\mathbb{R}^n), \text{ when } \Theta(p) = p. \tag{1.8}$$

Now we state our main theorems.

Theorem 1.1. *Let $\Theta(p) = p$. Assume $u_0 \in L^\infty$ and $\omega_0 = \mathrm{rot}\ u_0 \in Y_{ul}^{\Theta}(\mathbb{R}^2)$. Then there exists a weak solution to (E) on $[0,\infty)$ in the class*

$$\begin{aligned} &u \in C([0,\infty); L^\infty) \\ &\mathrm{rot}\ u \in L_{loc}^\infty([0,\infty); Y_{ul}^{\Theta}) \end{aligned} \tag{1.9}$$

with

$$\|u(t)\|_\infty \le (\|u_0\|_\infty + 1)\exp\{C\exp(Ct(\|u_0\|_\infty + 1)(\|\omega_0\|_{Y_{ul}^{\Theta}} + 1)^2)\} \quad \textit{for all } t \ge 0.$$

From (1.8), obviously we obtain

Corollary 1.2. *Assume $u_0 \in L^\infty$ and $\omega_0 \in$ **bmo**. Then there exists a weak solution to (E) on $[0,\infty)$ in the class*

$$\begin{aligned} &u \in C([0,\infty); L^\infty) \\ &\mathrm{rot}\ u \in L_{loc}^\infty([0,\infty); Y_{ul}^{\Theta}) \quad (\Theta(p) = p) \end{aligned} \tag{1.10}$$

with

$$\|u(t)\|_\infty \le (\|u_0\|_\infty + 1)\exp\{C\exp(Ct(\|u_0\|_\infty + 1)(\|\omega_0\|_{\mathbf{bmo}} + 1)^2)\} \quad \textit{for all } t \ge 0.$$

Remarks 1. (i) We can generalize Theorem 1.1 as follows. For

$$\Theta(p) = p \cdot \underbrace{\log(e+p) \cdot \log(e + \log(e+p)) \cdot \log(e + \log(e + \log(e+p))) \cdots}_{k \text{ times iterated}},$$

Theorem 1 holds with the estimate:

$$\|u(t)\|_\infty \le C\|u_0\|_\infty \underbrace{\exp C \exp C \cdots \exp}_{k+2 \text{ times}}\{Ct(\|u_0\|_\infty + 1)(\|\omega_0\|_{Y_{ul}^{\Theta}} + 1)^2\}. \tag{1.11}$$

We note that if $\Theta(p) = 1$ ($Y_{ul}^{\Theta} = L^\infty$), then there holds the *single* exponential estimate

$$\|u(t)\|_\infty \le \|u_0\|_\infty \exp(t\|\omega_0\|_\infty), \tag{1.12}$$

which was already proved by Serfati[30]. We can show that these estimates (1.11) and (1.12) hold for the solution to the Navier-Stokes equations, too. (See [32] and [28]).

(iii) We should note what condition on ω_0 guarantees $u_0 \in L^\infty$. If we assume that $\omega_0 \in Y_{ul}^{\Theta} \cap \dot{B}^{-1}_{\infty,1}$, then u_0 belongs to $\in L^\infty$. For example, if $\omega_0(x) = \sin(\sqrt{2}x_1) + \sum_{k\in \mathbb{Z}^2}(-1)^{k_1}(-1)^{k_2}\log^+(1/|x-k|) + (1+|x|^2)^{-6}$, then $u_0 \in L^\infty$ and $\omega_0 \in \mathbf{bmo}$.
The following lemma plays crucial roll in proving Theorems 1.1.

Lemma 1.3 (Uniformly local L^p estimate for the vorticity equation). *Let $0 \le \nu \le 1$, $a \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^2))$ with $\nabla \cdot a = 0$ and let $v \in L^\infty(0,T;L^\infty(\mathbb{R}^2))$ be a solution to the 2 dimensional vorticity equation*

$$(1.13) \qquad \frac{\partial}{\partial t}v - \nu\Delta v + a\cdot\nabla v = 0, \quad \text{in } \mathbb{R}^2\times(0,T), \quad v|_{t=0} = v_0.$$

Then there holds for all $t \in [0,T]$ and all $p \ge 2$

$$(1.14) \quad \|v(t)\|_{L^p_{ul}} \le eC^{1/p}\left(1 + t(1+\sup_{0<\tau<t}\|a(\tau)\|_\infty)\right)^{2/p}\|v_0\|_{L^p_{ul}} \quad (2\le p\le\infty),$$

where C is an absolute constant (independent of ν, p, t, T, a and v).

Proposition 1.4. *(i) Let $\phi \in \mathcal{S}$. then there holds*

$$(1.15) \qquad \|\phi * f\|_\infty \le C\,|\!|\!|\, f \,|\!|\!|_{1,1} \quad \text{for all } f \in L^1_{ul},$$

where C is independent of f.
(ii) If $m \ge 1$, then

$$|\!|\!|\, f \,|\!|\!|_{q,m\lambda} \le (2m^2)^{1/q}\,|\!|\!|\, f \,|\!|\!|_{q,\lambda} \quad \text{for all } f \in L^q_{ul}(\mathbb{R}^2), \lambda > 0.$$

Proposition 1.5. *Let $1 \le q \le \infty$, $j = 0, \pm1, \pm2, \cdots$, $\phi \in \mathcal{S}$ and let $f \in L^q_{ul}(\mathbb{R}^2)$. Then there holds*

$$(1.16) \qquad \|\phi_j * f\|_\infty \le \begin{cases} C2^{2j/q}\,|\!|\!|\, f \,|\!|\!|_{q,1} & \text{for all } j \ge 0, \\ C\,|\!|\!|\, f \,|\!|\!|_{q,1} & \text{for all } j \le -1, \end{cases}$$

where C is independent of q, j and f. Here $\phi_j(\cdot) = 2^{2j}\phi(2^j\cdot)$.

2. Proof of L^p_{ul} estimate for the 2-D Vorticity equation

In this section we sketch the proof of Lemma 1.3. We first fix $x \in \mathbb{R}^2$, $\lambda \ge 1$. Let $\rho \in C_0^\infty(\mathbb{R}^2)$ with $\rho(y) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2 \end{cases}$, $\|\nabla\rho\|_\infty \le 2$, $\|\Delta\rho\|_\infty \le 4$ and let $\rho_{x,\lambda}(y) = \rho(\frac{y-x}{\lambda})$. We easily see that

$$(2.17) \quad \frac{\partial}{\partial t}(\rho_{x,\lambda}v) - \nu\Delta(\rho_{x,\lambda}v) + a\cdot\nabla(\rho_{x,\lambda}v) = -\nu 2\nabla\rho_{x,\lambda}\cdot\nabla v - \nu(\Delta\rho_{x,\lambda})v + a\cdot(\nabla\rho_{x,\lambda})v.$$

Taking inner product in $L^2(\mathbb{R}^2)$ between (2.17) and $|\rho_{x,\lambda}v|^{p-2}\rho_{x,\lambda}v$, we have

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\rho_{x,\lambda}v\|_p^p + \nu(p-1)\int_{\mathbb{R}^2}|\nabla(\rho_{x,\lambda}v)|^2|\rho_{x,\lambda}v|^{p-2}dy \\
\le& 2\nu\|\nabla\rho_{x,\lambda}\|_\infty\||\nabla v||\rho_{x,\lambda}v|^{(p-2)/2}\|_2\||\rho_{x,\lambda}v|^{p/2}\|_2 \\
&+(\nu\|\Delta\rho_{x,\lambda}\|_\infty + \|a\cdot\nabla\rho_{x,\lambda}\|_\infty)\int_{\operatorname{supp}\rho_{x,\lambda}}|v|^p dy \\
\le& 2\epsilon\nu\int_{\mathbb{R}^2}|\nabla v|^2|\rho_{x,\lambda}v|^{p-2}dy \\
&+(\frac{2\nu}{\epsilon}\|\nabla\rho_{x,\lambda}\|_\infty^2 + \nu\|\Delta\rho_{x,\lambda}\|_\infty + \|a\cdot\nabla\rho_{x,\lambda}\|_\infty)\int_{\operatorname{supp}\rho_{x,\lambda}}|v|^p dy \\
\le& 2\epsilon\nu\int_{B(x,2\lambda)}|\nabla v|^2|v|^{p-2}dy + (\frac{8\nu}{\epsilon\lambda^2}+\frac{4\nu}{\lambda^2}+\frac{2\|a\|_\infty}{\lambda}) \,|\!|\!|\, v \,|\!|\!|_{p,2\lambda}^p \text{ (for all } \epsilon>0).
\end{aligned}
\tag{2.18}
$$

Then we have

$$
\begin{aligned}
&\|\rho_{x,\lambda}v(t)\|_p^p + \nu p(p-1)\int_0^t\int_{B(x,\lambda)}|\nabla v|^2|v|^{p-2}dy \\
\le& \|v_0\|_{p;B(x,2\lambda)}^p + 2p\epsilon\nu\int_0^t\int_{B(x,2\lambda)}|\nabla v(\tau)|^2|v|^{p-2}dyd\tau \\
&+p(\frac{8\nu}{\epsilon\lambda^2}+\frac{4\nu}{\lambda^2}+\frac{2\sup_{0<\tau<t}\|a(\tau)\|_\infty}{\lambda})\int_0^t |\!|\!|\, v(\tau) \,|\!|\!|_{p,2\lambda}^p d\tau.
\end{aligned}
$$

Taking supremum of the above inequality over $x\in\mathbb{R}^2$,

$$
\begin{aligned}
&|\!|\!|\, v(\tau) \,|\!|\!|_{p,\lambda}^p + \nu p(p-1)\sup_{x\in\mathbb{R}^2}\int_0^t\int_{B(x,\lambda)}|\nabla v|^2|v|^{p-2}dy \\
\le& 8\Big\{ |\!|\!|\, v_0 \,|\!|\!|_{p,\lambda}^p + 2p\epsilon\nu\sup_{x\in\mathbb{R}^2}\int_0^t\int_{B(x,\lambda)}|\nabla v(\tau)|^2|v|^{p-2}dyd\tau \\
&+p(\frac{8\nu}{\epsilon\lambda^2}+\frac{4\nu}{\lambda^2}+\frac{2\sup_{0<\tau<t}\|a(\tau)\|_\infty}{\lambda})\sup_{x\in\mathbb{R}^2}\int_0^t |\!|\!|\, v(\tau) \,|\!|\!|_{p,\lambda}^p d\tau\Big\},
\end{aligned}
\tag{2.19}
$$

since

$$\sup_{x\in\mathbb{R}^2}\left(\int_0^t\int_{B(x,2\lambda)}|f|dyd\tau\right) \le 8\sup_{x\in\mathbb{R}^2}\left(\int_0^t\int_{B(x,\lambda)}|f|dyd\tau\right).$$

Hence, letting $\epsilon = 1/16$, we obtain

$$|\!|\!|\, v(\tau) \,|\!|\!|_{p,\lambda}^p \le 8\Big\{ |\!|\!|\, v_0 \,|\!|\!|_{p,\lambda}^p + p(\frac{132\nu}{\lambda^2}+\frac{2\sup_{0<\tau<t}\|a(\tau)\|_\infty}{\lambda})\int_0^t |\!|\!|\, v(\tau) \,|\!|\!|_{p,\lambda}^p d\tau\Big\},$$

which and Gronwall's inequality yield

$$|\!|\!|\, v(t) \,|\!|\!|_{p,\lambda}^p \le 8 \,|\!|\!|\, v_0 \,|\!|\!|_{p,\lambda}^p \exp\Big\{8\,p\,t(\frac{132\nu}{\lambda^2}+\frac{2\sup_{0<\tau<t}\|a(\tau)\|_\infty}{\lambda})\Big\}.$$

By Proposition 1.4 (ii) we have

$$\| v(t) \|_{p,1} \leq \| v(t) \|_{p,\lambda} \leq (16\lambda^2)^{1/p} \| v_0 \|_{p,1} \exp\left\{ 8t\left(\frac{132\nu}{\lambda^2} + \frac{2\sup_{0<\tau<t} \|a(\tau)\|_\infty}{\lambda}\right)\right\}$$

for all $t \in (0,T)$ and all $\lambda \geq 1$. Now let $\lambda = 1 + 8t(132\nu + 2\sup_{0<\tau<t} \|a(\tau)\|_\infty)$. Then there holds

$$\| v(t) \|_{p,1} \leq e(16)^{1/p}\{1 + 8t(132\nu + 2\sup_{0<\tau<t} \|a(\tau)\|_\infty)\}^{2/p} \| v_0 \|_{p,1},$$

which is the desired estimate (1.14).

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We show the existence of solution to (E) by establishing an a-priori estimate for an approximate solution sequence generated by mollifying initial data. In Step 1, we establish an a-priori estimate for regular solution. In Step 2, we discuss the limit of approximate solution sequence. (We can also show the existence of solution to (E) by the zero diffusion limit of the Navier-Stokes equations, i.e., the vanishing viscosity limit. See Remark 2 in this section.)

Step 1. Let $m = 1, 2, \cdots$, and $0 < \alpha < 1$. Serfati[29] proved that there exist the unique global solution $(u, \nabla \mathbf{p})$ of (E) in $C([0,\infty); C^{m+\alpha}) \times C([0,\infty); C^{m+\alpha})$ for the initial data $u_0 \in C^{m+\alpha}$ with $\mathbf{p}(x)/|x| \to 0$ as $|x| \to 0$.

We first assume $u_0 \in C^{2+\alpha}(\mathbb{R}^2)$ and construct an a–priori estimate for the global solution with $u_0 \in C^{2+\alpha}(\mathbb{R}^2)$. Let $(u, \mathbf{p})$ be the global regular solution to (E) with $u_0 \in C^{2+\alpha}(\mathbb{R}^2)$ given by Serfati[29]. Then it satisfies

$$\frac{\partial}{\partial t}u + u \cdot \nabla u + \nabla \mathbf{p} = 0 \text{ on } t \in [0,\infty)$$

and hence

$$u(t) - u(s) = -\int_s^t [(u \cdot \nabla u)(\tau) + \nabla \mathbf{p}(\tau)] d\tau \quad \text{in } L^\infty. \tag{3.20}$$

Moreover Serfati[29] proved that $\mathbf{p}$ have the following representation:

$$\nabla \mathbf{p} = \frac{1}{2\pi}(\nabla(\rho \log|\cdot|)) * \partial_i\partial_j u^i u^j + \frac{1}{2\pi}(\partial_i\partial_j\nabla(1-\rho)\log|\cdot|) * u^i u^j$$

for $\rho \in C_0^\infty$ with $\rho(x) = 1$ near the origin $x = 0$. Now, using the Riesz operator R_k on $\dot{B}^0_{\infty,1}$, (see (1.6)) we define the Helmholtz operator

$$P = (P_{ij})_{i,j=1,2} = (\delta_{ij} + R_i R_j).$$

The boundedness of R_k in $\dot{B}^0_{\infty,1}$ implies that P is also bounded in $\dot{B}^0_{\infty,1}$. Since $u \cdot \nabla u = \nabla \cdot (u \otimes u) \in C([0,\infty); \dot{B}^0_{\infty,1})$, we easily see that

$$u(t) = u(s) - \int_s^t P(u \cdot \nabla u)(\tau) d\tau \quad \text{in } L^\infty. \tag{3.21}$$

The Littlewood-Paley decomposition (1.2) is easily generalized as follows:

$$1 = \hat{\psi}_N(\xi) + \sum_{j=N}^{\infty} \hat{\phi}_j(\xi) \quad (\xi \in \mathbb{R}^2, N = 0, \pm 1, \pm 2, \cdots,). \tag{3.22}$$

Here $\psi_N(x) = 2^{2N}\psi(2^N x)$. Then by (3.22) we have

$$\|u\|_\infty \le \|\psi_N * u\|_\infty + \sum_{j=N}^{\infty} \|\phi_j * u\|_\infty \equiv I_1 + I_2 \quad (N = 0, \pm 1, \pm 2, \cdots,). \tag{3.23}$$

From (3.21) we see

$$\psi_N * u(t) = \psi_N * u(s) + \int_s^t \psi_N * (P\nabla \cdot (u \otimes u))(\tau) d\tau,$$

where $u \otimes u = (u^i u^j)_{i,j=1,2}$ and $\nabla \cdot (u \otimes u) = \sum_{i=1}^{2} \partial_i (u^i u)$. Hence we obtain

$$\begin{aligned} I_1 = \|\psi_N * u(t)\|_\infty &\le \|\psi_N * u(s)\|_\infty + \int_s^t \|(\nabla P \psi_N) * (u \otimes u)(\tau)\|_\infty d\tau \\ &\le C_0 \|u(s)\|_\infty + C_1 2^N \int_s^t \|u(\tau)\|_\infty^2 d\tau, \ (0 \le s \le t) \end{aligned} \tag{3.24}$$

since $\|\nabla P \psi_N\|_{L^1} \le \|\nabla P \psi_N\|_{\mathcal{H}^1} \le \|\nabla \psi_N\|_{\mathcal{H}^1} \le C 2^N$. Here $C_0 = \|\psi\|_{L^1} (\ge \hat{\psi}(0) = 1)$, $C_1 = \|\nabla \psi\|_{\mathcal{H}^1}$ and $\mathcal{H}^1$ denotes Hardy space. (The above estimate for I_1 is essentially due to [30], although the Littlewood-Paley decomposition was not utilized in [30]. See also [32].)

To estimate $I_2 = \sum_{j=N}^{\infty} \|\phi_j * u(t)\|_\infty$, we use the Biot-Savart Law:

$$\left(\frac{\partial}{\partial x_2}\omega, -\frac{\partial}{\partial x_1}\omega\right) = -\Delta u,$$

which yields

$$\begin{aligned} \phi_j * u &= \left((-\Delta)^{-1} \frac{\partial}{\partial x_2} \phi_j * \omega, \ -(-\Delta)^{-1} \frac{\partial}{\partial x_1} \phi_j * \omega\right) \\ \|\phi_j * u\|_\infty &\le C 2^{-j} \|\phi_j * \omega\|_\infty. \end{aligned}$$

Let $\varepsilon > 0$ and let $p \ge 2 + \varepsilon$. Then by the above inequality and Proposition 1.5 we have

$$\begin{aligned} \sum_{j=N}^{\infty} \|\phi_j * u(t)\|_\infty &\le C \sum_{j=N}^{\infty} 2^{-j} \|\phi_j * \omega(t)\|_\infty \\ &\le \begin{cases} C 2^{-(1-2/p)N} \|\omega(t)\|_{L^p_{ul}} \text{ if } N \ge 0 \\ C 2^{-N} \|\omega(t)\|_{L^p_{ul}} \text{ if } N \le -1 \end{cases} \\ &\le C 2^{-N} \max\{2^{2N/p}, 1\} \|\omega(t)\|_{L^p_{ul}}, \end{aligned} \tag{3.25}$$

where the constant C depends only on ε. Hence by Lemma 1.3 we observe that

(3.26)
$$\sum_{j=N}^{\infty}\|\phi_j * u(t)\|_\infty \le C2^{-N}\max\{2^{2N/p},1\}\left(1+(t-s)(1+\sup_{s\le\tau\le t}\|u(s)\|_\infty)\right)^{2/p}\|\omega(s)\|_{L^p_{ul}}$$

for all $0\le s\le t$, all $p\ge 2+\varepsilon$ and all $N=0,\pm1,\pm2,\cdots$. Gathering the estimates (3.24) and (3.26) with (3.23), we obtain

$$\begin{aligned}\|u(t)\|_\infty \le\; & C_0\|u(s)\|_\infty + C_1 2^N\int_s^t \|u(s)\|_\infty^2 ds\\ & + C2^{-N}\max\{2^{2N/p},1\}\left(1+(t-s)(1+\sup_{s\le\tau\le t}\|u(s)\|_\infty)\right)^{2/p}\|\omega(s)\|_{L^p_{ul}} \qquad (3.27)\\ & (0\le s\le t,\ p\ge 2+\varepsilon,\ N=0,\pm1,\pm2,\cdots).\end{aligned}$$

Let $g(t)\equiv 1+\sup_{0\le s\le t}\|u(s)\|_\infty$. Then (3.27) yields

$$g(t)\le C_0 g(s)+C_1 2^N g(t)^2(t-s)+C2^{-N}\max\{2^{2N/p},1\}g(t)^{2/p}\|\omega(s)\|_{L^p_{ul}} \qquad (3.28)$$

for all $0\le s\le t\le s+1$.

Now we fix $s\in[0,\infty)$. Since $g(t)$ is a continuous function and since $2C_0>1$, there holds $g(t)\le 2C_0 g(s)$ for t which is sufficiently close to s. Let

$$T_1(s)\equiv \sup\{\tau\ge 0\ ;\ g(s+\tau)\le 2C_0 g(s)\}.$$

(Since g is a nondecreasing function, $g(t)\le 2C_0 g(s)$ for all $s\le t\le s+T_1(s)$.) Then (3.28) yields

$$g(t)\le C_0 g(s)+4C_0^2 C_1 2^N g(s)^2(t-s)+C_2 2^{-N}\max\{2^{2N/p},1\}g(s)^{2/p}\|\omega(s)\|_{L^p_{ul}} \qquad (3.29)$$

for all $s\le t\le s+\min\{1,T_1(s)\}$ and all $p\ge 2+\varepsilon$. Here the constant $C_2=C_2(\varepsilon)$ depends only on ε. Now we choose N suitably as follows.

When $\|\omega(s)\|_{L^p_{ul}}=0$, letting $N\to-\infty$, we have $g(t)\le C_0 g(s)$ for all $t\in[s,s+\min\{1,T_1(s)\}]$.

When $\frac{C_0 g(s)^{1-\frac{2}{p}}}{8C_2\|\omega(s)\|_{L^p_{ul}}}\le 1$, we choose $N\ge 0$ such as $2^{-(1-2/p)N}\sim \frac{C_0 g(s)^{1-\frac{2}{p}}}{8C_2\|\omega(s)\|_{L^p_{ul}}}$.

When $\frac{C_0 g(s)^{1-\frac{2}{p}}}{8C_2\|\omega(s)\|_{L^p_{ul}}}> 1$, we choose $N\le -1$ such as $2^{-N}\sim \frac{C_0 g(s)^{1-\frac{2}{p}}}{8C_2\|\omega(s)\|_{L^p_{ul}}}$.

Then we have

$$g(t)\le C_0 g(s)+C_3(\varepsilon)\max\left\{\|\omega(s)\|_{L^p_{ul}}^{\frac{p}{p-2}},\ \|\omega(s)\|_{L^p_{ul}}g(s)^{\frac{2}{p}}\right\}g(s)(t-s)+\frac{C_0}{8}g(s)$$

for all $s\le t\le s+\min\{1,T_1(s)\}$ and hence

(3.30)
$$g(t)<\frac{13}{8}C_0 g(s)$$

$$\text{for all } s\le t\le s+\min\left\{\frac{C_0}{2C_3\|\omega(s)\|_{L^p_{ul}}^{p/(p-2)}},\ \frac{C_0}{2C_3 g(s)^{2/p}\|\omega(s)\|_{L^p_{ul}}},\ 1,\ T_1(s)\right\}.$$

Here the constant C_3 depends only on ε. Obviously from the definition of $T_1(s)$ we obtain

$$T_1(s) > \min\left\{ \frac{C_0}{2C_3\|\omega(s)\|_{L^p_{ul}}^{p/(p-2)}}, \frac{C_0}{2C_3 g(s)^{2/p}\|\omega(s)\|_{L^p_{ul}}}, 1 \right\}$$

and hence
(3.31)

$$g(t) \le 2C_0 g(s) \text{ for all } s \le t \le s + \min\left\{ \frac{C_0}{2C_3(\varepsilon)\|\omega(s)\|_{L^p_{ul}}^{p/(p-2)}}, \frac{C_0}{2C_3(\varepsilon) g(s)^{2/p}\|\omega(s)\|_{L^p_{ul}}}, 1 \right\}.$$

We note that this estimate holds for all $s \ge 0$ and $p \ge 2+\varepsilon$.

Let $\varepsilon = 2$. Set $p_k = k+4$ and $\{t_k\}_{k=0}^{\infty}$ be the increasing sequence defined by

$$t_0 \equiv 0$$

$$t_{k+1} - t_k \equiv \min\left\{ \frac{C_0}{2C_3\|\omega(t_k)\|_{L^{p_k}_{ul}}^{p_k/(p_k-2)}}, \frac{C_0}{2C_3 g(t_k)^{2/p_k}\|\omega(t_k)\|_{L^{p_k}_{ul}}}, 1 \right\}.$$

Then by (3.31) we have

$$g(t_k) \le 2C_0 g(t_{k-1}) \le \cdots \le (2C_0)^k g(0) \tag{3.32}$$

and hence

$$g(t_k)^{2/p_k} \le ((2C_0)^k g(0))^{2/(k+4)} \le C g(0)^{2/(k+4)} \le C g(0)^{1/2}. \tag{3.33}$$

Since $t_k \le k$, by Lemma 1.3 and (3.33) we see

$$\begin{aligned} \|\omega(t_k)\|_{L^{p_k}_{ul}} &\le C\left(1 + t_k(1 + \sup_{0\le\tau\le t_k}\|u(\tau)\|_\infty)\right)^{2/p_k}\|\omega_0\|_{L^{p_k}_{ul}} \\ &\le C[2kg(t_k)]^{2/p_k}\|\omega_0\|_{L^{p_k}_{ul}} \\ &\le C g(0)^{1/2}\|\omega_0\|_{L^{p_k}_{ul}} \le C g(0)^{1/2} p_k \|\omega_0\|_{Y^{\Theta}_{ul}}. \end{aligned}$$

Therefore we observe that

$$t_{k+1} - t_k \ge \frac{C}{g(0)(\|\omega_0\|_{Y^{\Theta}_{ul}} + 1)^2} \min\left\{ \frac{1}{p_k^{p_k/(p_k-2)}}, \frac{1}{p_k} \right\} \ge \frac{C}{g(0)(\|\omega_0\|_{Y^{\Theta}_{ul}} + 1)^2}\left(\frac{1}{k+4}\right)^{\frac{k+4}{k+2}},$$

$$t_k = \sum_{j=1}^{k}(t_j - t_{j-1}) + t_0 \ge \frac{C}{g(0)(\|\omega_0\|_{Y^{\Theta}_{ul}} + 1)^2}\sum_{j=1}^{k}\left(\frac{1}{j+3}\right)^{\frac{j+3}{j+1}} \ge \frac{C\log k}{g(0)(\|\omega_0\|_{Y^{\Theta}_{ul}} + 1)^2}$$

and hence

$$g\left(\frac{C\log k}{g(0)(\|\omega_0\|_{Y^{\Theta}_{ul}} + 1)^2}\right) \le g(t_k) \le (2C_0)^k g(0) \text{ for all } k = 1, 2, \cdots,$$

which implies

$$\|u(t)\|_\infty \le (\|u_0\|_\infty + 1)\exp(C\exp Ct(\|u_0\|_\infty + 1)(\|\omega_0\|_{Y^{\Theta}_{ul}} + 1)^2). \tag{3.34}$$

Moreover from Lemma 1.3 we easily obtain

$$\|\omega(t)\|_{Y_{ul}^{\Theta}} \leq C(1+t(1+\sup_{0\leq\tau\leq t}\|u(\tau)\|_\infty))\|\omega_0\|_{Y_{ul}^{\Theta}} \tag{3.35}$$

for $u_0 \in C^{2+\alpha}$. Here C is an absolute constant.

Step 2. Next, we consider the case $u_0 \notin C^{2+\alpha}$. That is, we assume only $u_0 \in L^\infty$ and $\omega_0 \in Y_{ul}^{\Theta}$. Let $\rho \in C_0^\infty(B(0,1))$, $\rho \geq 0$, $\int_{\mathbb{R}^2}\rho dy = 1$ and $\rho_\epsilon = \epsilon^{-2}\rho(x/\epsilon)$. Then we easily see that $\|\rho_\epsilon * f\,\|_{p,1} \leq 8\,\|\, f\,\|_{p,1}$ for $0 < \epsilon < 1$. Let $u_0^\epsilon = \rho_\epsilon * u_0$, $\omega_0^\epsilon = \text{rot }(u_0^\epsilon) = \rho_\epsilon * \omega_0^\epsilon$. Then $u_0^\epsilon \in C^{2+\alpha}$ and

$$\|u_0^\epsilon\|_\infty \leq \|u_0\|_\infty, \quad \|\omega_0^\epsilon\|_{Y_{ul}^{\Theta}} \leq 8\|\omega\|_{Y_{ul}^{\Theta}}. \tag{3.36}$$

Now we consider the solution u^ϵ to (E) with the initial data $u_0^\epsilon \in C^{2+\alpha}$ and the limit of u^ϵ as $\epsilon \to 0$. From (3.34), (3.35) and (3.36) we see that there exists locally bounded function $M(t) \in C([0,\infty))$ depending only on $\|u_0\|_\infty$ and $\|\omega_0\|_{Y_{ul}^{\Theta}}$ such that

$$\|u^\epsilon(t)\|_\infty + \|\omega^\epsilon(t)\|_{Y_{ul}^{\Theta}} \leq M(t)$$

for all $t \in [0,\infty)$. Here $\omega^\epsilon = \text{rot }(u^\epsilon)$. Let $0 < \delta < 1$. Since $C^\alpha = B^\alpha_{\infty,\infty}$ for $0 < \alpha < 1$, (see [34],) in the same way as in (3.25) we have for all $0 < \epsilon < 1$

$$\begin{aligned} \|u^\epsilon(t)\|_{C^{1-\delta}} &\leq C\|u^\epsilon(t)\|_{B^{1-\delta}_{\infty,\infty}} \leq C(\|u^\epsilon(t)\|_\infty + \|\,\omega^\epsilon(t)\,\|_{2/\delta,1}) \\ &\leq C(\|u^\epsilon(t)\|_\infty + \|\omega^\epsilon(t)\|_{Y_{ul}^{\Theta}}) \leq CM(t). \end{aligned} \tag{3.37}$$

Let the vector $(-\omega u^2, \omega u^1)$ be denoted by $\omega \times u$, for simplicity. Since $\phi_j * P(u^\epsilon \cdot \nabla u^\epsilon) = \phi_j * P(\omega^\epsilon \times u^\epsilon)$, we have

$$\begin{aligned} \|P(u^\epsilon(t)\cdot\nabla u^\epsilon(t))\|_{B^{-\delta}_{\infty,\infty}} &= \|P\nabla\psi * u^\epsilon(t)\otimes u^\epsilon(t)\|_\infty + \sup_{j\geq 0} 2^{-\delta j}\|\phi_j * P(\omega^\epsilon(t)\times u^\epsilon(t))\|_\infty \\ &\leq C(\|u^\epsilon(t)\|_\infty^2 + \|u^\epsilon(t)\|_\infty\,\|\,\omega^\epsilon(t)\,\|_{2/\delta,1}) \quad (\because \|P\nabla\psi\|_{\mathcal{H}^1} < \infty) \\ &\leq CM(t)^2. \end{aligned}$$

Since u^ϵ satisfies

$$u^\epsilon(t) = u^\epsilon(s) - \int_s^t P(u^\epsilon\cdot\nabla u^\epsilon)(\tau)d\tau, \tag{3.38}$$

we obtain

$$\|u^\epsilon(t) - u^\epsilon(s)\|_{B^{-\delta}_{\infty,\infty}} \leq C|t-s|\sup_{s\leq\tau\leq t} M(\tau)^2. \tag{3.39}$$

Now we recall that

$$\|vf\|_{B^\gamma_{\infty,\infty}} \leq C(v,\gamma)\|f\|_{B^\gamma_{\infty,\infty}} \tag{3.40}$$

for all $\gamma \in \mathbb{R}$, $v \in C_0^\infty$ and $f \in B^\gamma_{\infty,\infty}$, see [34, p203, Theorem 4.2.2]. Let $r \in \mathbb{R}_+$ and $\varphi \in C_0^\infty(B(0,2r))$ with $\varphi = 1$ on $B(0,r)$. Then by (3.37), (3.39) and (3.40) we see that

$$\{\varphi u^\epsilon(\cdot)\}_{\epsilon>0} \text{ is uniformly bounded in } B^{1-\delta}_{\infty,\infty}(= C^{1-\delta}) \text{ on } [0,T]$$

$$\{\varphi u^\epsilon(\cdot)\}_{\epsilon>0} \text{ is equicontinuous in } B^{-\delta}_{\infty,\infty} \text{ on } [0,T]$$

for all $0 < T < \infty$. Hence we observe that there exist a subsequence $\{u^{\epsilon'}\}$ of $\{u^{\epsilon}\}$ and u such that

(3.41) $$u^{\epsilon'} \to u \text{ weak-* in } L^\infty(0,T;L^\infty)$$

(3.42) $$\varphi u^{\epsilon'} \to \varphi u^{\epsilon'} \text{ strongly in } C([0,T];B^{-\delta}_{\infty,\infty})$$

by using the Ascoli theorem. Moreover, since

$$\|f\|_\infty \le \|f\|_{B^0_{\infty,1}} \le \rho\|f\|_{B^{1-\delta}_{\infty,\infty}} + C(\rho)\|f\|_{B^{-\delta}_{\infty,\infty}} \quad \text{for all } \rho > 0,$$

in the same way as in [33, p271, Theorem 2.1], we obtain that

$$\begin{aligned} &\varphi u^{\epsilon'} \to \varphi u \text{ strongly in } L^\infty(0,T;L^\infty) \text{ and hence} \\ &u^{\epsilon'} \to u \text{ strongly in } L^\infty(0,T;L^\infty(B(0,r))). \end{aligned}$$

Then by usual argument we conclude that there exists a subsequence $\{u^{\epsilon''}\}$ of $\{u^{\epsilon'}\}$ such that

$$u^{\epsilon''} \to u \text{ strongly in } L^\infty(0,T;L^\infty(B(0,r))) \quad \text{for all } r > 0.$$

This strong convergence implies that u satisfies (ii) of Definition 2, which proves Theorem 1.1. □

Remark 2. Let u^ν be a solution to the Navier-Stokes equations with viscosity $\nu(0 < \nu \le 1)$:
(N-S)

$$\begin{cases} \dfrac{\partial u^\nu}{\partial t} - \nu\Delta u^\nu + u^\nu \cdot \nabla u^\nu + \nabla \mathbf{p}^\nu = 0, \quad \text{div } u^\nu = 0 \quad \text{in } x \in \mathbb{R}^2,\ t \in (0,T), \\ u^\nu|_{t=0} = u_0 \in L^\infty. \end{cases}$$

Giga-Matsui-Sawada[18] proved the existence of the unique global-in-time solution u^ν which satisfies

(3.43) $$u^\nu(t) = e^{\nu t\Delta}u_0 - \int_0^t e^{\nu(t-s)\Delta}P(u^\nu \cdot \nabla u^\nu)(s)ds$$

for initial data $u_0 \in L^\infty$ without any integrability condition. They gave a special estimate for $\|u^\nu(s)\|_\infty$. It, however, depends on ν. On the other hand, we can construct new estimate independent of ν in the same way as in Step 1 of Section 3. Indeed, since $\omega^\nu = \text{rot } u^\nu$ satisfies the viscous vorticity equation: $\frac{\partial}{\partial t}\omega^\nu - \nu\Delta\omega^\nu + u^\nu \cdot \nabla\omega^\nu = 0$, $\omega^\nu|_{t=0} = \text{rot } u_0$, by Lemma 1.3 we have

$$\|\omega^\nu(t)\|_{L^p_{ul}} \le eC^{2/p}\left(1 + (t-s)\sup_{s<\tau<t}\|u^\nu(\tau)\|_\infty\right)^{2/p}\|\omega^\nu(s)\|_{L^p_{ul}} \quad (2 \le p \le \infty),$$

where C is independent of ν. Hence using the method in Step 1 of Section 3 with (3.21) replaced by (3.43), we have

(3.44) $$\|u^\nu(t)\|_\infty \le (\|u_0\|_\infty + 1)\exp(C\exp Ct(\|u_0\|_\infty + 1)(\|\omega_0\|_{Y^\Theta_{ul}} + 1)^2),$$

(3.45) $$\|\omega^\nu(t)\|_{Y^\Theta_{ul}} \le C(1 + t(1 + \sup_{0\le\tau\le t}\|u^\nu(\tau)\|_\infty))\|\text{rot } u_0\|_{Y^\Theta_{ul}},$$

where C is an absolute constant.

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