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WELL-POSEDNESS FOR THE BOUSSINESQ-TYPE SYSTEM RELATED TO THE WATER WAVE

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1. Introduction

This proceeding is a summary of the joint work [13] with Prof. Naoyasu Kita, Kyushu University.

We consider the initial value problem for the Boussinesq-type system:

$$\begin{cases} \partial_t u + \partial_x v + u\partial_x u = 0, & x, t \in \mathbf{R}, \\ \partial_t v - \partial_x^3 u + \partial_x u + \partial_x (uv) = 0, & x, t \in \mathbf{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \mathbf{R}. \end{cases} \tag{1}$$

This system was firstly proposed by Kaup [8] as a model for the dynamics of the water wave with the surface tension. In the above equations, u and v stand for the horizontal velocity of the fluid and the vertical displacement of the surface from the equilibrium state, respectively. For detail on the physical background, see e.g., Kaup [8].

As far as we know, there is only one well-posedness result about (1) (Here, the well-posedness stands for the existence, uniqueness of the solution and continuous dependence on the initial data). Angulo [1] proved the local well-posedness of the solution in Sobolev space $H^{s,0} \times H^{s-1,0}$ with $s > 3/2$, where

$$H_x^{\sigma,\alpha} = \{ f \in \mathcal{S}'(\mathbf{R}); \| \langle x \rangle^\alpha \langle D_x \rangle^\sigma f \|_{L_x^2} < \infty \}$$

with $\langle x \rangle^\alpha = (1+x^2)^{\alpha/2}$ and $\langle D_x \rangle^\sigma = \mathcal{F}^{-1} \langle \xi \rangle^\sigma \mathcal{F}$. His idea is based on the energy method in terms of the a priori estimate like

$$\frac{d}{dt}(\|u(t)\|^2_{H_x^{s,0}} + \|v(t)\|^2_{H_x^{s-1,0}}) \le C \|\partial_x u(t)\|_{L_x^\infty} (\|u(t)\|^2_{H_x^{s,0}} + \|v(t)\|^2_{H_x^{s-1,0}}).$$

Therefore, one requires $s > 3/2$ at least so that $\|\partial_x u(t)\|_{L_x^\infty}$ is estimated by the Sobolev inequality. He also obtained the global well-posedness in $H^{s,0} \times H^{s-1,0}$ with $s \ge 2$. Furthermore, the stability of the solitary waves is also studied by assuming the local well-posedness holds in $H_x^{1,0} \times L_x^2$. (There is no proof given for the local well-posedness in this function space. The authors think that it is still open, and we are inspired to minimize the regularity of initial data.)

Our concern at present paper is to construct a solution to (1) in the function space with less regularity than the Angulo's assumption. The main theorem is

Theorem 1.1. (i) Let $(u_0, v_0) \in (H_x^{s,0} \times H_x^{s-1,0}) \cap (H_x^{s_1,\alpha_1} \times H_x^{s_1-1,\alpha_1}) \equiv X^s$ with $s > s_1 + \alpha_1$, $s_1 > 1/2$ and $\alpha_1 > 1/2$. Then, for some $T > 0$, there exists a unique solution to (1) such that $(u(t), v(t)) \in C([0,T]; X^s)$ and $\langle x \rangle^{\alpha_1} u \in L_x^2(L_T^\infty)$. Furthermore, this solution satisfies the smoothing properties :

$$\|D_x^{s-1/2}\partial_x u\|_{L_x^\infty(L_T^2)} + \|D_x^{s-1/2} v\|_{L_x^\infty(L_T^2)} < \infty.$$

(ii) Let $(u'(t), v'(t))$ be a solution to (1) for the initial data (u_0', v_0') with $\|(u_0', v_0') - (u_0, v_0)\|_{X^s} < \delta$. If $\delta > 0$ is sufficiently small, then there exists some $T' \in (0,T)$ such that

$$\begin{aligned} \|(u', v') - (u, v)\|_{L_{T'}^\infty(X^s)} &\le C\|(u_0', v_0') - (u_0, v_0)\|_{X^s}, \\ \|D_x^{s-1/2}\partial_x(u' - u)\|_{L_x^\infty(L_{T'}^2)} &\le C\|(u_0', v_0') - (u_0, v_0)\|_{X^s}, \\ \|D_x^{s-1/2}(v' - v)\|_{L_x^\infty(L_{T'}^2)} &\le C\|(u_0', v_0') - (u_0, v_0)\|_{X^s}. \end{aligned}$$

From the view of regularity, Theorem 1.1 is the generalization of Angulo's work and very close to the desired $H_x^{1,0} \times L_x^2$ well-posedness problem. Our idea to prove Theorem 1.1 is based on the contraction mapping principle of the integral equation after deforming (1) into the system of nonlinear Schrödinger equations which contains the derivatives of unknown functions in its nonlinearity (see section 2), and also we make use of the smoothing properties of Schrödinger group due to Kenig-Ponce-Vega [11]. We remark here that the direct application of this smoothing properties to the system will demand the smallness assumption of the initial data. This is because the nonlinear estimate like $\|uD_x^{s-1/2}\partial_x u\|_{L_x^1(L_T^2)}$ yields the quantity $\|u\|_{L_x^1(L_T^\infty)}$ by the inclusion $L_x^1(L_T^\infty) \cdot L_x^\infty(L_T^2) \subset L_x^1(L_T^2)$ and we can not expect to make this sufficiently small only by shrinking the time interval $[0,T]$. To remove this smallness assumption, we make further deformation called gauge transform (see section 3). This idea was firstly introduced by Hayashi [5].

The regularity and weight constraints on the initial data as in Theorem 1.1 are given by the estimate of (so called) the maximal function associated with Schrödinger group, i.e.,

$$\|U(t)\phi\|_{L_x^1(L_T^\infty)} \;\le\; C(\|\phi\|_{H^{s,0}} + \|\phi\|_{H^{s_1,\alpha_1}}),$$

where $U(t) = \exp(it\partial_x^2)$ is the Schrödinger one-parameter group. This estimate is almost optimal. Namely, we know that it fails if $s < 1$ (see remark in section 4).

It seems difficult to obtain the stability result stated in Theorem1.1(ii) only by the energy method which is the main idea in [1]. In our argument, however, we largely relies on the contraction mapping principle for constructing the solution and so Theorem 1.1 (ii) is derived as a by-product.

We close this section by introducing several notations. The quantity $\|\cdot\|_X$ denotes the norm of a Banach space X. $\mathcal{B}(X)$ denotes the bounded linear operators on X. Let $L_x^p(L_T^r)$ and $L_T^r(L_x^p)$ be the function spaces $L_x^p(\mathbf{R}; L^r(0,T))$ and $L^r(0,T; L_x^p(\mathbf{R}))$, respectively. The fractional order derivative D_x^σ stands for $\mathcal{F}^{-1}|\xi|^\sigma\mathcal{F}$.

We often use 2×1 vector valued functions like $\overrightarrow{f}(t,x) = (f_1(t,x), f_2(t,x))^t$ and we let $\|\overrightarrow{f}\|_X = \|f_1\|_X + \|f_2\|_X$. The projection P_j ($j = 1,2$) is defined by $P_j\overrightarrow{f} = f_j$. The inhomogeneous part $\int_0^t U(t-t')F(t')dt'$ is described as GF.

2. Transformation of the System

In this section, we transform the system (1) into the nonlinear Scroödinger system. Let us proceed in two steps.

(Step1) *Decomposition in the Fourier space.* Let $\eta(\xi) \in C_0^\infty(\mathbf{R})$ with

$$\eta(\xi) = \begin{cases} 1 & \text{if } |\xi| < 1, \\ 0 & \text{if } |\xi| > 2, \end{cases}$$

and let

$$\begin{aligned} v^{(\ell)} &= \mathcal{F}^{-1}\eta(\xi)\mathcal{F}v \qquad \text{(low frequency part of } v\text{)}, \\ v^{(h)} &= \mathcal{F}^{-1}(1-\eta(\xi))\mathcal{F}v \quad \text{(high frequency part of } v\text{)}. \end{aligned}$$

We easily see that $v = v^{(\ell)} + v^{(h)}$. Then, $(u, v^{(\ell)}, v^{(h)})$ satisfies

$$\begin{cases} \partial_t u + \partial_x v^{(h)} + u\partial_x u + \partial_x v^{(\ell)} = 0, \\ \partial_t v^{(h)} + (1 - \mathcal{F}^{-1}\eta\mathcal{F})(-\partial_x^3 u + \partial_x u + \partial_x(uv^{(\ell)} + uv^{(h)})) = 0, \\ \partial_t v^{(\ell)} + \mathcal{F}^{-1}\eta\mathcal{F}(-\partial_x^3 u + \partial_x u + \partial_x(uv^{(\ell)} + uv^{(h)})) = 0. \end{cases} \tag{2}$$

we write

$$w = \partial_x^{-1} v^{(h)} \left(\equiv \int_{-\infty}^{x} v^{(h)}(y)\, dy \right).$$

Then, the first two equations in (2) yield

$$\begin{cases} \partial_t u + \partial_x^2 w + u\partial_x u + f = 0, \\ \partial_t w - \partial_x^2 u + u\partial_x w + g = 0, \end{cases} \tag{3}$$

where

$$\begin{aligned} f &= \partial_x v^{(\ell)}, \\ g &= u + uv^{(\ell)} + \mathcal{F}^{-1}\eta\mathcal{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)})). \end{aligned}$$

Note that f and g do not cause the loss of derivative. Since the symbol of $\partial_x^{-1}\mathcal{F}^{-1}(1-\eta)\mathcal{F}$ does not have a singularity at $\xi = 0$, this operator is bounded on the weighted Sobolev spaces and so $w \in H_x^{s_1,\alpha_1}$ if $v \in H_x^{s_1,\alpha_1}$. This is why we made the decomposition in Fourier space.

(Step2) *Diagonalization.* We next diagonalize the system (3). Set

$$\begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv R \begin{pmatrix} u \\ w \end{pmatrix}.$$

Then (3) is transformed into the nonlinear Schrödinger system:

$$\partial_t \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} + \begin{pmatrix} -i\partial_x^2 & 0 \\ 0 & i\partial_x^2 \end{pmatrix} \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} + u\partial_x \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} + R \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4}$$

For the simple expression of (4), we let

$$\vec{u}^{(1)} = \begin{pmatrix} u^{(1)} \\ \overline{w^{(1)}} \end{pmatrix} \equiv Q \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix},$$

where $\overline{w}^{(1)}$ denotes the complex conjugate of $w^{(1)}$. Then, $\vec{u}^{(1)}$ satisfies

$$\partial_t \vec{u}^{(1)} - i\partial_x^2 \vec{u}^{(1)} + A(u)\partial_x \vec{u}^{(1)} + \vec{f}^{(1)} = \vec{0}, \tag{5}$$

where

$$A(u) = \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}, \vec{f}^{(1)} = QR \begin{pmatrix} f \\ g \end{pmatrix}.$$

Hence, Boussinesq-type system (1) is transformed into

$$\begin{cases} \partial_t \vec{u}^{(1)} - i\partial_x^2 \vec{u}^{(1)} + \underbrace{A(u)\partial_x \vec{u}^{(1)}}_{\text{higher order}} + \underbrace{\vec{f}^{(1)}}_{\text{lower order}} = 0. \\ \partial_t v^{(\ell)} - \underbrace{\partial_x \mathcal{F}^{-1}\eta\mathcal{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)}))}_{\text{lower order}} = 0, \\ \vec{u}^{(1)}(0,x) \equiv \vec{u}_0^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} u_0 + iw_0 \\ -i\overline{u_0} + \overline{w_0} \end{pmatrix}, \\ v^{(\ell)}(0,x) \equiv v_0^{(\ell)} = \mathcal{F}^{-1}\eta\mathcal{F}v_0. \end{cases} \tag{6}$$

3. Gauge Transform

If we simply apply the Kenig-Ponce-Vega's method [11] (Their proof is based on the contraction mapping principle via the associated integral equation) to (6), the smallness of the initial data will be required even for showing the local well-posedness. To overcome this difficulty, we introduce the gauge transform. Let $\varphi \in C_0^\infty(\mathbf{R})$ which will be taken close to u_0 in $H^{s,0} \cap H^{s_1,\alpha_1}$ later and

$$\overrightarrow{u}^{(2)} = \begin{pmatrix} e^{i\partial_x^{-1}\varphi/2} & 0 \\ 0 & e^{i\partial_x^{-1}\overline{\varphi}/2} \end{pmatrix} \overrightarrow{u}^{(1)} \equiv K(\varphi)\overrightarrow{u}^{(1)},$$

where

$$\partial_x^{-1}\varphi \equiv \int_{-\infty}^{x} \varphi(y)dy.$$

To explain how to control the nonlinearity, we, for a while, consider the following simple equation:

$$i\partial_t u^{(1)} + \partial_x^2 u^{(1)} + \underbrace{iu\partial_x u^{(1)}}_{\text{heavy}} = 0. \tag{7}$$

The equation (7) is equivalent to

$$i\partial_t u^{(1)} + \partial_x^2 u^{(1)} + \underbrace{i(u-\varphi)\partial_x u^{(1)}}_{\text{negligible}} + \underbrace{i\varphi\partial_x u^{(1)}}_{\text{heavy}} = 0. \tag{8}$$

Set $u^{(2)} = e^{i\partial_x^{-1}\varphi/2}u^{(1)}$. Then, multiplying $e^{i\partial_x^{-1}\varphi/2}$ to (8), we see that

$$\begin{aligned} &i\partial_t u^{(2)} + \partial_x^2 u^{(2)} - \frac{i}{2}\partial_x\varphi u^{(2)} + \frac{1}{4}\varphi^2 u^{(2)} - \underbrace{i\varphi e^{i\partial_x^{-1}\varphi/2}\partial_x u^{(1)}}_{\text{heavy}} \\ &+ \underbrace{i(u-\varphi)\partial_x u^{(2)}}_{\text{negligible}} + \frac{1}{2}\varphi u u^{(2)} - \frac{1}{2}\varphi^2 u^{(2)} + \underbrace{i\varphi e^{i\partial_x^{-1}\varphi/2}\partial_x u^{(1)}}_{\text{heavy}} = 0. \end{aligned}$$

Thus, the heavy term is canceled and we have

$$i\partial_t u^{(2)} + \partial_x^2 u^{(2)} + i(u-\varphi)\partial_x u^{(2)} + \underbrace{\left(-\frac{i}{2}\partial_x\varphi - \frac{1}{4}\varphi^2 + \frac{1}{2}\varphi u\right)u^{(2)}}_{\text{lower}} = 0.$$

Since φ is smooth, the last term in the above equation does not cause the loss of derivative. We can not replace φ by u_0 since one of our aim is to minimize the regularity of the initial data.

Let us return to our original case. By the precise computation, $\vec{u}^{(2)}$ and $v^{(\ell)}$ satisfy

$$\left\{\begin{array}{l} i\partial_t\vec{u}^{(2)} + \partial_x^2\vec{u}^{(2)} + iA(u-\varphi)\partial_x\vec{u}^{(2)} + \vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)}) = \vec{0}, \\ \partial_t v^{(\ell)} - \partial_x\mathcal{F}^{-1}\eta\mathcal{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)})) = 0, \end{array}\right. \tag{9}$$

where $\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)}) = B(\varphi,u)\vec{u}^{(2)} + iK(\varphi)\vec{f}^{(1)}$ with

$$A(u-\varphi) = \begin{pmatrix} u-\varphi & 0 \\ 0 & \overline{u-\varphi} \end{pmatrix},$$

$$B(\varphi,u) = \tfrac{1}{4}\begin{pmatrix} -2i\partial_x\varphi - \varphi^2 + 2\varphi u & 0 \\ 0 & -2i\partial_x\overline{\varphi} - \overline{\varphi}^2 + 2\overline{\varphi u} \end{pmatrix},$$

$$K(\varphi) = \begin{pmatrix} e^{i\partial_x^{-1}\varphi/2} & 0 \\ 0 & e^{i\partial_x^{-1}\overline{\varphi}/2} \end{pmatrix}.$$

Note that $\vec{f}^{(2)}$ does not cause the loss of derivative.

The relation between (u,v) and $(\vec{u}^{(2)},v^{(\ell)})$ is invertible. In fact, $(u,w) = R^{-1}Q^{-1}K(\varphi)^{-1}\vec{u}^{(2)}$, where

$$R^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad Q^{-1}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ \overline{g} \end{pmatrix},$$

and

$$K(\varphi)^{-1} = \begin{pmatrix} e^{-i\partial_x^{-1}\varphi/2} & 0 \\ 0 & e^{-i\partial_x^{-1}\overline{\varphi}/2} \end{pmatrix}.$$

Hence $(u,v) = (u, \partial_x w + v^{(\ell)}) \in C([0,T];X^s)$ if and only if $(\vec{u}^{(2)},v^{(\ell)}) \in C([0,T];H_x^{s,0}\cap H_x^{s_1,\alpha_1})$. Therefore, the solutions to (9) with the initial data

$$\begin{aligned} \vec{u}^{(2)}(0,x) &= K(\varphi)QR(u_0, \partial_x^{-1}\mathcal{F}^{-1}(1-\eta)\mathcal{F}v_0)^t, \\ v^{(\ell)}(0,x) &= \mathcal{F}^{-1}\eta\mathcal{F}v_0. \end{aligned}$$

is immediately transformed into the solution to (1). Hereafter, let us mainly seek for the solution to (9).

4. Derivative Loss and Smoothing Effect

The equation (9) is rewritten as the integral equation:

$$\begin{aligned} \vec{u}^{(2)} &= U(t)\vec{u}^{(2)}(0) - G\{A(u-\varphi)\partial_x\vec{u}^{(2)} - i\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})\}, \\ v^{(\ell)} &= v^{(\ell)}(0) + \int_0^t \partial_x\mathcal{F}^{-1}\eta\mathcal{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)}))(t')dt'. \end{aligned} \tag{10}$$

To overcome the regularity loss in the nonlinearity, we apply the smoothing effect of the linear Schrödinger group. This kind of smoothing effect is firstly shown by Kato [7] for the KdV equation. Later on, Kenig-Ponce-Vega

[11] (also Bekiranov-Ogawa-Ponce [2]) obtained the Schrödinger equation version described below.

Lemma 4.1. [2, 11] Let $p \in [2, \infty]$ and $q \in [2, \infty)$. Then, we have

$$\begin{aligned} \|D_x^{1/2-1/p}U(t)\psi\|_{L_x^p(L_T^2)} &\le CT^{1/p}\|\psi\|_{L_x^2}, \\ \|D_x^{1-2/q}GF\|_{L_x^q(L_T^2)} &\le CT^{1/q}\|F\|_{L_x^1(L_T^2)}, \\ \|\partial_x GF\|_{L_x^\infty(L_T^2)} &\le C\|F\|_{L_x^1(L_T^2)}. \end{aligned}$$

The next lemma states the estimate of maximal function associated with Schrödinger group. It determines how large regularity we have to impose on the initial data.

Lemma 4.2. Let $s > s_1 + \alpha_1 > 1, s_1 > 1/2, \alpha_1 > 1/2$ and $\mu > 0$ sufficiently small. Then, we have

$$\begin{aligned} \|\langle D_x\rangle^\mu \langle x\rangle^{\alpha_1} U(t)\psi\|_{L_x^2(L_T^\infty)} &\le C(\|\psi\|_{H_x^{s,0}} + \|\psi\|_{H_x^{s_1,\alpha_1}}), \\ \|\langle x\rangle^{\alpha_1} GF\|_{L_x^2(L_T^\infty)} &\le CT^{1/2}\|D_x^{s-1/2}F\|_{L_x^1(L_T^2)} \\ &\quad + C\|\langle D_x\rangle^{s_1}\langle x\rangle^{\alpha_1}F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}, \end{aligned}$$

where $\|f\|_{X+Y} = \inf\{\|g\|_X + \|h\|_Y; g + h = f\}$.

Proof of Lemma 4.2. We only prove the first inequality. The second one follows from the similar argument and simple application of the Strichartz estimate [18, 20]. Let $f(t,x) = U(t)\phi(x)$. Then, it satisfies

$$\begin{cases} i\partial_t f = -\partial_x^2 f, \\ f(0,x) = \phi(x). \end{cases} \tag{11}$$

Multiplying $\langle x\rangle^{\alpha_1}$ on both hand sides of (11), we have

$$i\partial_t(\langle x\rangle^{\alpha_1} f) = -\partial_x^2(\langle x\rangle^{\alpha_1} f) + 2(\partial_x\langle x\rangle^{\alpha_1})\partial_x f + (\partial_x^2\langle x\rangle^{\alpha_1})f.$$

Rewriting the above relation by Duhamel's principle, we see that

$$\langle x\rangle^{\alpha_1} U(t)\phi = U(t)\langle x\rangle^{\alpha_1}\phi - 2iG(\partial_x\langle x\rangle^{\alpha_1}\partial_x f) - iG(\partial_x^2\langle x\rangle^{\alpha_1} f). \tag{12}$$

According to (12), we see have

$$\begin{aligned} &\|\langle D_x\rangle^\mu\langle x\rangle^{\alpha_1}U(t)\phi\|_{L_x^2(L_T^\infty)} \\ \le\ & \|U(t)\langle D_x\rangle^\mu\langle x\rangle^{\alpha_1}\phi\|_{L_x^2(L_T^\infty)} + 2\|G\langle D_x\rangle^\mu(\partial_x\langle x\rangle^{\alpha_1})\partial_x U(t')\phi\|_{L_x^2(L_T^\infty)} \\ & + \|G\langle D_x\rangle^\mu(\partial_x^2\langle x\rangle^{\alpha_1})U(t')\phi\|_{L_x^2(L_T^\infty)} \\ \equiv\ & I_1 + I_2 + I_3. \end{aligned}$$

Let us use the well-known estimate (Constantin-Saut [4], Sjölin [15] and Vega [19])

$$\|U(t)\psi\|_{L^2_x(L^\infty_T)} \le C_T\|\psi\|_{H^{\sigma,0}_x} \quad \text{for } \sigma > 1/2.$$

Then, we have

$$\begin{aligned} I_1 &\le C\|\phi\|_{H^{s_1,\alpha_1}_x}, \\ I_3 &\le C\|(\partial_x^2\langle x\rangle^{\alpha_1})U(t)\phi\|_{L^\infty_T(H^{s_1,0}_x)} \\ &\le C\|\phi\|_{H^{s_1,\alpha_1}_x}. \end{aligned}$$

On the other hand, applying Lemma 4.1 to I_2 and making use of the fact that $[\langle D_x\rangle^{s_1}, \partial_x\langle x\rangle^{\alpha_1}]$ is the $s_1 - 1$th order pseudo-differential operator (see Stein [17], chapter VI), we see that

$$\begin{aligned} I_2 &\le CT^{1/2}\|\langle D_x\rangle^{1/2+\epsilon}(\partial_x\langle x\rangle^{\alpha_1})\partial_x U(t)\phi\|_{L^2_T(L^2_x)} \\ &\le C(\|(\partial_x\langle x\rangle^{\alpha_1})D_x^{1/2+\epsilon}\partial_x U(t)\phi\|_{L^2_x(L^2_T)} + \|\phi\|_{H^{s,0}_x}) \\ &\le C(\|D_x^{3/2+\epsilon}U(t)\phi\|_{L^q_x(L^2_T)} + \|\phi\|_{H^{s,0}_x}) \\ &\le C\|\phi\|_{H^{s,0}_x}, \end{aligned}$$

where $1/q > \alpha_1 - 1/2$. Hence, we obtain Lemma 4.2. □

Remark. The regularity condition in the first estimate of Lemma 4.2 is almost sharp. Indeed, we consider the smooth function $\phi \in C_0^\infty(-1,1)$. Set $\phi_n(x) = e^{inx}\phi(x)$. Then it is easy to show that $\|\phi_n\|_{H^s} = O(n^s)$ as n tends to ∞.

On the other hand, we have

$$\begin{aligned} \|U(t)\phi_n\|_{L^1_x(L^\infty_T)} &\le \|\int e^{-it\xi^2+ix\xi}\hat{\phi}(\xi-n)d\xi\|_{L^1_x(L^\infty_T)} \\ &\le \|\int e^{-it\xi^2+i(x-2nt)\xi}\hat{\phi}(\xi)d\xi\|_{L^1_x(L^\infty_T)}. \end{aligned}$$

We take $t = x/2n$. Note that $0 \le x \le 2nT$. Then it follows that

$$\begin{aligned} \|U(t)\phi_n\|_{L^1_x(L^\infty_T)} &\ge \int_0^{2nT}\left|\int e^{-ix\xi^2/2n}\hat{\phi}(\xi)d\xi\right|dx \\ &= 2n\int_0^{2T}\left|\int e^{-i\xi^2}\hat{\phi}(\xi)d\xi\right|dx \\ &= O(n^1) \qquad \text{as } n \to \infty. \end{aligned}$$

Consequently, the inequality

$$\|U(t)\phi\|_{L^1_x(L^\infty_T)} \le C\|\phi\|_{H^s},$$

fails if $s<1$. It is still open whether the case $s=1$ holds or fails.

5. Contraction Mapping Principle

In this section, we give the outline of the proof for Theorem 1.1. The main tool is the contraction mapping principle in terms of the smoothing properties of $U(t)$ and G. For simplicity, we only consider the case $s\in(1,3/2)$. Let us introduce the function spaces.

$$|||g|||_{Y_T} \equiv |||g|||_{initial}+|||g|||_{smooth}+|||g|||_{maxim},$$

where

$$\begin{aligned}|||g|||_{initial} &\equiv \|\langle D_x\rangle^s g\|_{L_T^\infty(L_x^2)}+\|\langle D_x\rangle^{s_1}\langle x\rangle^{\alpha_1} g\|_{L_T^\infty(L_x^2)}\\ |||g|||_{smooth} &\equiv \|\langle D_x\rangle^{s-1/2}\partial_x g\|_{L_x^\infty(L_T^2)}\\ |||g|||_{maxim} &\equiv \|\langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1} g\|_{L_x^2(L_T^\infty)}, \qquad (\mu>0 \textit{ is small}).\end{aligned}$$

We show that the map (Φ,Ψ) defined by

$$\begin{aligned}\vec{u}^{(2)} &= \Phi(\vec{u}^{(2)},v^{(\ell)})\\ &\equiv U(t)\vec{u}^{(2)}(0)-G\{A(u-\varphi)\partial_x\vec{u}^{(2)}-i\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})\},\\ v^{(\ell)} &= \Psi(\vec{u}^{(2)},v^{(\ell)})\\ &\equiv v_\ell(0)+\int_0^t \partial_x\mathcal{F}^{-1}\eta\mathcal{F}(\partial_x^2u-u-u(\partial_xw+v^{(\ell)}))(t')dt',\end{aligned}$$

is the contraction on $S_{u_0,v_0,\rho}$, where the closed set $S_{u_0,v_0,\rho}$ is given by

$$S_{u_0,v_0,\rho}=\left\{(\vec{u}^{(2)},v^{(\ell)});\quad \begin{array}{l}|||\vec{u}^{(2)}|||_{Y_T}+|||\langle D\rangle v^{(\ell)}|||_{initial}\le 2C(u_0,v_0),\\ \|\langle x\rangle^{\alpha_1}(u-\varphi)\|_{L_x^2(L_T^\infty)}\le\rho\\ \text{and } \|\langle D_x\rangle^{s-1/2}\partial_x\vec{u}^{(2)}\|_{L_x^\infty(L_T^2)}\le\rho\end{array}\right\},$$

with the metric $|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'}\equiv|||\vec{u}^{(2)}|||_{Y_T}+|||\langle D_x\rangle v^{(\ell)}|||_{initial}$. Note that $S_{u_0,v_0,\rho}\neq\phi$, if φ is sufficiently close to u_0 in $H^{s,0}\cap H^{s_1,\alpha_1}$ and $\rho>0$ is small enough.

We first show that the map (Φ,Ψ) is from $S_{u_0,v_0,\rho}$ into itself. In fact, Lemma 4.1 yields

$$\begin{aligned}\|D_x^s\Phi\|_{L_T^\infty(L_x^2)} \le\ & C\|\vec{u}^{(2)}\|_{H_x^{s,0}}+C\|D_x^{s-1/2}A(u-\varphi)\partial_x\vec{u}^{(2)}\|_{L_x^1(L_T^2)}\\ &+CT\|\vec{f}^{(2)}\|_{L_T^\infty(H_x^{s,0})}.\end{aligned}\tag{13}$$

The first term in (13) is bounded by $C\|(u_0,v_0)\|_{X^s}$, where the positive constant C does not diverge as $\varphi\to u_0$ in $H_x^{s,0}\cap H_x^{s_1,\alpha_1}$ (This convention will be continued in what follows). To estimate the second term in (13), we use the chain rule for the fractional order derivative (see Appendix in [12]), i.e.,

$$\|D_x^\sigma(fg)-(D_x^\sigma f)g-f(D_x^\sigma g)\|_{L_x^1(L_T^2)}\le C\|D_x^{\sigma_1}f\|_{L_x^{p_1}(L_T^{r_1})}\|D_x^{\sigma_2}g\|_{L_x^{p_2}(L_T^{r_2})},$$

where $\sigma,\sigma_1,\sigma_2\in(0,1)$, $\sigma=\sigma_1+\sigma_2$ and $p_j,r_j\in(1,\infty)$ (j=1,2) with $1/p_1+1/p_2=1$ and $1/r_1+1/r_2=1/2$. Let $f=A(u-\varphi)$ and $g=\partial_x h=\partial_x\vec{u}^{(2)}$. Then, for some $\theta\in(0,1)$, we have

$$\begin{aligned}
&\|D_x^{s-1/2}(f\partial_x h)\|_{L_x^1(L_T^2)}\\
&\le \|f(D_x^{s-1/2}\partial_x h)\|_{L_x^1(L_T^2)}+C\|D_x^{s-1/2}f\|_{L_x^{p_1}(L_T^{r_1})}\|D_x h\|_{L_x^{p_2}(L_T^{r_2})}\\
&\le C(\|\langle D_x\rangle^\mu f\|_{L_x^1(L_T^\infty)}+\|\langle D_x\rangle^\mu f\|^\theta_{L_x^1(L_T^\infty)}\|\langle D_x\rangle^{s-1/2}f\|^{1-\theta}_{L_x^\infty(L_T^2)}) \qquad (14)\\
&\quad\times(\|\langle D_x\rangle^{s-1/2}\partial_x h\|_{L_x^\infty(L_T^2)}+\|\langle D_x\rangle^\mu h\|^{1-\theta}_{L_x^1(L_T^\infty)}\|\langle D_x\rangle^{s-1/2}\partial_x h\|^\theta_{L_x^\infty(L_T^2)}),
\end{aligned}$$

where $s-1/2=\theta\mu/2+(1-\theta)(s-1/2-\mu/2)$, $1/p_1=\theta/1+(1-\theta)/\infty$, $1/p_2=(1-\theta)/1+\theta/\infty$, $1/r_1=\theta/\infty+(1-\theta)/2$ and $1/r_2=(1-\theta)/\infty+\theta/2$. Note that, to show (14), we used $L_x^{p_2}(L_T^{r_2})$-boundedness of the Hilbert transform (see Stein [16], Chapter II) and the interpolation inequalities. The third term in (13) is easily estimated as

$$\|\vec{f}^{(2)}\|_{L_T^\infty(H_x^{s,0})}\le C_\varphi(1+|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'})^2, \qquad (15)$$

where $C_\varphi>0$ may diverge as $\varphi\to u_0$ in $H_x^{s,0}\cap H_x^{s_1,\alpha_1}$. By the combination of (13)–(15), it turns out that

$$\begin{aligned}
\|\Phi\|_{L_T^\infty(H_x^{s,0})} &\le C\|(u_0,v_0)\|_{X^s}+CL(\varphi,T)C(u_0,v_0)\\
&\quad+TC_\varphi(1+2C(u_0,v_0))^2, \qquad (16)
\end{aligned}$$

where

$$\begin{aligned}
L(\varphi,T) &\equiv |||u-\varphi|||_{maxim}+|||u-\varphi|||^\theta_{maxim}(2C(u_0,v_0))^{1-\theta}\\
&\le \rho+\rho^\theta(2C(u_0,v_0))^{1-\theta}.
\end{aligned}$$

By Lemma 4.1 and the argument similar to the derivation of (16), we see that

$$\begin{aligned}
|||\Phi|||_{smooth} &\le C\|(u_0,v_0)\|_{X^s}+CL(\varphi,T)C(u_0,v_0)\\
&\quad+TC_\varphi(1+2C(u_0,v_0))^2. \qquad (17)
\end{aligned}$$

By Lemma 4.2 and the Strichartz type estimate in the weighted norm spaces, we have

$$\begin{aligned}
\|\Phi\|_{L_T^\infty(H_x^{s_1,\alpha_1})}+|||\Phi|||_{maxim} &\le C\|(u_0,v_0)\|_{X^s}+CL(\varphi,T)C(u_0,v_0)\\
&\quad+T^\beta C_\varphi(1+2C(u_0,v_0))^2. \qquad (18)
\end{aligned}$$

It is easy to see that

$$|||\langle D_x\rangle\Psi|||_{initial} \leq C\|(u_0,v_0)\|_{X^s}+TC(1+2C(u_0,v_0))^2. \quad (19)$$

Then, combining (16)–(19), we have

$$\begin{aligned}|||(\Phi,\Psi)|||_{Y'_T} \leq\ & C\|(u_0,v_0)\|_{X^s}+CL(\varphi,T)C(u_0,v_0)\\ &+T^\beta C_\varphi(1+2C(u_0,v_0))^2. \quad (20)\end{aligned}$$

Let $(\Phi_j,\Psi_j)=(\Phi(\vec{u}_j^{(2)},v_j^{(\ell)}),\Psi(\vec{u}_j^{(2)},v_j^{(\ell)}))$ $(j=1,2)$ for $(\vec{u}_j^{(2)},v_j^{(\ell)})\in S_{u_0,v_0,\rho}$. Then, similarly to (20), we gain

$$\begin{aligned}&|||(\Phi_1,\Psi_1)-(\Phi_2,\Psi_2)|||_{Y'_T}\\ &\leq (CM(\varphi,T)+C_\varphi T^\beta)|||(\vec{u}_1^{(2)},v_1^{(\ell)})-(\vec{u}_2^{(2)},v_2^{(\ell)})|||_{Y'_T}, \quad (21)\end{aligned}$$

where

$$\begin{aligned}M(\varphi,T) \equiv\ & |||\vec{u}_1^{(2)}|||_{smooth}+(2C(u_0,v_0))^{1-\theta}|||\vec{u}_1^{(2)}|||^\theta_{smooth}\\ &+|||u_2-\varphi|||_{maxim}+|||u_2-\varphi|||^\theta_{maxim}(2C(u_0,v_0))^{1-\theta}\\ \leq\ & 2\rho+2\rho^\theta(2C(u_0,v_0))^{1-\theta}.\end{aligned}$$

We next show that $|||P_1R^{-1}Q^{-1}K(\varphi)^{-1}\Phi-\phi|||_{maxim}\leq\rho$ and $|||\Phi|||_{smooth}\leq\rho$ and that we can take $\rho>0$ as small as we like by choosing $\varphi\in C_0^\infty(\mathbf{R})$ and $T>0$ suitably. This follows from the lemma given below (The proof is omitted).

Lemma 5.1 Let $(\vec{u}^{(2)},v^{(\ell)})\in S_{u_0,v_0,\rho}$ and $\varphi,\psi\in C_0^\infty(\mathbf{R})$. Then, there exist some $\theta\in(0,1)$ and $\beta>0$ such that

$$\begin{aligned}&|||P_1R^{-1}Q^{-1}K(\varphi)^{-1}\Phi-\varphi|||_{maxim}\\ &\leq C\|(u_0-\varphi,v_0-\psi)\|_{X^s}+C_\varphi T^\beta(1+C(u_0,v_0))^2,\\ &|||\Phi|||_{smooth}\\ &\leq C(\|(u_0-\varphi,v_0-\psi)\|_{X^s}+\|(u_0-\varphi,v_0-\psi)\|^\theta_{X^s}C(u_0,v_0)^{1-\theta})C(u_0,v_0)\\ &+C_\varphi T^\beta(1+C(u_0,v_0))^2.\end{aligned}$$

Lemma 5.1 suggests that we can choose $\rho>0$ sufficiently small by letting (φ,ψ) close to (u_0,v_0) in X^s and taking $T>0$ small enough. Hence, by (20) and (21), (Φ,Ψ) is the contraction map and the existence of the solution follows. The uniqueness and stability of the solution are obtained by the standard manner.

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