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On weak dividing in n-simple theories

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Weak dividing was originally defined by Shelah in [1]. After a long time, Dolich characterized that notion in simple context in [2]. Then Kim and Shi continued the investigation, in particular they proved that a theory T is stable if and only if weak dividing is symmetric in [3]. Recently, the class of simple theory was split into $\omega+1$ subclasses by Kolesnikov in [4]. He used the notion of n-simplicity for $n \leq \omega$. I studied his paper and had some consideration about the relation with weak dividing.
At first, we recall some definitions in [4].

For $n \geq 2$, let the symbol $\mathrm{Ind}(x; y_0, \ldots, y_{n-1})$ denote the type expressing that $y_0, \ldots, y_{n-1}$ are indiscernible over x.

Definition 1 Fix $1 \leq n \leq k < \omega$. Take a formula $\varphi(x, y_0, \ldots, y_{n-1})$ and a partial type $p(x)$. Define $D_n[p, \varphi, k] \geq \alpha$ by induction on α.
(1) $D_n[p, \varphi, k] \geq 0$ if p is consistent.
(2) for α limit, $D_n[p, \varphi, k] \geq \alpha$ if $D_n[p, \varphi, k] \geq \beta$ for all $\beta < \alpha$;
(3) $D_n[p, \varphi, k] \geq \alpha+1$ if for every finite $r \subseteq p(x)$ there is a sequence $\{a_i | i < \omega\}$ such that for all $\bar{\imath} \in [\omega]^n$
$D_n[r \cup \{\varphi(x, \bar{a}_{\bar{\imath}})\} \cup \mathrm{Ind}(x; \bar{a}_{\bar{\imath}}), \varphi, k] \geq \alpha$
and the set $\{\varphi(x, \bar{a}_{\bar{\imath}}) | \bar{\imath} \in [\omega]^n\}$ is $[k]^n$−contradictory.
The expressions $D_n[p, \varphi, k] = \alpha$, $D_n[p, \varphi, k] = -1$, and $D_n[p, \varphi, k] = \infty$ are defined as usual.

Definition 2 Let $\alpha \leq \omega$. We say that a complete thoery T is $\alpha-$*simple* if for all $n < \alpha$, for all $\varphi(x, y_0, \ldots, y_n)$ and $k > n+1$ the rank $D_{n+1}[x = x, \varphi, k]$ is bounded.(i.e. is less than ∞.)

Definition 3 (1) A formula $\varphi(x, y_0, \ldots, y_{n-1})$, a set of sequences $\{I_\eta | \eta \in ([\omega]^n)^{<\omega}\}$, and $k < \omega$ *witness the* $n-$*tree property* if for every $\eta \in ([\omega]^n)^\omega$, the type $\{\varphi(x, \bar{a}^{\eta\lceil l}_{\eta[l]}) | l < \omega\}$ is realized by $\bar{b}_\eta$ such that sequences $\bar{a}^{\eta\lceil l}_{\eta[l]}$ are

indiscernible over b_η for each $l < \omega$ and for every $\eta \in ([\omega]^n)^{<\omega}$ the set $\{\varphi(x, \bar{a}^{\eta}_{\bar{\imath}}) | \bar{\imath} \in [\omega]^n\}$ is $[k]^n-$contradictory where $\bar{a}^{\eta\lceil l}_{\eta[l]} := \{a^{\eta\lceil l}_{i_0}, \ldots, a^{\eta\lceil l}_{i_{n-1}}\}$ for $\eta[l] = i_0 \cdots i_{n-1}$.
(2) A theory T has the $n-$ *tree property* if there exist a formula, a set of parameters, and a number k witnessing the $n-$tree property.

The next proposition is proved by the definitions.

Proposition 4 ([4]) *A theory T is $\alpha-$ simple if and only if it does not have an $(n+1)-$tree property for any $n < \alpha$.*

Kolesnikov defined some notion of dividing for $n-$simple context.

Definition 5 For $n < \omega$, we say that a formula $\varphi(x, a_0, \ldots, a_{n-1})$ $n-$ *divides over* A if there is an indiscernible sequence $\{a_i | i < \omega\}$ over A and $b \models \varphi(x, a_0, \ldots, a_{n-1})$ such that $\{a_0, \ldots, a_{n-1}\}$ are indiscernible over b and the set $\{\varphi(x, \bar{a}_{\bar{\imath}}) | \bar{\imath} \in [\omega]^n\}$ is $[k]^n-$contradictory for some k.

Remark 6 It is clear that for $n = 1$ the definition is the same as that of dividing.

We recall the definition of weak dividing to make sure.

Definition 7 We say that $p(x) = \mathrm{tp}(a/B)$ *weakly divides over* A $(\subseteq B)$ if there is a formula $\psi(x_1, \ldots, x_n)$ over A such that $[p]^\psi := p(x_1) \cup \ldots \cup p(x_n) \cup \{\psi(x_1, \ldots, x_n)\}$ is inconsistent while $[q]^\psi$ is consistent where $q(x) = \mathrm{tp}(a/A)$.

The next facts are easily checked.

Fact 8 *Let $A \subset B$ and $\varphi(x_0, \ldots, x_{n-1}, b)$ be a formula over B. Suppose that there is an indiscernible sequence $\{a_i | i < \omega\}$ over A satisfying;*
$\models \varphi(a_0, \ldots, a_{n-1}, b)$ and $\{a_i | i < n\}$ are indiscernible over b. If the type $\{\varphi(x_0, \ldots, x_{n-1}, b)\} \cup$ Ind$(B; \{x_i | i < \omega\})$ is inconsistent, then there is a formula $\psi(a_0, \ldots, a_{n-1}, z)$ such that $\psi(a_0, \ldots, a_{n-1}, z)$ $n-$divides over A.

Fact 9 *Let $A \subset B$ and $p(x) = \mathrm{tp}(a/B)$. Suppose that there is a formula $\varphi(x_0, \ldots, x_{n-1})$ over A and an infinite indiscernible sequence $\{a_i | i < \omega\}$ over A with $\mathrm{tp}(a_0/A) = p\lceil A$ such that*
$\models \varphi(a_0, \ldots, a_{n-1})$ and
the type "$\{\varphi(x_0, \ldots, x_{n-1})\} \cup Ind(A; \{x_i : i < \omega\}) \cup \bigcup_{i<\omega} p(x_i)$" is inconsistent.
Then p weakly divides over A.
Moreover if T is simple, then p divides over A.

The case is problematic when realizations of the formula can not be extended to an infinite indiscernible sequence over the original parameters.
I tried to use the facts above for the argument of weak dividing in n-simple theories, but I have no result to show here.

We can define an analogy of weak dividing for n-dividing.

Definition 10 Let $A \subset B$. And let $p(x_0, \dots, x_{n-1})$ be a complete type over B such that $p(x_0, \dots, x_{n-1}) \vdash Ind(A; x_0, \dots, x_{n-1})$. We say that $p(x_0, \dots, x_{n-1})$"*weakly n−divides over A*" if there are $k < \omega$ and a formula $\psi(x_0, \dots, x_{k-1})$ over A such that $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{\imath} \in [k]^n} p(\bar{x}_{\bar{\imath}}) \lceil A$ is consistent while $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{\imath} \in [k]^n} p(\bar{x}_{\bar{\imath}})$ is inconsistent where $p(\bar{x}_{\bar{\imath}}) = p(x_{i_0}, \dots, x_{i_{n-1}})$ for $i_0 < i_1 < \cdots < i_{n-1} < k$ and $k > n$.

Remark 11 *When $n = 1$, "weak 1−dividing" is the same as "weak dividing".*

Notation
¿From now, we denote $[p]^\psi$ for the type $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{\imath} \in [k]^n} p(\bar{x}_{\bar{\imath}})$.

Fact 12 *Let $A \subset B \subset C$.*
(1) If $\mathrm{tp}(a/C)$ does not weakly n-divide over B, then $\mathrm{tp}(a/B)$ does not weakly n-divide over A.
(2) If $\mathrm{tp}(a/C)$ does not weakly n-divide over B and $\mathrm{tp}(a/B)$ does not weakly n-divide over A, then $\mathrm{tp}(a/C)$ does not weakly n-divide over A.

Fact 13 *Weak n-dividing has the local character.*

Fact 14 *If $\mathrm{tp}(b/Aa_0 \dots a_{n-1})$ n−divides over A, then $\mathrm{tp}(a_0 \dots a_{n-1}/Ab)$ weakly n−divides over A.*

Remark 15 *Naturally, we define weak n-dividing for complete types as follows:*
a complete type p weakly n-divides over A if its implies a formula which weakly n-divides over A.

Lemma 16 *Let $A \subset B$. And let $p(x_0, \dots, x_{n-1})$ be a complete type over B such that $p(x_0, \dots, x_{n-1}) \vdash Ind(A; x_0, \dots, x_{n-1})$.*
Then the following are equivalent;
(1) p does not weakly n−divide over A.
(2) For any set $C := \{a_i | i \in I\}$ satisfying that for any n−sequence $a_{i_0}, \dots, a_{i_{n-1}} \in C$ with $i_0 < i_1 < \dots < i_{n-1}, \models p\lceil A(a_{i_0}, \dots, a_{i_{n-1}})$, there is B' such that $\mathrm{tp}(B/A) = \mathrm{tp}(B'/A)$ and for any $a_{i_0}, \dots, a_{i_{n-1}} \in C$ with $i_0 < i_1 < \dots < i_{n-1}$, $\mathrm{tp}(B'/a_{i_0} \dots a_{i_{n-1}}A) = \mathrm{tp}(B/a_0 \dots a_{n-1}A)$.

The further characterization needs to investigate the relation between $n-$simple theories and $n-$dividing more.

References

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