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On CM-triviality

筑波大学数学系　米田郁生 (Ikuo Yoneda)
Institute of Mathematics, University of Tsukuba

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Abstract

Hrushovski's generic construction yields CM-trivial structures with weak elimination of imaginaries. Here, we would like to give questions on CM-trivality.

1 CM-triviality of generic structures.

In this section, we show weak elimination of imaginaries and CM-triviality of well-known stable generic structures.

Definition 1 We say that T is CM-trivial if

$$A_1 \underset{B}{\downarrow} A_2 \Rightarrow A_1 \underset{\mathrm{acl}^{eq}(A_1A_2)\cap B}{\downarrow} A_2$$

for $A_1, A_2, B \subset \mathcal{M}^{eq}$ algebraically closed sets.

Fact 2 *Let T be any theory of well-known stable generic structure. Then we have;*

1. *($\mathcal{M}$: big model of T) For any $A, B \subset \mathcal{M}$ algebraically closed sets,*

$$A \underset{A\cap B}{\downarrow} B \Leftrightarrow AB = A \otimes_{A\cap B} B \leq \mathcal{M},$$

2. *any type over algebraically closed sets in real sort is stationary.*

Explanations

1. The language is $L = \{R_i(X_1 \ldots X_{n_i}) : i < \omega\}$, where $R_i(X_1 \ldots X_{n_i})$ is an n_i-ary predicate.

2. We assume that any predicate is closed under permutations and $R_i(a_1 \ldots a_{n_i}) \Rightarrow a_i \neq a_j (i \neq j \leq n_i)$.

3. We defined predimension on finite L-structures.

$$\delta(A) = |A| - \sum_{i<\omega} \alpha_i \cdot |R_i^A|$$

   , where A is a finite L-structure, R_i^A is the set of tuples of A satisfying R_i (up to permutations) and $\alpha_0 > \alpha_1 > \ldots > \alpha_i (i < \omega)$ are fixed positive real numbers .

4. For finite L-structures A, B, we say A is *closed* in B (write $A \leq B$) if

$$\delta(XA) - \delta(A) \geq 0$$

   for any $X \subseteq B$. "A is closed in B" means that there are only suitably many (depending on α) sequences intersecting A and $B \backslash A$, and satisfying some predicates.

   For possibly infinite L-structures A, N, we say that A is closed in N (write $A \leq N$) if

$$A_0 \leq A_0 X$$

   for any $A_0 \subset_\omega A, X \subset_\omega N \setminus A$.

   There exists the smallest closed subset $\mathrm{cl}_N(A)$ of N containing A.

   In any well-known stable generic structure, $\mathrm{cl}_N(A) \subseteq \mathrm{acl}_N(A)$, in particular $\mathrm{acl}_N(A) \leq N$.

5. For L-structure A, B, C with $A \cap B \subseteq C$, we say that A and B are *freely joined* over C if there are no $i < \omega$ and $\bar{d} \in ABC$ such that $R_i(\bar{d})$, $\bar{d} \cap (A \setminus C) \neq \emptyset$ and $\bar{d} \cap (B \setminus C) \neq \emptyset$, and we write $ABC = A \otimes_C B$.

From now on, let T be as in Fact 2.

Proposition 3 *T has weak elimination of imaginaries.*

proof First we show the following claim.
Claim Let A, B, B_1, B_2 be algebraically closed. Suppose that $B_i \subseteq B$ and $A \downarrow_{B_i} B$ for $i = 1, 2$. Then $A \downarrow_{B_1 \cap B_2} B$.
The proof of this claim: Put $A_i = \mathrm{acl}(AB_i)$. Then $A_iB = A_i \otimes_{B_i} B \leq \mathcal{M}$ by Fact 2, for $i = 1, 2$. Intersecting the two sets yields $(A_1 \cap A_2)B = (A_1 \cap A_2) \otimes_{B_1 \cap B_2} B \leq \mathcal{M}$. Note that $A_1 \cap A_2$ and $B_1 \cap B_2$ are algebraically closed. So by Fact 2, $A_1 \cap A_2 \downarrow_{B_1 \cap B_2} B$; since $A \subseteq A_1 \cap A_2$ we get $A \downarrow_{B_1 \cap B_2} B$, as desired.
Now we show the weak elimination of imaginaries. Let $E(\bar{x}, \bar{y})$ be a definable equivalence relation over $\emptyset$, and consider $e = \bar{a}_E$, where $\bar{a}_E$ is the E-class of $\bar{a}$. Take $\bar{b}_1, \bar{b}_2$ such that $\bar{a}, \bar{b}_1, \bar{b}_2$ are independent over e, and $\bar{a}_E = (\bar{b}_1)_E = (\bar{b}_2)_E$. As $e \in \mathrm{acl}^{eq}(\bar{b}_i)$ we have $\bar{a} \downarrow_{\bar{b}_i} \bar{b}_1\bar{b}_2$, for $i = 1, 2$.
Put $B = \mathrm{acl}(b_1) \cap \mathrm{acl}(b_2)$, where the algebraic closure is taken in the real sort. Then $\bar{a} \downarrow_B \bar{b}_1\bar{b}_2$ by claim. As $\mathrm{tp}(\bar{a}/B)$ is stationary and $e \in \mathrm{dcl}^{eq}(\bar{a}) \cap \mathrm{dcl}^{eq}(\bar{b}_1\bar{b}_2)$, we get $e \in \mathrm{dcl}^{eq}(B)^*$. On the other hand, as $\bar{b}_1 \downarrow_e \bar{b}_2$, we have $B \subseteq \mathrm{acl}(e)$. By compactness we can find a finite tuple $\bar{b} \in B$ with $e \in \mathrm{dcl}^{eq}(\bar{b})$; clearly $\bar{b} \in \mathrm{acl}^{eq}(e)$, as desired.

Proposition 4 *T is CM-trivial.*

proof We use the following fundamental property.

1. If $ABC = A \otimes_C B, A \cap B \subseteq C' \subset C, A \setminus C = A \setminus C', B \setminus C = B \setminus C'$, then $ABC' = A \otimes_{C'} B$.

2. If $ABC = A \otimes_C B, B' \subset B, B'C \leq BC$, then $AB'C = A \otimes_C B' \leq ABC = A \otimes_C B$.

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*If $\mathrm{tp}(a/A)$ is stationary and $e \in \mathrm{dcl}^{eq}(a)$, then $\mathrm{tp}(e/A)$ is also stationary: Suppose $e \equiv_A e', e \downarrow_A B$ and $e' \downarrow_A B$. We need to show $e \equiv_B e'$. By $e \in \mathrm{dcl}^{eq}(a)$ and compactness, there exists a definable function f such that $f(a) = e$. We may assume $a \downarrow_{Ae} B$, so we have $a \downarrow_A B$. Take a' with $ea \equiv_A e'a'$. Again we may assume $a' \downarrow_{Ae'} B$, so $a' \downarrow_A B$ follows. As $a \equiv_A a'$, $a \equiv_B a'$ follows. On the other hand $e = f(a), e' = f(a')$, we see $e \equiv_B e'$.
If $\mathrm{tp}(a/A)$ is stationary, $a \downarrow_A B$ and $a \in \mathrm{dcl}^{eq}(B)$, then $a \in \mathrm{dcl}^{eq}(A)$: Note that $a \in \mathrm{acl}^{eq}(A)$. So, if $a' \equiv_A a$, then $a' \downarrow_A B$. By stationarity, we see $a \equiv_B a'$, so $a = a'$ follows.

By weak elimination of imaginaries, we may work in $\mathcal{M}$ not in $\mathcal{M}^{eq}$ to show the CM-triviality. Put $D = \mathrm{acl}(A_1A_2), \tilde{A}_i = \mathrm{acl}(A_iE)$. We need to show $A_1 \downarrow_E A_2 \Rightarrow A_1 \downarrow_{E\cap D} A_2$. By Fact 2 we see $\tilde{A}_1\tilde{A}_2 = \tilde{A}_1 \otimes_E \tilde{A}_2 \leq \mathcal{M}$. So, by 1 $D = (\tilde{A}_1 \cap D) \otimes_{D\cap E} (\tilde{A}_2 \cap D) \leq \mathcal{M}$ (†). Put $A_i' = \mathrm{acl}(A_i(D \cap E)) \subset \tilde{A}_i$. As $A_1 \downarrow_E A_2$, we see $A_1' \cap A_2' = D \cap E$. By $D \cap E \leq A_i' \leq \tilde{A}_i \cap D$,(†) and 2,

$$A_1'A_2' = A_1' \otimes_{D\cap E} A_2' \leq (\tilde{A}_1 \cap D) \otimes_{D\cap E} (\tilde{A}_2 \cap D)D \leq \mathcal{M}.$$

By Fact 2 again, we see $A_1 \downarrow_{E\cap D} A_2$.

2 Questions

We say that a theory (or structure) is strictly CM-trivial if it is CM-trivial but not one-based.

Question 1 Every strictly CM-trivial stable structure we know has weak elimination of imaginaries. On the other hand, Evans had CM-trivial *SU*-rank 1 structure without weak elimination of imaginaries in [E].
Does strictly CM-trivial strongly minimal set has weak elimination of imaginaries?
(It is well-known that if D is a strongly minimal sets with infinite $\mathrm{acl}(\emptyset)$, then D has weak elimination of imaginaries.)

Question 2 *Is there strictly CM-trivial stable structure except stable generic structures?*

Question 3 One-basedness coinsides with local modularity among strongly minamal sets. (This is not true among *SU*-rank 1 sets. See [V].)
Is there combinatorial geometric notion equivalent to CM-triviality among strongly minimal sets?

The following is a famous question.

Question 4 Evans showed supersimple $\aleph_0$-categorical CM-trivial theory must have finite SU-rank in [EW].
Is there supersimple $\aleph_0$-categorical theory with infinite SU-rank? Is there $\aleph_0$-categorical simple non-CM-trivial theory?

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