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Gap Number of Groups

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Finite gap number is introduced by J. C. Lennox and J. E. Roseblade in [LR]. We study groups of the small gap number.

1 ladder index

Let T be complete theory in an L, $\phi(\overline{x}, \overline{y})$ L-formula ($\overline{x}, \overline{y}$ are free variables).

Definition 1 *An n-**ladder** for ϕ is a sequence $(\overline{a}_0, \ldots, \overline{a}_{n-1}; \overline{b}_0, \ldots, \overline{b}_{n-1})$ of tuples in some model M of T, such that*

$$\forall i, j < n, M \models \phi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i \leq j.$$

*We say that ϕ is **stable** formula if there exists n such that no n-ladder for ϕ exists; otherwise it is **unstable**. The least such n is the **ladder index** of ϕ.*

Theorem 2 *The theory T is unstable if and only if there exists an unstable formula in L for T.*

Henceforth we consider the ladder index for the commutativity formula "$xy = yx$". The ladder index of a group G for the commutativity formula is denoted by $\ell(G)$.

2 gap number

Let G be a group.

Definition 3 *A group G has a finite gap number if for any subgroups $H_0, H_1, \ldots, H_n, \ldots$ of G, among the sequence*

$$C_G(H_0) \leq C_G(H_1) \leq \cdots \leq C_G(H_n) \leq \cdots$$

there exist at most m many strict inclusions. The most such m is the ***gap number*** *of G, and denoted by $g(G)$.*

Lemma 4 *Let $g(G) = n$. Suppose that the sequence*

$$C_G(H_0) > C_G(H_1) > \cdots > C_G(H_n)$$

gives gap number n. Then there exists $a_i (0 \leq i \leq n)$ in G such that $C_G(H_i) = C_G(\{a_0, a_1, \ldots, a_i\})$ for each i. In particular we may do $a_0 = 1$. Henceforth we abbreviate as $C_G(\{a_0, a_1, \ldots, a_i\}) = (a_0, a_1, \ldots, a_i)$.

By Lemma 4, we can prove the following:

Theorem 5 [ITT] $\ell(G) = g(G) + 2$.

Lemma 6 *Let $A, B \subset G$ with $A \subset B$. Then $(A) \supset (B)$.*

Lemma 7 *Let $A \subset G$. Then $(((A))) = (A)$.*

By the above lemma, the following holds:

Lemma 8 *Let $g(G) = n$. Suppose $(a_0, \ldots, a_n; b_0, \ldots, b_n)$ is $(n+1)$-ladder. Then*

$$\begin{aligned} ((a_0)) &= (b_n, \ldots, b_1, b_0); \\ ((a_0, a_1)) &= (b_n, \ldots, b_1); \\ &\vdots \\ ((a_0, \ldots, a_n)) &= (b_n). \end{aligned}$$

Lemma 9 *Let $g(G) = n$. Suppose that the sequece*

$$G > (a_1) > \cdots > (a_1, a_2, \ldots, a_n)$$

gives gap nuber n. Then $(a_1, a_2, \ldots, a_{n-1})$ is abelian.

3 Groups of gap number up to four

From now on we do not consider the ladder index but the gap number.

Theorem 10 [ITT] $g(G) = 0$ *if and only if G is abelian.*

Theorem 11 [ITT] *There exist no groups G of* $g(G) = 1$.

(proof) Let $g(G) \geq 1$. Then there exists $a \in G$ such that $G > (a)$. Since $(a) \neq G$, there exists $b \notin (a)$. Therefore, we have
$G > (a) > (a, b)$. Thus $g(G) \geq 2$.

Theorem 12 [ITT] $g(G) = 2$ *if and only if G is not abelian, and for any* $a, b \in G \setminus Z(G)$, *if* $(a) \neq (b)$ *then* $(a, b) = Z(G)$.

Example 13 $g(S_3) = g(D_n) = 2$ *(D_n is a dihedral group).*

Example 14 $g(SL(2, F)) = 2$ *(F is a field).*

Theorem 15 [ITT] *There exist no groups G of* $g(G) = 3$.

(proof) Let $g(G) \geq 3$. Then there exist $a_1, a_2 \in G$ such that
$G > (a_1) > (a_1, a_2) > Z(G)$.
Case 1: $a_1a_2 = a_2a_1$.
Since $(a_1) \neq (a_2)$, we may assume $(a_1) \setminus (a_2) \neq \emptyset$. Let $b \in (a_1) \setminus (a_2)$. As $a_1 \notin G$, there exists a $c \in G \setminus (a_1)$. Therefore, we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, b) > (a_1, a_2, b, c).$$

Thus $g(G) \geq 4$.
Case 2: $a_1a_2 \neq a_2a_1$.
There exists a $d \in (a_1, a_2) \setminus Z(G)$. Since $d \notin Z(G)$, we can find $e \notin G \setminus (d)$. Then we have

$$G > (d) > (d, a_1) > (d, a_1, a_2) > (d, a_1, a_2, e).$$

Thus $g(G) \geq 4$.

Example 16 $g(S_4) = g(S_5) = 4$.

4 Groups of gap number five

In this section, we investigate whether a group G of gap number 5 exists or not.

Let $g(G) = 5$ and let $(1, a_1, \ldots, a_5; b_0, \ldots, b_4, 1)$ be 6-ladder.

Case 1: $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1$ and $a_2a_3 = a_3a_2$.

Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_2) > (a_1, a_2, a_3, b_2, b_1) > Z(G).$$

Thus, $g(G) \geq 6$.

Case 2: $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1, a_2a_3 \neq a_3a_2$ and $a_1a_4 \neq a_4a_1$.

Then we have

$$G > (b_4) > (b_4, a_1) > (b_4, a_1, a_2) > (b_4, a_1, a_2, a_3) > (b_4, a_1, a_2, a_3, a_4) > Z(G).$$

Thus, $g(G) \geq 6$.

Case 3: $a_1a_2 = a_2a_1, a_1a_3 = a_3a_1, a_2a_3 \neq a_3a_2$ and $a_1a_4 = a_4a_1$.

Then we have

$$G > (a_1) > (a_1, b_3) > (a_1, b_3, a_2) > (a_1, b_3, a_2, a_3) > (a_1, b_3, a_2, a_3, a_4) > Z(G).$$

Thus, $g(G) \geq 6$.

Case 4: $a_1a_2 = a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 = a_3a_2$.

Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover $a_2a_1 = a_1a_2, a_2a_3 = a_3a_2$ and $a_1a_3 \neq a_3a_1$. By case 2, 3, $g(G) \geq 6$.

Case 5: $a_1a_2 = a_2a_1$ and $a_1a_3 \neq a_3a_1$.

Then we have

$$G > (b_4) > (b_4, b_3) > (b_4, b_3, a_1) > (b_4, b_3, a_1, b_1) > Z(G).$$

Moreover $b_4a_1 = a_1b_4$ and $b_3a_1 = a_1b_3$. By case 1, 4, $g(G) \geq 6$.

Therefore, by case 1 through 5, we hold $a_1a_2 \neq a_2a_1$.

Case 6: $a_1a_2 \neq a_2a_1$ and $a_1a_3 = a_3a_1$.
Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_2) > (a_1, a_3, a_2, a_4) > Z(G).$$

Moreover $a_1a_3 = a_3a_1$. Thus, $g(G) \geq 6$.

Case 7: $a_1a_2 \neq a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 = a_3a_2$.
Then we have

$$G > (a_2) > (a_2, a_1) > (a_2, a_1, a_3) > (a_2, a_1, a_3, a_4) > Z(G).$$

Moreover $a_2a_3 = a_3a_2$. By case 6, $g(G) \geq 6$.

Therefore, by case 1 through 7, we hold $a_1a_2 \neq a_2a_1, a_1a_3 \neq a_3a_1$ and $a_2a_3 \neq a_3a_2$.

Case 8: all of a_1, a_2, a_3, a_4 are noncommutative except $a_1a_4 = a_4a_1, a_2a_4 = a_4a_2$.
Then we have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_4) > (a_1, a_2, a_4, a_3) > Z(G).$$

Moreover $a_1a_4 = a_4a_1$. By case 6, $g(G) \geq 6$.

Case 9: all of a_1, a_2, a_3, a_4 are noncommutative except $a_1a_4 = a_4a_1, a_3a_4 = a_4a_3$.
Then we have

$$G > (a_1) > (a_1, a_3) > (a_1, a_3, a_4) > (a_1, a_3, a_4, a_2) > Z(G).$$

Moreover $a_1a_4 = a_4a_1$. By case 6, $g(G) \geq 6$.

Case 10: all of a_1, a_2, a_3, a_4 are noncommutative except $a_2a_4 = a_4a_2, a_3a_4 = a_4a_3$.
Then we have

$$G > (a_2) > (a_2, a_3) > (a_2, a_3, a_4) > (a_2, a_3, a_4, a_1) > Z(G).$$

Moreover $a_2a_4 = a_4a_2$. By case 6, $g(G) \geq 6$.

In the cases of remaining we understand the following:

Lemma 17 *Let* $G > (a_1) > (a_1, a_2) > \cdots > (a_1, a_2, a_3, a_4, a_5) = Z(G)$. *Then we can do as follows: all of* a_1, a_2, a_3, a_4 *are noncommutative except* $a_1a_4 = a_4a_1, a_2a_4 = a_4a_2, a_3a_4 = a_4a_3, a_1a_5 = a_5a_1, a_2a_5 = a_5a_2, a_3a_5 = a_5a_3$.

(proof) We have

$$G > (a_1) > (a_1, a_2) > (a_1, a_2, a_3) > (a_1, a_2, a_3, b_4) > (a_1, a_2, a_3, b_4, b_3).$$

Moreover all of a_1, a_2, a_3 are noncommutative, and all of b_4, b_3, b_2 are noncommutative, as desired.

Question 18 *Does there exist a group* G *of* $g(G) = 5$*?*

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