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On the existence of a Morley sequence with a certain property

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Abstract

Simplicity (in model theoretic sense) was originally defined by Shelah in [0]. Under simplicity, he showed the existence of a Morley sequence starting from a given element. Kim [1] showed various important results on simple theories. The existence of a Morley sequence played a key role in his proofs. In this note, we prove a result on the existence of a Morley sequence of a certain form.

T is a simple theory formulated in the language L. We work in an ω-saturated model M of T. Elements of M will be denoted by $a, b, a_i, a_{ij}, \ldots$ and so on. Finite tuples of elements of M will be denoted by $\bar{a}, \bar{a}_i, \ldots$ Subsets of M will be denoted by $A, B, \ldots$. A sequence $\{\bar{a}_i\}_{i\in\omega}$ is called an indiscernible sequence over A if whenever $\{i_1 < \ldots < i_n\}$ and $\{j_1 < \ldots < j_n\}$ are finite subsets of ω, and $\varphi(x_1, \ldots, x_n)$ is an $L(A)$-formula then

$$\varphi(a_{i_1}, \ldots, a_{i_n}) \leftrightarrow \varphi(a_{j_1}, \ldots, a_{j_n})$$

holds in M. An indiscernible sequence $\{\bar{a}_i\}_{i\in\omega}$ over A is called a Morley sequence over A if the sequence is an independent set over A. (See [0] and [1].)

In Kolesnikov's attempt ([2]) for dividing simple theories into countably many classes, he showed that for any finite independent set $\{a_1, \ldots, a_n\}$, there is a Morley sequence $\{\bar{b}_i : i \in \omega\}$ such that $b_{ii} = a_i$ for $i = 1, \ldots, n$, where b_{ij} is the j-th element of $\bar{b}_i$. In this short note, we extend his result slightly.

Proposition 1 *Let T be simple and $\{a_0, a_1, \ldots, a_{n-1}\}$ an independent set. For $i < n$, let I_i be a Morley sequence starting from a_i. Then there is an independent set $A = \{a_{ij} : i < n, j \in \omega\}$ with $a_{00} = a_0, \ldots, a_{0,n-1} = a_n$ such that for each $i < n$,*

1. $\{a_{ij} : j \in \omega\}$ *(the i-th row of the matrix A) is isomorphic to* I_i;

2. $\{\bar{a}_{ij} : j \in \omega\}$ *is a Morley sequence over* $a_{i+1}, ..., a_{n-1}$, *where* $\bar{a}_{ij} = a_{0j}, ..., a_{ij}$.

Proof: By induction on $i < n$, we shall choose $J_i = \{a_{ij} : j \in \omega\}$ satisfying the following conditions.

- $J_0 \cup ... \cup J_i \cup \{a_{i+1}, ..., a_n\}$ is an independent set;
- $a_{00} = a_0, ..., a_{i0} = a_i$;
- $J_i \cong I_i$.
- $\{\bar{a}_{ij} : j \in \omega\}$ is a Morley sequence over $a_{i+1}, ..., a_{n-1}$, where $\bar{a}_{ij} = a_{0j}, ..., a_{ij}$.

Suppose that we have chosen J_j's for $j < i$. So $K = \{\bar{a}_{i-1,j} : j \in \omega\}$ is a Morley sequence over $a_i, a_{i+1}, ..., a_{n-1}$. We can choose a copy $J_i = \{a_{ij} : j \in \omega\}$ of I_i over a_i such that J_i is a Morley sequence over $a_{i+1}, ..., a_{n-1}$. Let us consider the type $q_{a_i,a_{i+1},...,a_n} = \mathrm{tp}(K/a_i, a_{i+1}, ..., a_n)$. By our condition, K and $a_i, a_{i+1}, ..., a_{n-1}$ are independent. So the type $\bigcup_{b \in I_i} q_{b,a_{i+1},...,a_n}$ is consistent, and does not fork over $\emptyset$. So there is a copy K' of K realizing $\bigcup_{b \in I_i} q_{b,a_{i+1},...,a_n}$ such that K' and $J_i \cup \{a_{i+1}, ..., a_{n-1}\}$ are independent. Using an $a_i, ..., a_n$-isomorphism sending K' to K, we may assume that K and $J_i \cup \{a_{i+1}, ..., a_n\}$ are independent from the first. Now, by compactness argument using Ramsey's theorem, we may further assume that $\{\bar{a}_{i,j} : j \in \omega\}$ is an indiscernible sequence over $a_{i+1}, ..., a_n$, so it is a Morley sequence over $a_{i+1}, ..., a_n$.

Corollary 2 *Let T be simple and* $\{a_0, a_1, ..., a_{n-1}\}$ *an independent set. Then there is an independent set* $A = \{a_{ij} : i < n, j \in \omega\}$ *satisfying the following properties:*

1. *The columns of A form a Morley sequence, in other words, if we put* $\bar{a}_i = a_{0i}...a_{n-1,i}$, *then* $I = \{\bar{a}_i : i \in \omega\}$ *is a Morley sequence.*

2. *For any* $i_0, ..., i_{n-1} \in \omega$, $\mathrm{tp}(a_{0,i_0}, ..., a_{n-1,i_{n-1}}) = \mathrm{tp}(a_0, ..., a_{n-1})$.

Proof: Choose any $A = \{a_{ij} : i < n, j \in \omega\}$ satisfying conditions stated in the proposition above. It is sufficient to prove the second item. But it can be shown by the iterated use of the second item in the proposition.

References.

[0] S. Shelah; Simple unstable theories, Ann. Math. Logic 19 (1980) 177–203

[1] B. Kim, Forking in simple unstable theories, Journal of London Math. Soc. (2) 57 (1998) 257–267.

[2] A. Kolesnikov, n-simplicity paper.