数理解析研究所講究録 1390 巻 2004 年 49-53

NON-FINITE AXIOMATIZABILITY

BYUNGHAN KIM

MATHEMATICS DEPARTMENT, MIT, 77 MASS. AVE., CAMBRIDGE, MA 02139
bkim@math.mit.edu

ABSTRACT. We prove non-finite axiomatizability of some rank-1 ω-categorical structures.

Fix an ω-categorical infinite structure M having a simple theory T such that the universe itself is a solution set of (unique) rank-1 Lascar strong 1-type $p^1(x)$. Without loss of generality, we can assume the language $\mathcal{L}$ has only relational symbols. For $A\subseteq M$, $acl(A)$ is an algebraic closure of A in M, and $acl^{eq}(A)$ is that in M^{eq}.

Now given $n>1$, there are i_n-many n-(independent) complete types $p_1^n(x_1...x_n)$, ..., $p_{i_n}^n(x_1...x_n)$, such that each $p_j^n(x_1...x_n)$ implies $\{x_1, ..., x_n\}$ independent.

Obviously in M, given $p_j^n(x_1, ..., x_n)$, there is non-empty finite set $F(n,j)\subseteq\{1, ..., i_{n+1}\}$ such that $\exists x_{n+1}p_l^{n+1}(x_1, ..., x_{n+1})$ is equivalent to $p_j^n(x_1, ..., x_n)$ iff $l\in F(n,j)$.

Moreover for each $n>0$, there is a formula $\psi_n(x_1...x_n)$ such that $M\models\psi_n(a_1, ..., a_n)$ iff $\{a_1, ..., a_n\}$ is independent.

Definition 0.1. *Let N be a subset of M. We say that N is* k-generic substructure *of M for $k\geq 1$ if N is an algebraically closed subset of M such that, for any $m<k$, and any tuple $(a_1, ..., a_m)$ from N with $M\models p_j^m(a_1, ..., a_m)$, and $l\in F(m,j)$, there is $b\in N$ such that $M\models p_l^{m+1}(a_1, ..., a_m, b)$.*

Lemma 0.2. *There is a function $bd:\omega\rightarrow\omega$ (depending on T) satisfying the following: Let σ be a sentence in T having k quantifiers (in its Prenex normal form). Suppose that N is $bd(k)$-generic substructure of M. Then $N\models\sigma$.*

Proof. Let the function pbd be defined in such a way that for any $j\leq m<k$, and any tuple $\bar{e}=(e_1...e_m)$ from M, if $\bar{e}'=(e_{i_1}, ..., e_{i_j})$ is the maximal independent subtuple, then there are at most $pbd(k)$ many conjugate of $\bar{e}$ over $\bar{e}'$. (As T is ω-categorical, this is possible.) Define $bd(k)=k+pbd(k)$.

Now, to prove the lemma, it obviously suffices to prove the following.

Claim) For $m<k$, and $(a_1, ..., a_m)\in N$ and $(b_1, ..., b_m, c_1)\in M$, if $tp_M(a_1...a_m)=tp_M(b_1...b_m)$, then there is $d\in N$ so that $tp_M(a_1...a_md)=tp_M(b_1...b_mc_1)$:

Suppose that such $\bar{a}=(a_1, ..., a_m)\in N$ and $\bar{b}=(b_1, ..., b_m)$ are chosen. If $c_1\in acl(\bar{b})$, then as N itself is algebraically closed in M, we can find the desired d in N.

Hence we can assume that $c_1\notin acl(\bar{b})$. Now, there is maximal independent subtuple, say $\bar{b}'=(b_1, ..., b_j)$ of $\bar{b}=(b_1, ..., b_m)$, and $\bar{b}$ has $s(\leq pbd(k))$ conjugates over $\bar{b}'$, say $\bar{b}_1(=\bar{b}), ..., \bar{b}_s$. Then there is $\bar{c}=(c_1...c_s)$ independent over $\bar{b}'$ such that

$tp(\bar{b}c_1) = tp(\bar{b}_l c_l)$ $(l = 1, ..., s)$. Note that the independent tuple $\bar{b}'\bar{c}$ has length less than $bd(k)$. Hence by $bd(k)$-genericity of N, there is $\bar{d} = (d_1...d_s) \in N$ such that $tp(\bar{a}'\bar{d}) = tp(\bar{b}'\bar{c})$ $(\bar{a}' = (a_1...a_j))$. Then clearly, for some d_i, $tp(\bar{b}c_1) = tp(\bar{a}d_i)$. Hence the claim and the lemma are proved. □

As M has rank 1, M forms a pregeometry. We first consider the case when M forms a geometry such as a random graph, i.e. for $a \in M$, $acl(a) = \{a\}$.

Lemma 0.3. *Suppose that M is trivial and forms a geometry. Let N be a finite substructure of M. Then for k, there is M_k such that $N \subseteq M_k \subseteq M$ and M_k is k-generic while not m-generic for some $m > k$. (Hence from 0.2, T is* not *finitely axiomatizable.)*

Proof. If T is stable, any $acl(N \cup S)$ where S is some set of k-independent points serves the example of finite one (even for non-trivial pregeometry case !).

So, we freely assume that T is unstable. Then there must exist an integer e such that, say $p_1^e(x_1...x_e)$ has at least 2 independent extensions, say $p_1^{e+1}(x_1...x_e; x_{e+1})$ and $p_2^{e+1}(x_1...x_e; x_{e+1})$. Now pick up independent tuples $\bar{a}_i$ of M $(i = 1, ..., k)$, such that each $\bar{a}_i \models p_1^e$, and $\bar{a} = \bar{a}_1...\bar{a}_k$ is also independent. We can clearly assume that $N \cap \bar{a} = \emptyset$, and set $\bar{a}_0 = N$. Denote $S = \cup_{i \leq k}\bar{a}_i (\supseteq N)$.

Step 1.

Choose a $y_0 \in S$. Then $y_0 \in \bar{a}_{i_0}$ $(i_0 \leq k)$. Now find independent elements $\{x_j : j \in F(1,1)\}$ which is also independent from S such that $y_0 x_j \models p_j^2$ and, for each j and $1 \leq i (\neq i_0) \leq k$, $\bar{a}_i x_j \models p_1^e$. This is possible by the Independence Theorem. Let $S_1 = S \cup \{x_j : j \in F(1,1)\}$. Then repeat the step 1 for another point $y_1 \in \bar{a}_{i_1} \subseteq S \setminus \{y_0\}$ by finding points $\{x'_j : j \in F(1,1)\}$ independent from S_1 such that $p_j^2(y_1 x'_j)$ and $p_1^e(\bar{a}_i, x'_j)$ for any $1 \leq i \neq i_1 \leq k$. Then eventually we can find $U_2(\supset ...S_1 \supset S \supset N)$ such that, for each $x \in S$ 2-genericity is witnessed inside U_2, whereas

$$\cup\{p_2^{e+1}(\bar{a}_i; z) | 1 \leq i \leq k\} \text{ is not realized} \quad (*)$$

inside U_2.

Step 2.

Now by modifying Step 1, find $U_3(\supset U_2)$ to witness 3-genericity for S inside U_3 while to satisfy (*). Namely for given independent $x, y \in \bar{a}_{i_2} \cup \bar{a}_{i_3} \subseteq S$ $(i_2, i_3 \leq k)$ realizing p_i^2, choose independent points $\{w_j | j \in F(2,i)\}$ independent from U_2 such that $p_j^3(x, y, w_j)$ and $p_1^e(\bar{a}_i, w_j)$ for $1 \leq i \neq i_2, i_3 \leq k$. By repeating the process, we can obtain $U_k(... \supset U_3 \supset ...S \supset N)$ inside which k-genericity for S is witnessed where as above (*) holds.

Step 3.

Rename U_k as W_1, and repeat the previous steps for W_1. Continuing in this way we obtain a chain of spaces $S \subset W_1 \subset W_2 \subset \dots W_i \subset \dots$ such that, inside W_{i+1}, k-genericity of W_i is witnessed whereas (*) holds. Let $M_k = \cup_i W_i$. Then by construction M_k is the desired substructure. Therefore the theorem is proved. □

Theorem 0.4. *If M is trivial, then T is not finitely axiomatizable.*

Proof. In M^{eq}, we have the geometry D of M. For n, clearly there is k such that whenever A is a set of independent $i(< n)$ points of M, then A has at most k conjugates over unique $B \subseteq D$ with $acl^{eq}(A) = acl^{eq}(B)$. Now by previous lemma, there is $(n+k)$-generic $D' \subseteq D$ which is not m-generic for some $m > n+k$. Then we can find $C \subseteq M$ such that $acl(C) \cap M = C$, and C, D' are interalgebraic.

We claim that C is n-generic, but not m-generic. (This finishes the proof.): Let $\bar{a} = a_1 \dots a_i$ be a set of independent $i(< n)$ points of C, and let b_1 be a point in M independent from $\bar{a}$. We want to find $c \in C$ so that $tp(\bar{a}c) = tp(\bar{a}b_1)$. Now, there is $\bar{d} = d_1 \dots d_i$ in D such that $\bar{d}$ and $\bar{a}$ are interalgebraic. Then over $\bar{d}$, there are $s(\leq k)$ conjugates of $\bar{a}$, say $\bar{a}_1(= \bar{a}), \dots, \bar{a}_s$. Then there is $\bar{b} = (b_1 \dots b_s)$ independent over $\bar{a}$ such that $tp(\bar{a}_j b_j/\bar{d}) = tp(\bar{a}_1 b_1/\bar{d})$ $(j = 1, \dots, s)$. Moreover there exist corresponding $\bar{e} = (e_1 \dots e_s) \in D$ such that b_j and e_j are interalgebraic. Now by $n+k$-genericity of D', there is $\bar{e}'$ in D' such that $tp(\bar{e}/\bar{d}) = tp(\bar{e}'/\bar{d})$. Then as C, D' are interalgebraic, $\bar{b}' = (b'_1 \dots b'_s) \in C$ where $tp(\bar{b}\bar{e}/\bar{d}) = tp(\bar{b}'\bar{e}'/\bar{d})$. Then clearly, for some b'_i, $tp(\bar{a}b_1) = tp(\bar{a}b'_i)$. Therefore C is n-generic. Finally as D' is not m-generic for some $m > n$, obviously C can not be m-generic, either. We have proved the theorem. □

Now we begin to prove the same result for the structure $N = (M, P)$ in $\mathcal{L}_P = \mathcal{L} \cup \{P\}$ where P is generic unary predicate in the sense of Pillay and Chazidakis [1]. (For this we assume that $T = Th(M)$ has quantifier elimination.) We can understand the predicate P as a P-coloring on M.

We can similarly define k-genericity of substructures of N. (We know that the independence and algebraic closedness in N coincide with those notions in M.)

Definition 0.5. *Let N_1 be a subset of N. We say that N_1 is* k-generic substructure *of N if N_1 is an algebraically closed subset of N such that, for any $m < k$, and independent tuple $\bar{a} = (a_1, \dots, a_m)$ from N_1 and $b \in N \setminus N_1$, there is $b_1 \in N_1$ such that $tp_N(\bar{a}b) = tp_N(\bar{a}b_1)$.*

Similarly the reader can show the following.

Lemma 0.6. *Let σ be a sentence in T' having k quantifiers. Then for sufficiently large n, whenever N_1 is n-generic substructure of N, then $N_1 \models \sigma$.*

Now the following proposition shows non-finite axiomatizability of N.

Proposition 0.7. *For k, there substructure N' of N which is k-generic, but not k'-generic for some $k' > k$.*

Proof. When the geometry is trivial, the proof is be left to the reader. (It will be almost the same as the proof of 0.3.) Hence we assume that the geometry is non-trivial, so that there is independent $(e_1, ...e_n, e_{n+1})$ which is non-trivial, i.e. there is $e \in acl(e_1...e_{n+1})$ with $e \notin acl(e_1...e_n) \cup acl(e_{n+1})$ (†). Let $q = tp_{\mathcal{L}}(e_1, ..., e_n)$ and $q' = tp_{\mathcal{L}}(e_1...e_{n+1})$. The rest of the proof will also be similar to the proof of 0.3. Now pick up independent tuples $\bar{a}_i$ of M $(i = 1, ..., k)$, such that each $\bar{a}_i \models q$, and $\bar{a} = \bar{a}_1...\bar{a}_k$ is also independent. Let $A = acl(\bar{a})$. There definitely is $b \notin A$ such that $\bar{a}_i b \models q'$ for all i, and moreover, $acl(Ab) \setminus A$ is entirely *not* P-colored (‡).

Now we proceed in a series of steps to construct the desired k-generic N' containing A such that $tp_N(b/\bar{a})$ is not realized in N'.

Step 1.

Choose a $y_0 \in A$. Clearly, y_0 is independent from some $\bar{a}_{i_0}$. Now find independent set $\{x_j\}_j$ which is also independent from A witnessing 2-genericity for y_0 (i.e. every independent 2-complete type extending $tp_N(y_0)$ in T' is realized by some $y_0 x_j$). Let $A_1 = acl(A \cup \{x_j\}_j)$. Now, moreover by the character of N, we can further assume that $A_1 \setminus (A \bigcup \cup_j acl(y_0 x_j))$ is entirely colored by P (*). We claim that $tp_N(b/\bar{a})$ is not realized in A_1 (Call this property, (**) for A_1.):

Suppose not, say $tp_N(b/\bar{a})$ is realized by $p \in A_1$. Then by (*) and (‡), $p \notin A_1 \setminus (A \bigcup \cup_j acl(y_0 x_j))$. Hence $p \in acl(y_0 x_{j_0}) \setminus acl(y_0)$ for some j_0 (⋆). Since $\bar{a}_{i_0} p \models q'$, there is $z \in acl(\bar{a}_{i_0} p) \subseteq A_1$ witnessing non-triviality of $\bar{a}_{i_0} p$. We shall show that $z \in A_1 \setminus (A \bigcup \cup_j acl(y_0 x_j))$. (Then it contradicts to (*) and (‡). Hence the claim is verified.) Firstly, by (†), $z \notin A$. Secondly, to show $z \notin acl(y_0 x_{j_0})$, we note that by (⋆), $acl(y_0 p) = acl(y_0 x_{j_0})$ and $acl(p) = acl(\bar{a}_{i_0} p) \cap acl(y_0 p)$ as $\bar{a}_{i_0}$ is independent from y_0 over p. Then $z \notin acl(y_0 x_{j_0}) = acl(y_0 p)$ since otherwise $z \in acl(p)$ contradicting to (†). Similarly one can see that $z \notin acl(y_0 x_j)$ for any j. Therefore we have proved the claim (**) for A_1.

Now repeat the step 1 for another point $y_1 (\in A)$ independent from some $\bar{a}_{i_1}$. Namely, find points $\{x'_j\}_j$ independent from A_1 witnessing 2-genericity of T' for y_1 such that $A_2 \setminus (A_1 \bigcup \cup_j acl(y_1 x'_j))$ is entirely colored by P, where $A_2 = acl(A_1 \cup \{x'_j\}_j)$. Then by the same argument, $tp_N(b/\bar{a}_{i_1})$ is not realized in $A_2 \setminus A_1$. Eventually we can find $U_2 (... A_2 \supset A_1 \supset A)$ such that, for each $x \in A$ 2-genericity is witnessed inside U_2, whereas (**) for U_2 holds.

Step 2.

For $m < k$, and any independent $\bar{c} = (c_1, ..., c_m) \in A$, clearly some $\bar{a}_{i_2}$ is independent from $\bar{c}$. Now then by modifying Step 1, find $U_{m+1} (... \supset U_2)$ to witness $(m+1)$-genericity for any $\bar{c} \in A$ inside U_{m+1} while to hold (**) for U_{m+1}. Namely choose independent points $\{w_j\}_j$ independent from U_m witnessing genericity for $\bar{c}$ such that $U'_m \setminus (U_m \bigcup \cup_j acl(\bar{c} w_j))$ is entirely colored by P and $U'_m (= acl(U_m \cup \{w_j\}_j)) \setminus U_m$ omits $tp_N(b/\bar{a}_{i_2})$. U_{m+1} will contain U'_m.

Step 3.

Rename U_k as W_1, and repeat the previous steps for W_1. Continuing in this way we obtain a chain of spaces $A \subset W_1 \subset W_2 \subset \ldots W_i \subset \ldots$ such that, inside W_{i+1}, k-genericity of W_i is witnessed whereas (**) for W_{i+1} holds. Let $N' = \cup_i W_i$. Then by construction N' is k-generic while omits $tp_N(b/\bar{a})$. Therefore the theorem is proved. □

References

[1] Z. Chatzidakis and A. Pillay, 'Generic structures and simple theories', *Ann. of Pure and Applied Logic* 95 (1998) 71-92.

[2] T. de Piro and B. Kim, 'The geometry of 1-based minimal types', *Transactions of American Math. Soc.* 355 (2003) 4241-4263.

[3] F. O. Wagner, *Simple theories*, Kluwer Academic Publishers, Dordrecht (2000).