数理解析研究所講究録 1393 巻 2004 年 72-78

Extendability of symplectic torus actions with isolated fixed points

大阪大学大学院理学研究科　宅間俊志 (Shunji TAKUMA)
Department of Mathematics, Osaka Univesity

1 Abstract

I itemise the abstract of my talk in the following.

Problem : When does or doesn't a given effective and symplectic torus group action on a compact connected symplectic manifold with isolated fixed points extend to an effective and symplectic action of a higher dimensional torus group ?

Method : Translating geometrical objects into graphical objects (due to recent works of V. Guillemin and C. Zara [5, 6, 7, 8]) and considering the above problem on the graphical level.

Result : I obtained a necessary condition for the torus action to extend. That is, if the torus action extends, then a certain obstruction must vanish.

2 From geometric data to graphical data

Here I introduce the graphical object obtained from geometric data under suitable assumptions, which is introduced and studied by V.Guillemin and C.Zara.

Consider

(M^{2d},ω,J): a cpt.conn.symp.mfd.with a compatible alm.cpx.str.

T^n: an n-dim.torus acting on M effectively and preserving ω and J.

Assume

(1) M^T discrete, and

(2) isotropy weights at $\forall p \in M^T$ are pairwise linearly independent.

With this assumption, we obtain the following graphical objects:
(Γ, θ, α):Goreskey-Kottwitz-MacPherson graph.
("GKM graph" or "GKM 1-skeleton")
which consists of three data
Γ : a d-valent graph
θ : a connection on the graph
α : an axial function

Let's describe these three data in detail in the following.

The graph:

$\Gamma = (V_\Gamma, E_\Gamma)$

$V_\Gamma = M^T$(the fixed point set),

$$E_\Gamma \subset V_\Gamma \times \left\{ \begin{array}{c} \text{embedded } \mathbb{CP}^1\text{'s fixed by} \\ \text{codimension one subtori} \end{array} \right\}$$

consisting of pairs (p, Σ) with $p \in \Sigma^T$.

Notation:

$e = (p, \Sigma) = (p, q)$ $(\Sigma^T = \{p, q\})$
$\overline{e} = (q, \Sigma) = (q, p)$
$E_p = \{e \in E_\Gamma | i(e) = p\}$

The connection:

$\theta =$ a collection of bijections $\{\theta_e\}_{e \in E_\Gamma}$:

$\theta_e : E_p \to E_q$ $(e = (p, q))$

satisfying

(1) $\theta_e(e) = \overline{e}$, and (2) $\theta_{\overline{e}} = (\theta_e)^{-1}$

The axial function:

$\alpha =$ a map from E_Γ to $\mathfrak{t}^*$ satisfying

(1) $\alpha(\overline{e}) = -\alpha(e)$, for $\forall e \in E_\Gamma$

(2) $\alpha(\theta_e(f)) = \alpha(f) + c(f, e)\alpha(e)$, for $\forall f \in E_{i(e)}$, $\forall e \in E_\Gamma$.

Moreover, I give the following additional assumption which Guillemin and Zara did not assume in[].

The effectiveness condition:

For any p, let $E_p = \{f_1, \dots, f_d\}$. Then

$$\gcd\left\{ \begin{vmatrix} \alpha(f_{i_1}) \\ \vdots \\ \alpha(f_{i_n}) \end{vmatrix} \;\middle|\; 1 \le i_1 < \cdots < i_n \le d \right\} = 1$$

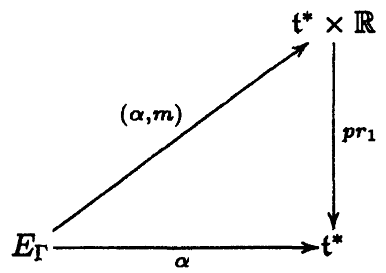
3 Translation of the Extendability

From now on we assume $k = d - n \geq 1$ (this number is called the **complexity** of the torus action).

The T-action extends to a $T \times S^1$-action.

⇓ implies

The axial function α lifts to the axial function (α, m) commuting the following diagram.



In this way we obtain an extra function m. Since I will discuss whether the extra function exists or not, I abstract the properties of the fuction and redefine it as follows.

Definition 1 We call the map $m : E_\Gamma \to \mathbb{Z} \subset \mathbb{R}$ satisfying the following conditions an **extra weight**:

(1) $m(\theta_e(f)) = m(f) + c(f, e)m(e)$, for $\forall f \in E_{i(e)}$, $\forall e \in E_\Gamma$, and

(2) (α, m) satisfies the effectiveness condition.

4 Construction of the Obstruction

Let G_p be the d-dimemsional torus acting standardly on $T_pM \cong \mathbb{C}^d$, and let $\mathfrak{g}_p$ be its Lie algebra.

Note that $Map(E_p, \mathbb{R}) \cong \mathfrak{g}_p$, $Map(E_p, \mathbb{Z}) \cong L(\mathfrak{g}_p)$, where $L(\mathfrak{g}_p)$ is the lattice of $\mathfrak{g}_p$.

Parallel transport along an edge

For an edge $e = (p, q)$ we define a map $e : L(\mathfrak{g}_p) \to L(\mathfrak{g}_q)$, $m \mapsto e(m)$ by the equation

$$e(m)(\theta_e(f)) = m(f) + c(f, e)m(e)$$

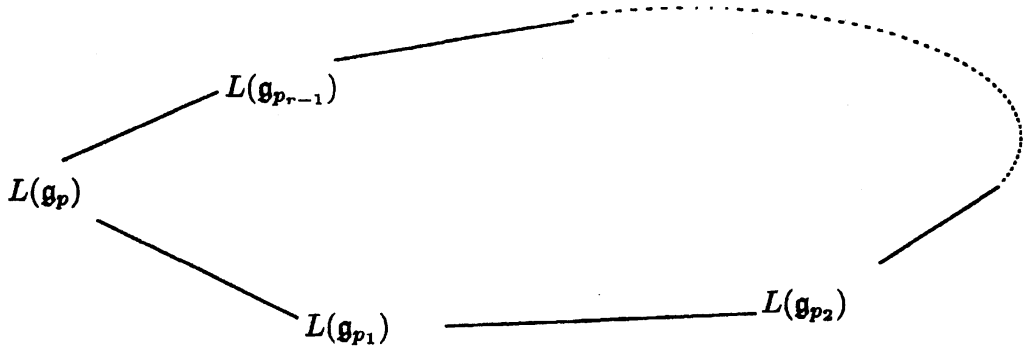
for $\forall f \in E_p$.

A loop in the graph

We call a **loop** a sequence of edges $e_1 \cdots e_r$ with $i(e_1) = t(e_r) = p$ and $t(e_j) = i(e_{j+1})$ $(j = 1, \ldots, r-1)$. (denoted also by γ)
A loop is called **trivial** if it is written in the form $e_1 \cdots e_s \bar{e}_s \cdots \bar{e}_1$.
Let $\mathcal{L}(\Gamma, p)$ the set of all loops based at p. $\mathcal{L}(\Gamma, p)$ has a group structure if we quotient it by trivial loops.

Parallel transport along a loop

For a loop $\gamma = e_1 \cdots e_r \in \mathcal{L}(\Gamma, p)$, we define $\gamma : L(\mathfrak{g}_p) \to L(\mathfrak{g}_p)$ by
$\gamma(m) = e_r(\cdots(e_1(m))\cdots)$.
This leads us to the $\mathcal{L}(\Gamma, p)$-action on $L(\mathfrak{g}_p)$.



The obstruction

The axial function at p, $\alpha_p \in Map(E_p, \mathfrak{t}^*)$, is considered as a linear map $L(\mathfrak{t}) \xrightarrow{\alpha_p} L(\mathfrak{g}_p)$. Then we have the following short exact sequence

$$0 \longrightarrow L(\mathfrak{t}) \xrightarrow{\alpha_p} L(\mathfrak{g}_p) \xrightarrow{\pi_p} L(\mathfrak{g}_p)/\mathrm{Im}\alpha_p \longrightarrow 0.$$

Here we have the following important lemma.

Lemma 1 *For $\Delta \in L(\mathfrak{g}_p)/Im\alpha_p$ and $m \in (\pi_p)^{-1}(\Delta)$,*
1.(α_p, m) is effective $\Longleftrightarrow$ Δ is ***primitive****.*
2. For a loop γ based at p, $\gamma(m) - m$ depends only on Δ, does not depend on the choice of m.

Definition 2 We define a map $eo(p,\Delta) : \mathcal{L}(\Gamma,p) \to L(\mathfrak{g}_p)$ by

$$eo(p,\Delta)(\gamma) = \gamma(m) - m$$

for $m \in (\pi_p)^{-1}(\Delta)$.

From above we have our result:

Theorem 1 *There exists an extra weight for α if and only if there exists a primitive Δ such that $eo(p,\Delta) \equiv 0$.*

5 Some Properties

1. (Calculation rule)
For two loops γ and $\delta \in \mathcal{L}(\Gamma,p)$,

$$eo(p,\Delta)(\gamma\delta) = eo(p,\Delta)(\gamma) + eo(p,\gamma(\Delta))(\delta) \tag{1}$$

Note that $\mathcal{L}(\Gamma,p)$ acts also on the quotient space $L(\mathfrak{g}_p)/\mathrm{Im}\alpha_p$.

In the case that the complexity is one,
2. The $\mathcal{L}(\Gamma,p)$-action on $L(\mathfrak{g}_p)/\mathrm{Im}\alpha_p \cong \mathbb{Z}$ is very easy to understand. This action is given by the homomorphism

$$\begin{array}{ccc} \mathcal{L}(\Gamma,p) & \to & Isom(L(\mathfrak{g}_p)/\mathrm{Im}\alpha_p) \cong \mathbb{Z}_2 \\ \gamma & \longmapsto & \mathrm{sgn}(\gamma), \end{array}$$

where

$$\mathrm{sgn}(\gamma) = \mathrm{sgn}(\theta_\gamma) \times (-1)^{|\gamma|}. \tag{2}$$

θ_γ : the holonomy map $E_p \to E_p$ along γ w.r.t the connection θ.
$|\gamma|$: the number of the edges of which γ consists. The length of γ.

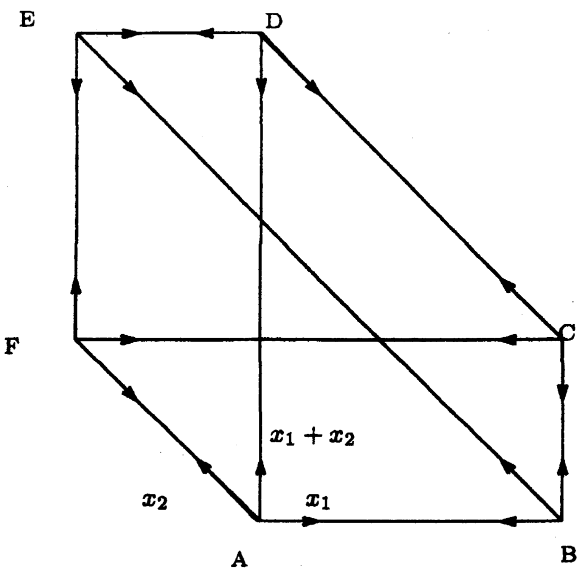
3. (Calculation rule)

$$eo(p,\Delta)(\gamma\delta) = eo(p,\Delta)(\gamma) + \mathrm{sgn}(\gamma)eo(p,\Delta)(\delta) \tag{3}$$

6 Example

1. 3-flag variety $SL(3;\mathbb{C})/B$ with standard T^2-action

The graph Γ is visualized as follows.



Let the base vertex $p = A$ and let $E_p = \{f_1 = AB, f_2 = AF, f_3 = AD\}$.
All loops are of the length even. The holonomy along each loop is trivial. Thus $\mathrm{sgn}(\gamma) = 1$ for all loops γ. Hence the action of $\mathcal{L}(\Gamma, p)$ on $L(\mathfrak{g})_p/\mathrm{Im}(\alpha_p)$ is trivial.
$x_1 = (1, 0)$, $x_2 = (-1, 1)$ and $x_1 + x_2 = (0, 1)$ are weights at p.
$\mathcal{L}(\Gamma, p)$ is generated by four loops, e.g., ABCDA, ADEFA, ABCFA, ABEFA. $\mathfrak{eo}(p, 1)(\gamma)$ for these generators are written in the following table in which the columns express the values of $\mathfrak{eo}(p, 1)(\gamma)$.

	$ABCDA$	$ADEFA$	$ABCFA$	$ABEFA$
f_1	-1	1	-2	-1
f_2	-1	1	1	2
f_3	-2	2	-1	1

By calculation rule (1) or (3), for example, we have

$$\begin{aligned} \mathfrak{eo}(ABCDEFA) &= \mathfrak{eo}(ABCDA \cdot ADEFA) \\ &= \mathfrak{eo}(ABCDA) + \mathfrak{eo}(ADEFA) \\ &= {}^t(-1, -1, -2) +^t (1, 1, 2) \\ &= (0, 0, 0). \end{aligned}$$

In this case eo is an additive function.

References

[1] M. Audin: *The topology of torus actions on symplectic manifolds*, Progress in Mathematics **93**, Birkhäuser Verlag, 1991.

[2] T. Delzant: *Hamiltoniens périodiques et image comvexe de l'application moment*, Bull. Soc. Math. France **116** (1998), no. 3, 315-339.

[3] M. Goreskey, R. Kottwitz and R. MacPherson: *Equivariant cohomology, Koszul duality and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25-83.

[4] V. Guillemin: *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, Progress in Mathematics **122**, Birkhäuser Verlag, 1994.

[5] V. Guillemin and C. Zara: *1-skeleta, Betti numbers and equivariant cohomology*, Duke Math. J. **107**(2001), no. 2, 283–349.

[6] V. Guillemin and C. Zara: *The existence of generating families for the cohomology ring of a graph*, Advances in Mathematics **174** (2003), 115-153.

[7] V. Guillemin and C. Zara: *G-actions on graphs*. Internat. Math. Res. Notices (2001), no. 10, 519-542

[8] V. Guillemin and C. Zara: *Combinatorial formulas for products of Thom classes*. Geometry, mechanics, and dynamics, 363–405, Springer, New York, 2002.

[9] Y. Karshon: *Periodic Hamiltonian flows on four dimensional manifolds*, Mem. Amer. Math. Soc. **672** (1999).

[10] Y. Karshon: *Maximal Tori in the Symplectictomorphism Groups of Hirzeburch Surfaces*, Math. Res. Lett. **10** (2003), no. 1, 125-132.

[11] Y. Karshon and S. Tolman: *Centered complexity one Hamiltonian torus actions*, Trans. Amer. Math. Soc. **353** (2001), 4831-4861.

[12] Y. Karshon and S. Tolman: *Tall complexity one Hamiltonian torus actions*. Preprint, math SG/0202167 (18 Feb. 2002).

[13] Y. Karshon and S. Tolman: *Complete invariants for Hamiltonian torus actions with two dimensional quotients*, Preprint, arXiv:math.SG/0202167 v2 14 Dec 2003.

Shunji Takuma
Department of Mathematics,
Graduate School of Science, Osaka University
Toyonaka, Osaka 560-0043
Japan
e-mail:takuma@math.sci.osaka-u.ac.jp