数理解析研究所講究録 1393 巻 2004 年 102-105

DEFINABLE C^rG TRIVIALITY OF G INVARIANT PROPER DEFINABLE C^r MAPS

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ABSTRACT. Let G be a compact definable C^r group and $1 \leq r < \infty$. We prove that every G invariant proper definable C^r onto submersion from an affine definable C^rG manifold to $\mathbb{R}$ is definably C^rG trivial.

1. INTRODUCTION

M. Coste and M. Shiota [1] proved that a proper Nash onto submersion from an affine Nash manifold to $\mathbb{R}$ is Nash trivial. This Nash category is a special case of the definable C^r category and it coincides with the definable C^∞ category based on $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ [16]. General reference on o-minimal structures are [2], [5], see also [15]. Further properties and constructions of them are studied in [3], [4], [6], [12] and there are uncountably many o-minimal expansions of $\mathcal{R}$ [13]. Equivariant definable category is studied in [7], [9], [10], [11].

Let G be a definable C^r group, X a definable C^rG manifold and $1 \leq r < \infty$. Suppose that f is a G invariant definable C^r function from X to $\mathbb{R}$. We say that f is *definably C^rG trivial* if there exist a definable C^rG manifold F and a definable C^rG map $h : X \to F$ such that $H = (f, h) : X \to \mathbb{R} \times F$ is a definable C^rG diffeomorphism. If f is definably C^rG trivial, then for any $y \in \mathbb{R}$, $f^{-1}(y)$ is definably C^rG diffeomorphic to F and there exists a definable C^rG diffeomorphism $\phi : X \to \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ \phi$, where $p : \mathbb{R} \times f^{-1}(y) \to \mathbb{R}$ denotes the projection.

A map $\psi : M \to N$ between topological spaces is *proper* if for any compact set $C \subset N$, $\psi^{-1}(C)$ is compact.

We consider an equivariant definable C^r version of [1] and an equivariant version of [1].

Theorem 1.1. *Let G be a compact definable C^r group and X an affine definable C^rG manifold and $1 \leq r < \infty$. Then every G invariant proper definable C^r onto submersion $f : X \to \mathbb{R}$ is definably C^rG trivial.*

Let $X = \{y = 0\} \cup \{xy = 1\} \subset \mathbb{R}^2, Y = \{y = 0\} \subset \mathbb{R}^2$ and $f : X \to Y, f(x, y) = x$. Then f is a polynomial onto submersion and it is not definably trivial. Thus proper condition is necessary.

The projection onto S^n of the tangent bundle of the standard n-dimensional sphere S^n with the standard $O(n+1)$ action for $n \geq 8$ is not piecewise definably C^rG trivial. Thus G invariant condition is necessary.

2000 Mathematics Subject Classification 14P10, 14P20, 57R22, 58A05, 03C64
Keywords and Phrases. o-minimal, definable C^r manifolds, proper definable C^r functions, definable C^rG trivial, Nash G trivial.

Corollary 1.2. *Let G be a finite group and X an affine Nash G manifold. Then every G invariant proper Nash onto submersion from X to $\mathbb{R}$ is Nash G trivial.*

2. Proof of results

The following is a result on piecewise definable C^rG triviality of G invariant submersive surjective definable C^r maps [9].

Theorem 2.1 (1.1 [9]). *(Piecewise definable C^rG triviality). Let X be an affine definable C^rG manifold, Y a definable C^r manifold and $1 \le r < \infty$. Suppose that $f : X \to Y$ is a G invariant submersive surjective definable C^r map. Then there exist a finite decomposition $\{T_i\}_{i=1}^k$ of Y into definable C^r submanifolds and definable C^rG diffeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \le i \le k)$, where p_i denotes the projection $T_i \times f^{-1}(y_i) \to T_i$ and $y_i \in T_i$.*

The following is existence of a definable C^rG tubular neighborhood of a definable C^rG submanifold of a representation of G when $1 \le r < \infty$.

Proposition 2.2 ([8]). *If $1 \le r < \infty$, then every definable C^rG submanifold X of a representation Ω of G has a definable C^rG tubular neighborhood (U, θ) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta : U \to X$ is a definable C^rG map with $\theta|X = id_X$.*

Note that if $r = \infty$ or ω, then Proposition 2.2 is already known in [11].

Proof of Theorem 1.1. Applying Theorem 2.1, we have a partition $-\infty = a_0 < a_1 < a_2 < \cdots < a_j < a_{j+1} = \infty$ of $\mathbb{R}$ and definable C^rG diffeomorphisms $\phi_i : f^{-1}((a_i, a_{i+1})) \to (a_i, a_{i+1}) \times f^{-1}(y_i)$ with $f|f^{-1}((a_i, a_{i+1})) = p_i \circ \phi_i$, $(0 \le i \le j)$, where p_i denotes the projection $(a_i, a_{i+1}) \times f^{-1}(y_i) \to (a_i, a_{i+1})$ and $y_i \in (a_i, a_{i+1})$.

Now we prove that for each a_i with $1 \le i \le j$, there exist an open interval I_i containing a_i and a definable C^rG map $\pi_i : f^{-1}(I_i) \to f^{-1}(a_i)$ such that $F_i = (f, \pi_i) : f^{-1}(I_i) \to I_i \times f^{-1}(a_i)$ is a definable C^rG diffeomorphism. By Proposition 2.2, we have a definable C^rG tubular neighborhood (U_i, π_i) of $f^{-1}(a_i)$ in X. Since f is proper, there exists an open interval I_i containing a_i such that $f^{-1}(I_i) \subset U_i$. Note that if f is not proper, then such an open interval does not always exist. Hence shrinking I_i, if necessary, $F_i = (f, \pi_i) : f^{-1}(I_i) \to I_i \times f^{-1}(a_i)$ is the required definable C^rG diffeomorphism.

By the above argument, we have a finite family of $\{J_i\}_{i=1}^l$ of open intervals and definable C^rG diffeomorphisms $h_i : f^{-1}(J_i) \to J_i \times f^{-1}(y_i)$, $(1 \le i \le l)$, such that $y_i \in J_i$, $\cup_{i=1}^l J_i = \mathbb{R}$ and the composition of h_i with the projection $J_i \times f^{-1}(y_i)$ onto J_i is $f|f^{-1}(J_i)$.

Now we glue these trivializations to get a global one. We can suppose that $i \ge 2$, $U_{i-1} \cap J_i = (a, b)$ and $k_{i-1} : f^{-1}(U_{i-1}) \to U_{i-1} \times f^{-1}(y_1)$ is a definable C^rG diffeomorphism with $f|f^{-1}(U_{i-1}) = proj_{i-1} \circ k_{i-1}$, where $U_{i-1} = \cup_{s=1}^{i-1} J_s$ and $proj_{i-1}$ denotes the projection $U_{i-1} \times f^{-1}(y_1) \to U_{i-1}$. Take $z \in (a, b) = U_{i-1} \cap J_i$. Then since $f^{-1}(y_1) \cong f^{-1}(z) \cong f^{-1}(y_i)$, $f^{-1}(y_1)$ is definably C^rG diffeomorphic to $f^{-1}(y_i)$. Hence we may assume that h_i is a definable C^rG diffeomorphism from $f^{-1}(J_i)$ to $J_i \times f^{-1}(y_1)$. Then we have a definable C^rG diffeomorphism

$$k_{i-1} \circ h_i^{-1} : (a, b) \times f^{-1}(y_1) \to (a, b) \times f^{-1}(y_1), (t, x) \mapsto (t, q(t, x)).$$

Take a C^r Nash function $u : \mathbb{R} \to \mathbb{R}$ such that $u = \frac{a+b}{2}$ on $(-\infty, \frac{3}{4}a + \frac{1}{4}b]$ and $u = id$ on $[\frac{1}{4}a + \frac{3}{4}b, \infty)$. Let

$$H : (a,b) \times f^{-1}(y_1) \to f^{-1}((a,b)), H(t,x) = k_{i-1}^{-1}(t, q(u(t),x)).$$

Then H is a definable C^rG diffeomorphism such that $H = h_i^{-1}$ if $\frac{1}{4}a + \frac{3}{4}b \leq t \leq b$ and $H = k_{i-1}^{-1} \circ (id \times \psi)$ if $a \leq t \leq \frac{3}{4}a + \frac{1}{4}b$, where $\psi : f^{-1}(y_1) \to f^{-1}(y_1), \psi(x) = q(\frac{a+b}{2}, x)$. Thus we can define

$$k_i : f^{-1}(U_i) \to U_i \times f^{-1}(y_1),$$

$$k_i(x) = \begin{cases} (id \times \psi)^{-1} \circ k_{i-1}(x), & f(x) \leq \frac{3}{4}a + \frac{1}{4}b \\ H^{-1}(x), & \frac{3}{4}a + \frac{1}{4}b \leq f(x) \leq b \\ h_i(x), & f(x) > b \end{cases}.$$

Then k_i is a definable C^rG diffeomorphism. Therefore k_l is the required definable C^rG diffeomorphism. □

By [14] and 4.3 [9], we have the following proposition.

Proposition 2.3. *Let G be a finite group, f a C^r Nash G map between affine Nash G manifolds and $1 \leq r < \infty$. Then f is approximated by a Nash G map.*

Proof of Corollary 1.2. By Theorem 1.1, we have a C^r Nash G diffeomorphism $F = (f, \phi) : X \to \mathbb{R} \times f^{-1}(y)$ such that $f = p \circ F$, where $p : \mathbb{R} \times f^{-1}(y) \to \mathbb{R}$ denotes the projection. By Proposition 2.3, we have a Nash G map $\psi : X \to f^{-1}(y)$ as an approximation of ϕ. If this approximation is sufficiently close, then $H = (f, \psi) : X \to \mathbb{R} \times f^{-1}(y)$ is the required Nash G diffeomorphism. □

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