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TWISTED SECOND COHOMOLOGY GROUP OF A FINITELY PRESENTED GROUP

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Abstract: For a finitely presented group G and G-module M, using combinatorial group theory, a new calculation of a twisted second cohomology group $H^2(G,M)$ is introduced. We apply our method to some well-known groups and calculate their second cohomology groups.

Keywords: twisted second cohomology group

1. INTRODUCTION

For a finitely presented group $G=\langle X\,|\,S\rangle$, let F be a free group on X and R the normal closure of S in F. If we regard $\mathbf{Z}$ as a trivial G-module, then we have the second homology group

$$H_2(G,\mathbf{Z})\simeq(R\cap[F,F])/[F,R]$$

of G by Hopf's formula. (See [2].) On the other hand, if G acts on M non-trivially, then a computation of twisted second (co)homology group $H^2(G,M)$ is much more complicated. In this paper, for a finitely presented group G and a G-module M, we introduce one of methods of a calculation of the second cohomology group $H^2(G,M)$ using combinatorial group theory. Furthermore, we apply our method to some well-known groups, for example, the dihedral group D_n, the special linear group $SL(2,\mathbf{Z})$ and the braid group B_3 of index three.

In this paper, we use the following notation. Let G be a group and M a G-module. We denote the group ring of G over $\mathbf{Z}$ by $\mathbf{Z}[G]$. For any $\alpha\in\mathbf{Z}[G]$, we put

$$M^{\alpha}=\{m\in M\,|\,\alpha\cdot m=m\},$$
$$\alpha M=\{\alpha\cdot m\in M|m\in M\},$$

where $\alpha\cdot m$ denotes the action of α on m.

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2. THE REIDEMEISTER-SCHREIER METHOD

In this section, we review the Reidemeister-Schreier method. This is one of methods to obtain a presentation for a subgroup H of a given presented group $G = \langle X \mid S \rangle$. We use the Reidemeister-Schreier method to calculate the second cohomology groups in later sections.

Let F be the free group on X and K a subgroup of F. A subset $T \subset F$ is called Schreier transversal for K in F if T satisfies the following properties

(1) T is a right coset representative system for K in F,
(2) $1 \in T$, where 1 is the identity element of F,
(3) (Schreier property) T contains all initial segments of all elements of T, that is,

$$t = x_{\mu_1}^{e_1} x_{\mu_2}^{e_2} \cdots x_{\mu_n}^{e_n} \in T \Rightarrow x_{\mu_1}^{e_1} x_{\mu_2}^{e_2} \cdots x_{\mu_{n-1}}^{e_{n-1}} \in T$$

where $t = x_{\mu_1}^{e_1} x_{\mu_2}^{e_2} \cdots x_{\mu_n}^{e_n}$ is a reduced word and $e_i \in \{\pm 1\}$, $(1 \leq i \leq n)$.

Let H be a subgroup of G and H' the inverse image of H under the natural homomorphism $\varphi : F \to G$. We denote a Schreier transversal for H' in F by T. For any $w \in F$, we define $\overline{w} \in T$ by the rule $H'w = H'\overline{w}$. A map

$$\overline{} : F \to T \quad w \mapsto \overline{w}$$

is called a right coset representative function for F modulo H'. For any $t \in T$ and $x \in X$ we put

$$(t, x) := tx(\overline{tx})^{-1}, \quad (t, x^{-1}) := (\overline{tx^{-1}}, x)^{-1} \in H'.$$

Let $X^{-1} = \{x^{-1} \mid x \in X\}$. For any word $w = y_1 y_2 \cdots y_n \in F$, $y_i \in X \cup X^{-1}$, we put

$$\tau(w) := (1, y_1)(\overline{y_1}, y_2) \cdots (\overline{y_1 \cdots y_{i-1}}, y_i) \cdots (\overline{y_1 \cdots y_{n-1}}, y_n).$$

The map τ is called the Reidemeister-Schreier rewriting process for H'.

Proposition 2.1. *With the above notation, if we put*

$$X' = \{(t, x) \in H' \mid t \in T, \ x \in X \ (t, x) \neq 1\},$$
$$S' = \{\tau(tst^{-1}) \in H' \mid t \in T, \ s \in S\},$$

then we have

(1) *H' is the free group on X',*
(2) *$\ker(\varphi|_{H'})$ is the normal closure of S' in H'.*

Hence, H has a presentation $H = \langle X' \mid S' \rangle$.

This proposition is well-known fact. For details, see [4].

3. A calculation of the second cohomology of a finitely presented group

Let G be a group and M a G-module. We assume that G has a finite presentation $G = \langle X \mid S \rangle$. Let F be the free group on X, R the normal closure of S in F and T a Schreier transversal for R in F. From the spectral sequence of the group extension

$$1 \to R \to F \to G \to 1,$$

we have an exact sequence

$$0 \to H^1(G,M) \to H^1(F,M) \xrightarrow{\text{res}} H^1(R,M)^G \to H^2(G,M) \to H^2(F,M).$$

Since F is the free group, $H^2(F,M) = 0$. Hence, to calculate $H^2(G,M)$, it suffices to calculate the group $H^1(R,M)^G$.

Now, R is a free group. If we can obtain a free basis X' of R, then we can determine a basis of $H^1(R,M)$ as a free abelian group. Furthermore, we see that

$$\begin{aligned} &H^1(R,M)^G \\ &\quad = \{f \in H^1(R,M) \mid f(\sigma^{-1}x'\sigma) = f(x'),\ \forall\sigma \in X,\ \forall x' \in X'\}. \end{aligned}$$

In this paper, to obtain a free basis X' of R, we use the Reidemeister-Schreier method. Then, considering the restriction map res : $H^1(F,M) \to H^1(R,M)^G$, we obtain $H^2(G,M)$.

In this method, it is important to construct a Schreier transversal for R in F. The difficulty of the construction of a Schreier transversal depends on not only a given group G but also a presentation for the group G. Hence it is necessary to find a suitable presentation for G.

4. The cyclic group C_n

It is well-known that the (co)homology groups of the cyclic group are completely determined. We, however, dare to apply our method in this case. It is the best way to use a simple example to understand our method. Let C_n be a cyclic group of degree $n \geq 2$. The group C_n has a finite presentation

$$C_n = \langle x \mid x^n = 1 \rangle.$$

Let F be the free group on $\{x\}$ and R the normal closure of $\{x^n\}$ in F.

Lemma 4.1. *The group R is a free group with basis $\{x^n\}$.*

Proof. Since F is an abelian group, it is clear that $\{x^n\}$ is a free basis of R. However, to understand our method, we apply the Reidemeister-Schreier method to this case.

First, we see that $T = \{1, x, \ldots, x^{n-1}\}$ is a Schreier transversal for R in F. Hence, a free basis

$$X^* = \{(t,x) \mid t \in T,\ x \in X,\ (t,x) \neq 1\}$$

of R is calculated as follows:

- For $t = x^i$, $(0 \le i \le n-2)$,
$$(t,x) = tx(\overline{tx})^{-1} = x^{i+1}(\overline{x^{i+1}})^{-1} = 1.$$
- For $t = x^{n-1}$,
$$(t,x) = x^n(\overline{x^n})^{-1} = x^n.$$

Hence we obtain $X^* = \{x^n\}$. □

Lemma 4.2. *Let M be C_n-module. Then $H^1(R,M)^{C_n} \simeq M^{C_n}$.*

Proof. Since R acts on M trivially and R is a free group with basis $\{x^n\}$, we obtain an isomorphism
$$\rho : H^1(R,M) \to M$$
defined by $\rho(f) \mapsto f(x^n)$.

Now, for any $y = x^i \in C_n$, and $f \in H^1(R,M)$, the action of y on f is given by
$$\begin{aligned}(y \cdot f)(x^n) &= yf(yx^n y^{-1})\\ &= yf(x^i x^n x^{-i})\\ &= yf(x^n).\end{aligned}$$
This shows that ρ is a C_n-isomorphism. Hence we have $H^1(R,M)^{C_n} \simeq M^{C_n}$.
□

Proposition 4.1. *For any C_n-module M, we have*
$$H^2(C_n,M) \simeq M^{C_n}/(1+x+\cdots+x^{n-1})M$$

Proof. It suffices to show that the image of
$$\psi := \rho \circ \mathrm{res} : H^1(F,M) \to M^{C_n}$$
is $(1+x+\cdots+x^{n-1})M$. For any $[f] \in H^1(F,M)$, we have
$$\begin{aligned}\psi([f]) &= f(x^n)\\ &= (1+x+\cdots+x^{n-1})f(x)\end{aligned}$$
where $[f]$ denotes the equivalence class of a crossed homomorphism f. This shows $\mathrm{Im}(\psi) = (1+x+\cdots+x^{n-1})M$. □

We also obtain the following results. For details, see [6].

5. The dihedral group D_n

For any $n \ge 1$, let D_n be the dihedral group of order $2n$. The group D_n has a finite presentation
$$D_n = \langle \sigma, \tau \mid \sigma^n = \tau^2 = 1,\ \tau\sigma\tau = \sigma^{-1} \rangle.$$
Let F be the free group on $\{\sigma, \tau\}$ and R the normal closure of $\{\sigma^n, \tau^2, \tau\sigma\tau\sigma\}$ in F.

Lemma 5.1. *The group R is a free group with basis*

$$\left\{x,\ y_k,\ z_k \;\middle|\; 0 \le k \le n-1\right\}$$

where

$$\begin{aligned} x &= \sigma^n, \\ y_0 &= \tau\sigma\tau^{-1}\sigma^{-(n-1)}, \\ y_k &= \sigma^k\tau\sigma\tau^{-1}\sigma^{-(k-1)}, \quad (1 \le k \le n-1), \\ z_k &= \sigma^k\tau^2\sigma^{-k} \quad (0 \le k \le n-1). \end{aligned}$$

Proof. It is easily seen that

$$T = \left\{1, \sigma, \dots, \sigma^{n-1}, \tau, \sigma\tau, \dots, \sigma^{n-1}\tau\right\}$$

is a Schreier transversal for R in F. Using the Reidemeister-Schreier method, we show this lemma. □

Lemma 5.2. *Let M be any D_n-module. Then we have*

$$H^1(R, M)^{D_n} \simeq L$$

where

$$L = \left\{(a,b,c) \in M^\sigma \oplus M^\sigma \oplus M^\tau \;\middle|\; nb = (\tau - (n-1))a, \right.$$
$$\left. (\tau - 1)a + (\tau - 1)b + (\sigma - 1)c = 0\right\}.$$

Proposition 5.1. *For any D_n-module M, we have*

$$H^2(D_n, M) \simeq L/K$$

where

$$K = \left\{((1 + \sigma + \cdots + \sigma^{n-1})s, \right.$$
$$\left. (1-\sigma^{n-1})t + (\tau - (1+\sigma+\cdots+\sigma^{n-2}))s,\ (1+\tau)t) \in L \;\middle|\; s, t \in M\right\}.$$

6. The group $PSL(2, \mathbf{Z})$

Let $PSL(2,\mathbf{Z})$ be the projective special linear group over $\mathbf{Z}$. The group $PSL(2,\mathbf{Z})$ has a finite presentation

$$PSL(2,\mathbf{Z}) = \langle \sigma,\ \tau \mid \sigma^3 = 1,\ \tau^2 = 1\rangle.$$

Let F be the free group on $\{\sigma, \tau\}$ and R the normal closure of $\{\sigma^3, \tau^2\}$ in F. To calculate a Schreier transversal for R in F, we prepare the following notations. For $m \ge 1$, $e_i \in \{1, 2\}$ $(1 \le i \le m)$ and $k \in \{0, 1\}$, put

$$\begin{aligned} \alpha_k(e_1, \dots, e_m) &= \tau^k\sigma^{e_1}\tau\sigma^{e_2}\tau\cdots\tau\sigma^{e_m}, \\ \beta_k(e_1, \dots, e_m) &= \tau^k\sigma^{e_1}\tau\sigma^{e_2}\tau\cdots\tau\sigma^{e_m}\tau. \end{aligned}$$

Lemma 6.1. *Let*

$$T_1 = \left\{\alpha_k(e_1,\dots,e_m),\ \beta_k(e_1,\dots,e_m)\,\middle|\, k\in\{0,1\},\ m\geq 1,\ e_i=1,2\right\},$$

and $T_2=\{1,\tau\}$. *Then* $T=T_1\cup T_2$ *is a Schreier transversal for* R *in* F.

For $m\geq 1$, $e_i\in\{1,2\}$ $(1\leq i\leq m)$ and $k\in\{0,1\}$, put

$$\begin{aligned} v &= \tau^2,\\ w_k &= \tau^k\sigma^3\tau^{-k},\\ x_k(e_1,\dots,e_m) &= \tau^k\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau^2\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-k},\\ y_k(e_1,\dots,e_m) &= \tau^k\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau\sigma^3\tau^{-1}\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-k}.\end{aligned}$$

Lemma 6.2. *The group* R *is a free group with basis*

$$\left\{v,\ w_k,\ x_k(e_1,\dots,e_m),\ y_k(e_1,\dots,e_m)\,\middle|\, k\in\{0,1\},\ m\geq 1,\ e_i=1,2\right\}.$$

Lemma 6.3. *Let* M *be any* $PSL(2,\mathbf{Z})$*-module. Then*

$$H^1(R,M)^{PSL(2,\mathbf{Z})}\simeq M^\tau\oplus M^\sigma.$$

Proposition 6.1. *For any* $PSL(2,\mathbf{Z})$*-module* M,

$$H^2(PSL(2,\mathbf{Z}),M)\simeq \left(M^\tau\big/(1+\tau)M\right)\oplus\left(M^\sigma\big/(1+\sigma+\sigma^2)M\right).$$

7. The group $SL(2,\mathbf{Z})$

Let $SL(2,\mathbf{Z})$ be the special linear group over $\mathbf{Z}$. The group $SL(2,\mathbf{Z})$ has a finite presentation

$$SL(2,\mathbf{Z}) = \langle\sigma,\ \tau\,|\,\sigma^3=\tau^2,\tau^4=1\rangle.$$

The elements σ and τ correspond to

$$\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} \text{ and } \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

respectively. Let F be the free group on $\{\sigma,\tau\}$ and R the normal closure of $\{\sigma^3\tau^{-2},\tau^4\}$ in F. To calculate a Schreier transversal for R, we prepare the following notations. For $m\geq 1$, $e_i\in\{1,2\}$ $(1\leq i\leq m)$ and k $(0\leq k\leq 3)$, put

$$\begin{aligned}\alpha_k(e_1,\dots,e_m) &= \tau^k\sigma^{e_1}\tau\sigma^{e_2}\tau\cdots\tau\sigma^{e_m}\\ \beta_k(e_1,\dots,e_m) &= \tau^k\sigma^{e_1}\tau\sigma^{e_2}\tau\cdots\tau\sigma^{e_m}\tau\\ \gamma_k &= \tau^k.\end{aligned}$$

Lemma 7.1. *Let*

$$T=\bigcup_{k\in\mathbf{Z}}\left\{\alpha_k(e_1,\dots,e_m),\ \beta_k(e_1,\dots,e_m),\ \gamma_k\,\middle|\, m\geq 1,\ e_i=1,2\right\}.$$

Then T *is a Schreier transversal for* R *in* F.

For $m \geq 1$, $e_i \in \{1,2\}$ $(1 \leq i \leq m)$ and k $(0 \leq k \leq 3)$, put

$$\begin{aligned}
v &= \tau^4, \\
w_0 &= \sigma^3\tau^{-2}, \\
w_1 &= \tau\sigma^3\tau^{-3}, \\
w_2 &= \tau^2\sigma^3, \\
w_3 &= \tau^3\sigma^3\tau^{-1}, \\
x_0(e_1,\dots,e_m) &= \sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau^2\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-2}, \\
x_1(e_1,\dots,e_m) &= \tau\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau^2\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-3}, \\
x_2(e_1,\dots,e_m) &= \tau^2\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau^2\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}, \\
x_3(e_1,\dots,e_m) &= \tau^3\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau^2\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-1}, \\
y_0(e_1,\dots,e_m) &= \sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau\sigma^3\tau^{-1}\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-2}, \\
y_1(e_1,\dots,e_m) &= \tau\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau\sigma^3\tau^{-1}\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-3}, \\
y_2(e_1,\dots,e_m) &= \tau^2\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau\sigma^3\tau^{-1}\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}, \\
y_3(e_1,\dots,e_m) &= \tau^3\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau\sigma^3\tau^{-1}\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-1}.
\end{aligned}$$

Lemma 7.2. *The group R is a free group with basis*

$$\bigcup_{0\leq k\leq 3}\left\{v,\ x_k(e_1,\dots,e_m),\ y_k(e_1,\dots,e_m),\ z_k \,\middle|\, m\geq 1,\ e_i=1,2\right\}.$$

Lemma 7.3. *Let M be any $SL(2,\mathbf{Z})$-module. Then*

$$H^1(R,M)^{SL(2,\mathbf{Z})} \simeq N$$

where

$$N \simeq \left\{(a,d)\in M^\tau \oplus M \,\middle|\, (1-\sigma)a = -(1-\sigma)(1+\sigma^3)d\right\}.$$

Proposition 7.1. *For any $SL(2,\mathbf{Z})$-module M, we have*

$$H^2(SL(2,\mathbf{Z}),M) \simeq N/L,$$

where

$$L = \left\{\left((1+\tau+\tau^2+\tau^3)t,\ (1+\sigma+\sigma^2)s-(1+\tau)t\right) \,\middle|\, s,t\in M\right\}.$$

8. The braid group B_3 of index three

Let B_3 be the braid group of index three. B_3 has a finite presentation

$$B_3 = \langle \sigma,\ \tau \mid \sigma^3 = \tau^2 \rangle.$$

Let F be the free group on $\{\sigma,\tau\}$ and R the normal closure of $\{\sigma^3\tau^{-2}\}$ in F. To calculate a Schreier transversal for R, we prepare the following notations.

For $m \geq 1$, $e_i \in \{1,2\}$ $(1 \leq i \leq m)$ and $k \in \mathbf{Z}$, put

$$\alpha_k(e_1,\ldots,e_m) = \tau^k\sigma^{e_1}\tau\sigma^{e_2}\tau\cdots\tau\sigma^{e_m}$$
$$\beta_k(e_1,\ldots,e_m) = \tau^k\sigma^{e_1}\tau\sigma^{e_2}\tau\cdots\tau\sigma^{e_m}\tau$$
$$\gamma_k = \tau^k.$$

Lemma 8.1. *Let*

$$T = \bigcup_{k\in\mathbf{Z}}\left\{\alpha_k(e_1,\ldots,e_m),\ \beta_k(e_1,\ldots,e_m),\ \gamma_k \middle| m \geq 1,\ e_i = 1,2\right\}.$$

Then T is a Schreier transversal for R in F.

For $m \geq 1$, $e_i \in \{1,2\}$ $(1 \leq i \leq m)$ and $k \in \mathbf{Z}$, put

$$x_k(e_1,\ldots,e_m) = \tau^k\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau^2\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-(k+2)},$$
$$y_k(e_1,\ldots,e_m) = \tau^k\sigma^{e_1}\tau\cdots\tau\sigma^{e_m}\tau\sigma^3\tau^{-1}\sigma^{-e_m}\tau^{-1}\cdots\tau^{-1}\sigma^{-e_1}\tau^{-(k+2)},$$
$$z_k = \tau^k\sigma^3\tau^{-(k+2)}.$$

Lemma 8.2. *The group R is a free group with basis*

$$\bigcup_{k\in\mathbf{Z}}\left\{x_k(e_1,\ldots,e_m),\ y_k(e_1,\ldots,e_m),\ z_k \middle| m \geq 1,\ e_i = 1,2\right\}.$$

Lemma 8.3. *Let M be any B_3-module. Then*

$$H^1(R,M)^{B_3} \simeq M.$$

Proposition 8.1. *For any B_3-module M, we have*

$$H^2(B_3,M) \simeq M\Big/(1+\sigma+\sigma^2)M + (1+\tau)M.$$

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