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2-ELEMENTS OUTSIDE OF THE DRESS SUBGROUP OF TYPE 2

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1. Introduction

Let G be a finite group. We denote by $\pi(G)$ the set of prime divisors of the order of G. For a prime p, we denote by the symbol $O^p(G)$, called the *Dress subgroup of G of type p*, the smallest normal subgroup of G such that $\pi(G/O^p(G)) \subseteq \{p\}$. We denote by $\mathcal{P}(G)$ the set of subgroups P of G of prime power order, possibly 1 and by $\mathcal{L}(G)$ the set of subgroups H of G containing the Dress subgroup $O^p(G)$ of type p for some prime p.

We say that a G-module V is *$\mathcal{L}(G)$-free* if $\dim V^{O^p(G)} = 0$ holds for any prime p. Here a G-module means a $\mathbb{R}[G]$-module which is finite dimensional over $\mathbf{R}$. We denote by $\mathcal{D}(G)$ the set of all pairs (P, H) of subgroups of G such that $P < H \leq G$ and P is of prime power order. A G-module V is called a *gap G-module* if V is $\mathcal{L}(G)$-free and the number

$$\dim V^P - 2\dim V^H$$

is positive for any pair $(P, H) \in \mathcal{D}(G)$. A finite group G is called a *gap group* if there exists a gap G-module and is called a *nongap group* otherwise.

A finite group G is an *Oliver group*, if G has no isthmus series of subgroups of the form

$$P \triangleleft H \triangleleft G$$

where $|\pi(P)| \leq 1$, $|\pi(G/H)| \leq 1$ and H/P is cyclic. A finite group G has a fixed point free smooth action on a disk if and only if G is an Oliver group ([5]). Furthermore, Oliver has completely decided which a smooth compact manifold is the fixed point set of a smooth action on a disk ([6]). On the other hand, Laitinen and Morimoto ([2]) has shown that a finite group G has a smooth one fixed point action of a sphere

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if and only if G is an Oliver group. We do not know which a smooth manifold of positive dimension is the fixed point set of a smooth action on a sphere. For an Oliver group G which is a gap group, one can apply equivariant surgery to convert an appropriate smooth action of G on a disk D into a smooth action of G on a sphere S with $S^G = M = D^G$, where $\dim M > 0$ (cf. [3, Corollary 0.3]). Thus it is important to ask whether a given group G is a gap group.

2. Centralizers of 2-elements outside of the Dress subgroup of type 2

Let G be a finite group. An element x of G is a *2-element* if the order of x is a power of 2 or equals to 1. Let K be a normal subgroup of G with $K \geq O^2(G)$.

For an element x of G, we denote by $\psi(x)$ the set of odd primes q such that there exists a subgroup N of G satisfying $x \in N$ and $O^q(N) \neq N$. We define a subset $E_2(G,K)$ of $G \setminus K$ as the set of involutions (elements of order 2) x such that either $|\psi(x)| > 1$ or $|\pi(C_G(x))| = |\pi(O^2(C_G(x)))| = 2$ holds, and define $E_4(G,K)$ as the subset of 2-elements x of $G \setminus K$ of order ≥ 4 with $|\psi(x)| > 0$. Set $E(G,K) = E_2(G,K) \cup E_4(G,K)$ (cf. [8]). Note that $E_2(G,K) = \emptyset$ if $K \neq O^2(G)$. We define sets $E_2^g(G,K)$, $E_4^g(G,K)$ and $E^g(G,K)$ as follows. The set $E_4^g(G,K)$ consists of 2-elements x of $G \setminus K$ of order > 2 such that $C_G(x)$ is not a 2-group. The set $E_2^g(G,K)$ consists of involutions x of $G \setminus K$ such that $|\pi(O^2(C_G(x)))| \geq 2$ holds. Set $E^g(G,K) = E_2^g(G,K) \cup E_4^g(G,K)$. Note that the sets $E_2^g(G,K)$, $E_4^g(G,K)$ and $E^g(G,K)$ are subsets of $E_2(G,K)$, $E_4(G,K)$ and $E(G,K)$ respectively.

We set

$$\mathcal{D}^2(G) = \Big\{ (P,H) \in \mathcal{D}(G) \;\Big|\; [H:P] = [O^2(G)H : O^2(G)P] = 2 \text{ and } O^q(G)P = G \text{ for all odd primes } q \Big\}.$$

(cf. [4]) and set

$$\mathcal{D}^2(G,K) = \Big\{ (P,H) \in \mathcal{D}^2(G) \;\Big|\; H \not\leq K \Big\}.$$

According to Laitinen and Morimoto [2], we denote by $V(G)$ the G-module

$$(\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G/O^p(G)] - \mathbb{R}).$$

If G is a group of prime power order, then $V(G) = \{0\}$ holds. Laitinen and Morimoto [2, Theorems 2.3 and B] have shown that $V(G)$ is an $\mathcal{L}(G)$-free G-module such that

$$\dim V(G)^P - 2\dim V(G)^H$$

is nonnegative for any pair $(P, H) \in \mathcal{D}(G)$ and is zero only if either $(P, H) \in \mathcal{D}^2(G, \varnothing)$ or $P \in \mathcal{L}(G)$. Note that $P \notin \mathcal{L}(G)$ for $(P, H) \in \mathcal{D}(G)$ if $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint.

Theorem 1. *Let G be a finite group such that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint. Let K be a subgroup of G with index 2. Then the following claims are equivalent.*

(1) *$E^g(G, K)$ is empty.*

(2) *$E(G, K)$ is empty.*

(3) *There exist pairs $(P_j, H_j) \in \mathcal{D}^2(G, K)$ such that*

$$\sum_j \left(\dim V^{P_j} - 2\dim V^{H_j}\right) = 0$$

for any $\mathcal{L}(G)$-free G-module V.

Corollary 2. *If $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint, then either sets $E(G, O^2(G))$ and $E^g(G, O^2(G))$ are both empty or both nonempty.*

3. Nongap groups

Let G be a finite group such that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint. The group G is a gap group if and only if any subgroup K of G with $K > O^2(G)$ is a gap group. Therefore it is easy to see the following result by Theorem 1.

Theorem 3. *Let G be a finite group and let K be a gap subgroup of G with index 2. Then the following claims are equivalent.*

(1) *$E^g(G, K)$ is empty.*

(2) *$E(G, K)$ is empty.*

(3) *G is a nongap group.*

Now, assume that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. Recall that if $\mathcal{P}(G) \cap \mathcal{L}(G) \neq \varnothing$, then G is a nongap group.

Proposition 4. *Let G be a finite group such that $O^2(G) \neq G$ and $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, and let K be a subgroup of G such that $[G, K] = 2$. Suppose that $E^g(G, K) = \emptyset$. Let G_2 be a Sylow 2-subgroup of G. Then it holds the followings.*

(1) *If two elements x and y of G_2 outside of K are conjugate in G, then they are conjugate in G_2.*

(2) $\displaystyle\sum_{(x)_G} \frac{2}{|C_{G_2}(x)|} = 1$, *where $(x)_G$ runs over conjugacy classes in G represented by elements of G_2 outside of K.*

(3) $\displaystyle\sum_{(C)_G} \frac{|C|}{|N_{G_2}(C)|} = 1$, *where $(C)_G$ runs over conjugacy classes in G represented by cyclic groups C of G_2 with $CK = G$.*

Proof. For an element x of $G \setminus K$, we denote by x_2 the involution of the cyclic subgroup generated by x. As $E^g_2(G)$ is empty, x_2 is an element outside of K. Recall that if two elements x and y of $G \setminus K$ are conjugate in G, namely $x = g^{-1}yg$, for some $g \in G$, then $x_2 = g^{-1}y_2g$ and thus $g \in C_G(x_2)$. Since $E^g_2(G, K)$ is empty and

$$\sum_{(x)_G \subseteq G \setminus K} \frac{|G|}{|C_G(x)|} = |G| - |K| = \frac{|G|}{2},$$

we have

$$\begin{aligned}
1 &= \sum_{(x)_G \subseteq G \setminus K} \frac{2}{|C_G(x)|} = \left(\sum_{\substack{(x)_G \subseteq G \setminus K \\ |x|=2}} + \sum_{\substack{(x)_G \subseteq G \setminus K \\ |x| \neq 2^*}} + \sum_{\substack{(x)_G \subseteq G \setminus K \\ |x|=2^*>2}} \right) \frac{2}{|C_G(x)|} \\
&= \sum_{\substack{(y)_G \subseteq G \setminus K \\ |y|=2}} \left(\frac{2}{|C_G(y)|} + \sum_{\substack{(x)_G \subseteq G \setminus K \\ x_2=y;\ |x| \neq 2^*}} \frac{2}{|C_G(x)|} \right) + \sum_{\substack{(x)_G \subseteq G \setminus K \\ |x|=2^*>2}} \frac{2}{|C_G(x)|} \\
&= \sum_{\substack{(y)_G \subseteq G \setminus K \\ |y|=2}} \left(\frac{2}{|C_G(y)|} + \sum_{\substack{(x)_G \subseteq G \setminus K \\ x_2=y;\ |x| \neq 2^*}} \frac{2}{|C_{C_G(y)}(x)|} \right) + \sum_{\substack{(x)_G \subseteq G \setminus K \\ |x|=2^*>2}} \frac{2}{|C_G(x)|} \\
&= \sum_{\substack{(y)_G \subseteq G \setminus K \\ |y|=2}} \sum_{\substack{(x)_G \subseteq G \setminus K \\ x_2=y;\ |x^2|:\ \text{odd}}} \frac{2}{|C_{C_G(y)}(x)|} + \sum_{\substack{(x)_G \subseteq G \setminus K \\ |x|=2^*>2}} \frac{2}{|C_G(x)|}.
\end{aligned}$$

Set $L(y) = O^2(C_G(y))\langle y \rangle \cong O^2(C_G(y)) \times \langle y \rangle$. Let $\mathcal{B}(y)$ (resp. $C(y)$) be the set of conjugacy classes in $C_G(y)$ which are represented by elements of $L(y) \setminus O^2(C_G(y))$ (resp. $O^2(C_G(y))$). Note that if two elements x and x' of G outside of K with $x_2 = x'_2$

are conjugate in G, then they are conjugate in $C_G(x_2)$. Therefore we obtain that

$$
(5)\quad
\begin{aligned}
1 &= \sum_{\substack{(y)_G\subseteq G\setminus K\\ |y|=2}} \sum_{(x)_{C_G(x)}\in\mathcal{B}(y)} \frac{2}{|C_{C(y)}(x)|} + \sum_{\substack{(x)_G\subseteq G\setminus K\\ |x|=2^*>2}} \frac{2}{|C_G(x)|} \\
&= \sum_{\substack{(y)_G\subseteq G\setminus K\\ |y|=2}} \sum_{(x)_{C_G(y)}\in\mathcal{B}(y)} \frac{2}{|C_{C_G(y)}(x^2)|} + \sum_{\substack{(x)_G\subseteq G\setminus K\\ |x|=2^*>2}} \frac{2}{|C_G(x)|} \\
&= \sum_{\substack{(y)_G\subseteq G\setminus K\\ |y|=2}} \sum_{(z)_{C_G(y)}\in C(y)} \frac{2}{|C_{C_G(y)}(z)|} + \sum_{\substack{(x)_G\subseteq G\setminus K\\ |x|=2^*>2}} \frac{2}{|C_G(x)|}.
\end{aligned}
$$

Let $\mathcal{A}$ be the set of conjugacy classes $(x)_{G_2}$ in G_2 represented by elements of $G_2 \setminus (G_2 \cap K)$. As $E_4^g(G,K)$ is empty, we have $C_G(x)$ for $x \in G \setminus K$ with $|x| = 2^* > 2$ is a 2-group. Furthermore by using the assumption that $E_2^g(G,K)$ is empty again, the last number at (5) equals to

$$
(6)\quad
\begin{aligned}
&\sum_{\substack{(y)_G\subseteq G\setminus K\\ |y|=2}} \frac{2|O^2(C_G(y))|}{|C_G(y)|} + \sum_{\substack{(x)_G\subseteq G\setminus K\\ |x|=2^*>2}} \frac{2}{|C_G(x)|} \\
&= \sum_{\substack{(y)_G\subseteq G\setminus K\\ |y|=2}} \frac{2}{|C_G(y)_2|} + \sum_{\substack{(x)_G\subseteq G\setminus K\\ |x|=2^*>2}} \frac{2}{|C_G(x)_2|} \leq \sum_{(y)_{G_2}\in\mathcal{A}} \frac{2}{|C_{G_2}(y)|} = 1,
\end{aligned}
$$

where $C_G(x)_2$ (resp. $C_G(y)_2$) is a Sylow 2-subgroup of $C_G(x)$ (resp. $C_G(y)$). Therefore any inequality or equality in (6) must be equality and thus if $x, y \in G_2$ are conjugate in G, then they are conjugate in G_2. □

Theorem 7. *Let G be a nongap group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and that $[G : O^2(G)] = 2$. Let G_2 be a Sylow 2-subgroup of G. Suppose the order of G is divisible by 4. Then it holds the followings.*

(1) *If x and y are involutions of $G_2 \setminus K$, then $xy \in [G_2, G_2]$.*
(2) *There exists an element x of $G_2 \setminus K$ such that $|x| > 2$.*
(3) *The group generated by all involutions of G_2 outside of K is a proper subgroup of G_2.*

Theorem 8. *Let G be a finite group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and that $G/[G,G]$ is not a 2-group. If G is a nongap group, then $O^2(G)$ is of odd order.*

Proof. If G is perfect, then G is a gap group. Suppose that $G/[G,G]$ is of even order. Let K be a subgroup of G such that $K > O^2(G)$, $[K : O^2(G)] = 2$ and $O^2(K/O^2(K))$ is isomorphic to $O^2(G/O^2(G))$. If G is a nongap group, then K is also a nongap group. There exist no 2-elements, not involutions, of K outside of $O^2(K)$. If there might exist such an element x, then x lies in $E(K, O^2(K))$ which implies that K is a gap group by Theorem 1. Therefore, the group generated by all involutions of K_2 outside of K is just K_2, where K_2 is a Sylow 2-subgroup of K. By Theorem 7 (3), the order of K is not divisible by 4. Since $[K : O^2(K)] = 2$, the order of $O^2(K) = O^2(G)$ is odd. □

Corollary 9. *Let G be a finite group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and that $G/[G,G]$ is not a 2-group. If G is a nongap group, then G is solvable.*

Proof. By Theorem 8, the Dress group $O^2(G)$ of type 2 is of odd order. Recall that $G/O^2(G)$ is a 2-group. By Burnside's theorem, $O^2(G)$ and $G/O^2(G)$ are both solvable. Thus G is solvable. □

Note that a finite group G such that $\mathcal{P}(G) \cap \mathcal{L}(G) \neq \emptyset$ is solvable.

4. Direct product

Lemma 10. *Let G be a finite group such that $O^2(G) \neq G$ and $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, and let K be a subgroup of G such that $[G, K] = 2$. If all elements of H outside of K are 2-elements, then*

$$\sum_{(C)_G} |N_G(C)/C|^{-1} \left|(H\backslash G)^C\right| = 1$$

where $(C)_G$ runs over conjugacy classes in G represented by cyclic groups C of G with $CK = G$.

We define $E^d(G, K)$ as the set of 2-elements x of G outside of K such that $C_G(x)$ is not a 2-group, Note that $E^g(G, K)$ is a subset of $E^d(G, K)$. There exist finite groups G so that $[G : O^2(G)] = 2$ and $E^d(G, O^2(G))$ is empty. A solvable group SmallGroup(1920, 239651) and a nonsolvable group SmallGroup(1344, 11427) both satisfy such conditions. (cf. [1])

Proposition 11. *Let G be a finite group such that $O^2(G) \neq G$ and $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, and let K be a subgroup of G such that $[G, K] = 2$. Suppose that $E^d(G, K) = \emptyset$. Let G_2 be a Sylow 2-subgroup of G and let C be a cyclic subgroup of G with $CK = G$. Then it holds the followings.*

(1) *If a subgroup of G_2 intersects with any conjugacy class $(x)_G$ represented by elements of G_2 outside of K, then it is just G_2.*

(2) $\left|(G_2\backslash G)^C/N_G(C)\right| = 1$ *holds. In particular,* $(G_2\backslash G)^C = G_2\backslash G_2N_G(C)$, *if* $C < G_2$.

Proof. Let C be a cyclic subgroup of G with $CK = G$. By assumption, $(H\backslash G)^C$ is nonempty. By Proposition 4 (3), we obtain that

$$\sum_{(C)_G} |N_G(C)/C|^{-1} \left|(H\backslash G)^C\right| \geq \sum_{(C)_G} \frac{|C|}{|N_{G_2}(C)|} = 1,$$

where $(C)_G$ runs over conjugacy classes in G represented by cyclic groups C of G_2 with $CK = G$. Furthermore as C is a 2-group, we obtain that

$$\sum_{(C)_G} |N_G(C)/C|^{-1} \left|(H\backslash G)^C\right| = \sum_{(C)_G} \frac{|C|}{|N_{G_2}(C)|} = 1$$

by Lemma 10 and thus

$$\left|(H\backslash G)^C\right| = 1.$$

Take an element $a \in G$ such that $aCa^{-1} \leq H$. Then we have

$$(H\backslash G)^C \supseteq H\backslash N_G(H)a.$$

Supposing that $H \neq G_2$, it holds $N_G(H) \neq H$, which implies $\left|(H\backslash G)^C\right| \geq 2$. □

Theorem 12. *Let G be a finite group satisfying that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint, $|O^2(G)|$ is even and $G/O^2(G)$ is cyclic. Let K be a subgroup of G with index 2. Then the following claims are equivalent.*

(1) *$E^d(G,K)$ is nonempty.*

(2) *$G \times G$ is a gap group.*

(3) *$G^k = \underbrace{G \times \cdots \times G}_{k\ times}$ is a gap group for $k \geq 2$.*

Note that G^k is a nongap group for any $k \geq 1$ if $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are not disjoint, since $\mathcal{P}(G^k)$ and $\mathcal{L}(G^k)$ are not disjoint. The assumption that $|O^2(G)|$ is even is need.

Remark 13. Let p, q and r be odd primes with $p \neq q$. Let $G = D_{2pq} \times C_r$ be the direct product group of a dihedral group D_{2pq} of order $2pq$ and a cyclic group C_r of order r. Then it holds that $E^d(G, O^2(G))$ is nonempty, $O^2(G)$ is of order odd and G^k is a nongap group for any $k \geq 1$.

Corollary 14. *Let G be a finite group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, $|O^2(G)|$ is even and $[G : O^2(G)] = 2$. Let $k > 1$ be an integer. Then we have the following claims:*

(1) *G and G^k are gap groups $\Longleftrightarrow E^g(G, O^2(G)) \neq \emptyset$.*

(2) *G^k is a gap group and G is a nongap group $\Longleftrightarrow E^g(G, O^2(G)) = \emptyset$ and $E^d(G, O^2(G)) \neq \emptyset$.*

(3) *G^k (and G) are nongap groups $\Longleftrightarrow E^d(G, O^2(G)) = \emptyset$.*

5. Wreath product

Let K and L be finite groups. We denote by $K \int L$ the semidirect product group $K^{|L|} \rtimes L$ such that L acts on $K^{|L|}$ by permutation:

$$1 \to K^{|L|} \to K \int L \to L \to 1$$

Proposition 15. *Let G be a finite group satisfying that $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$ and that $G/O^2(G)$ is cyclic. Let K be a subgroup of G with index 2. If $G \int C_n$ is a gap group for a 2-power integer n, then $E^d(G, K)$ is nonempty, where C_n is a cyclic group of order n.*

Let G = SmallGroup(1344, 11427). It is a nonsolvable group satisfying that $[G : O^2(G)] = 2$ and $E^d(G, O^2(G)) = \emptyset$. By Corollary 9, $G \int C_n$ is a gap group for any integer $n > 1$, not a 2-power.

Theorem 16. *Let G be a finite group satisfying that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint. For any subgroup K, $O^2(G) \triangleleft K \leq G$, possessing a cyclic quotient $K/O^2(G)$, the set $E(K, K_0)$ is nonempty, if and only if G is a gap group, where K_0 is a subgroup of K with index 2.*

Corollary 17. *Let G be a finite group satisfying that $\mathcal{P}(G)$ and $\mathcal{L}(G)$ are disjoint and $[G : O^2(G)] = 2$. The set $E(G, O^2(G))$ is nonempty if and only if G is a gap group.*

Before closing this section, we show the following theorem:

Theorem 18. *Let G be a finite group satisfying that $G/O^2(G)$ is cyclic, $\mathcal{P}(G) \cap \mathcal{L}(G) = \emptyset$, $E^d(G,K) \neq \emptyset$ and that $O^2(G)$ is of even order, where K is a subgroup of G with index 2. For any nontrivial finite group L, the wreath product group $G \int L$ is a gap group.*

First we show the assertion in the case where $L = C_2$:

Lemma 19. *Let G and K be finite groups as in Theorem 18. For a cyclic subgroup $C = C_2$ of order 2, the wreath product group $G \int C$ is a gap group.*

Proof. Let $\pi\colon G \int C \to \left(G \int C\right) / O^2(G \int C) \cong (G/O^2(G)) \int C$ be an epimorphism. If $\pi^{-1}(\pi(\langle x\rangle))$ is a gap group for any nontrivial 2-element x of $(G/O^2(G)) \int C$, then $G \int C$ is a gap group. Note that $O^2(G \int C) = O^2(G)^2 = O^2(G) \times O^2(G)$. Let f be a generator of C. Let h be a 2-element of G outside of K such that $C_G(h)$ is not a 2-group. Recall that $G \times G$ is a gap group by Theorem 12. It suffices to show that

$$N := \left\langle O^2(G)^2,\ (h_1,h_2)f \right\rangle$$

is a gap group for any elements h_1 and h_2 of $\langle h\rangle$. Note that

$$((h_1,h_2)f)^2 = (h_1h_2, h_2h_1).$$

We obtain that

$$C_{G_2}((h_1,h_2)f) = \left\langle (h_1,h_2)f,\ (a, h_1^{-1}ah_1) \;\middle|\; a \in C_{O^2(G)}(h_1h_2) \right\rangle.$$

As $[G : O^2(G)] = 2$, the group $C_{O^2(G)}(h)$ is not a 2-group. Thus $C_{G_2}((h_1,h_2)f)$ is not a 2-group by $C_{O^2(G)}(h_1h_2) \geq C_{O^2(G)}(h)$. Let

$$N_0 := \left\langle O^2(G)^2,\ (h_1h_2, h_2h_1) \right\rangle$$

be a subgroup of N with index 2. We show that $E^g(N,N_0)$ is nonempty. If $(h_1,h_2)f$ is not an involution, then $(h_1,h_2)f$ lies in $E^g_4(N,N_0)$. Suppose that $(h_1,h_2)f$ is an involution. Then it follows $h_1 = h_2$ which is an involution. In this case, $C_{G_2}((h_1,h_2)f)$ is isomorphic to $O^2(G)$ and thus $(h_1,h_2)f$ lies in $E^g_2(N,N_0)$. Therefore $E^g(N,N_0)$ is nonempty. Since N_0 is a subgroup of $G \times G$ with 2-power index, N_0 is a gap group. Then N is a gap group by combining Theorems 1 and 16. □

Proof of Theorem 18. Let $\pi\colon G \int L \to L$ be an epimorphism. If $\pi^{-1}(\pi(\langle x\rangle))$ is a gap group for any 2-element x of $G \int L$ outside of $O^2(G \int L)$, then $G \int L$ is a gap group. As $G^{|L|}$ is a gap group by Theorem 12, it suffices to show that $\pi^{-1}(C)$ is a gap group for any nontrivial cyclic group C. Let $C = C_n$ be a cyclic subgroup of L of order $n > 1$. Note that $|O^2(G \int C)|$ is even and $\mathcal{P}(G \int C) \cap \mathcal{L}(G \int C) = \emptyset$ since there is a subgroup of $G \int C$ isomorphic to G. Thus if n is not a 2-power integer, then $G \int C$ is a gap group by Corollary 9.

Assume that n is a 2-power integer, say 2^k. We show that $G \int C$ is a gap group by induction on k. In the case where $n = 2$, the assertion follows from Lemma 19. Let $m = 2^{k-1} \geq 2$ and let C_m be a cyclic subgroup of C with index 2.

Suppose that $G \int C_m$ is a gap group for any G as in Theorem 18. Note that $\rho^{-1}(C_m) = G^2 \int C_m$, where $\rho\colon G \int C \to C$ is an epimorphism. $\rho^{-1}(C_m)$ is isomorphic to a subgroup of the gap group $(G \int C_m)^2$ with 2-power index and thus is a gap group.

Let h be a 2-element of G outside of K such that $C_G(h)$ is not a 2-group. Let h_j be an element of $\langle h \rangle$ for each $j = 1, \ldots, n$ and let f be a generator of C. Consider the subgroup

$$N := \left\langle O^2(G)^n,\ (h_1, \ldots, h_n)f \right\rangle.$$

Let N_0 be a subgroup of N with index 2. As N_0 is a subgroup of $\rho^{-1}(C_m)$ with 2-power index, it is a gap group. Thus it suffices to show that $E^g(N, N_0)$ is nonempty. We show that $(h_1, \ldots, h_n)f$ lies in $E^g(N, N_0)$. We have

$$\begin{aligned} &C_{O^2(G)^n}((h_1, \ldots, h_n)f) \\ &\quad = \Big\langle (a, h_1^{-1}ah_1,\ (h_1h_2)^{-1}a(h_1h_2), \ldots, (h_1 \ldots h_{n-1})^{-1}a(h_1 \ldots h_{n-1})) \\ &\qquad\qquad \Big|\ a \in C_{O^2(G)}(h_1h_2 \ldots h_n) \Big\rangle. \end{aligned}$$

The group $C_{O^2(G)}(h_1h_2 \ldots h_n)$ contains the group $C_{O^2(G)}(h)$ and thus it is not a 2-group. As the element $(h_1, \ldots, h_n)f$ is not an involution, it lies in $E^g(N, N_0)$ and then N is a gap group.

The group $G \int C$ is a gap group, since any subgroup N, $O^2(G)^n \triangleleft N \leq G \int C$, possessing a cyclic quotient $N/O^2(G)^n$ is a gap group. □

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