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An application of localization formula to the moduli space of stable vector bundles over Riemann surfaces

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1 Introduction

There is a well known principle called "localization principle". Roughly speaking this principle can be said as follows:

If a manifold M adomits some group action, then information about M can be determined by information near its fixed point set.

There are many applications of this principle, for example G-signatrure theorem, Lefchetz fixed point formula and so on. Our method is anology for a finite group action with Duistermaat-Heckman's formula for Hamiltonian S^1-action. That is to say, we use Gysin map in Borel cohomology and a equivariant cohomology class that is a lift of symplectic class with respect to a fiber bundle arising from Borel construction. Then we can get relations between characteristic numbers (symplectic volume etc) of M and its fixed point set.

We apply this method to the moduli space of stable vector bundles over Riemann surface with a cyclic group action and we can obtain relations about charactreristic numbers of the moduli space.

In our case, it turns out that the fixed point set in the moduli space can be identified with some other moduli spaces, so relations we got between the moduli space and its fixed point set gives relations between the moduli space and many other moduli spaces.

These characteristic numbers contained in these relations have investigated by Jeffry, Kirwan, Thaddeus, Witten (see [4][5][9][10]), but by using our method, we can get relations "between" these numbers.

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2 Topological classification of equivariant vector bundles over Riemann surfaces

In this section, we classify the lifted actions on a complex vector bundle over a surface with a prime order cyclic group action.

This classification is necessaly in section 3 to determine the fixed point set of the action on the moduli space.

2.1 Notation

Let Σ be a closed oriented connected surface of genus g. Assume $G = \mathbb{Z}/p = \langle\sigma\rangle$ (p : prime) acts on Σ by a subgroup of the orientation preserving diffeomorphism group (i.e $\sigma : \Sigma \to \Sigma$ is a orientation preserving diffeo and $\sigma^p = \mathrm{id}$) and we denote its fixed point set by $\Sigma^G = \{p_1, \cdots, p_N\}$. For a fixed point $p_i \in \Sigma^G$ we denote the weight of the isotoropy representaion of G at p_i by δ_i and its inverse (in $\mathbb{Z}/p$) by $\bar{\delta}_i$.

Let $(E, \tilde{\sigma})$ be a G-equivariant vector bundle of rank n over (Σ, σ), i.e $E \to \Sigma$ is a complex vector bundle of rank n and $\tilde{\sigma} : E \to E$ is a bundle isomorphism s.t $\tilde{\sigma}$ covers $\sigma : \Sigma \to \Sigma$ and $\tilde{\sigma}^p = \mathrm{id}$. We define a map $f_{(E,\tilde{\sigma})} : \Sigma^G \to R_n(G)$ by $f_{(E,\tilde{\sigma})}(p_i) := (E|_{p_i}, \tilde{\sigma}|_{p_i})$. Where $R_n(G)$ is a set of isomorphism classes of rank n complex representation of G.

2.2 Results of classification

First, we can prove following proposition about "existence" condition of a lift.

Proposition 2.2.1. *For given $n \in \mathbb{Z}_{\geq 0}$, $d \in \mathbb{Z}$ and $f : \Sigma^G \to R_n(G)$, there exists G-equivariant vector bundle $(E, \tilde{\sigma})$ s.t $n =$rank$E$, $d = \deg E (= \int_\Sigma c_1(E))$ and $f = f_{(E,\tilde{\sigma})}$ if and only if the following relation holds.*

$$\prod_{p_i \in \Sigma^G} (\det f(p_i))^{\bar{\delta}_i} = \xi^d \in R_1(G).$$

Where $\xi^d \in R_1(G)$ is a weight d rank 1 irreducible representation of G and $\det, \prod$ and $\cdot^{\bar{\delta}_i}$ are operations for representations (highest rank exterior power and tensor product).

To prove this propsition,we nead following 3 steps:

(1)In the case $n = 1$, construct a expected G-equivariant line bundle for a given date satisfing that condition by "divisor consturuction" around fixed points.

(2)Show that for a G-equivariant line bundle $(L, \tilde{\sigma})$, the date $f_{(L,\tilde{\sigma})}$ and $\deg L$ satisfies that condition.

(3)Show that any G-equivariant vector bundle can be consturucted as a direct sum of G-equivariant line bundles.

Remark 2.1. This "existence" condition can be rewritten in terms of the weights of $(E|_{p_i}, \tilde{\sigma}|_{p_i})$ in the following way:

$$\sum_{p_i \in \Sigma^G} \sum_{j=1}^{n} \varepsilon_i^{(j)} \bar{\delta}_i \equiv d \bmod p.$$

Where $(\varepsilon_i^{(j)})_{1 \leq j \leq n}$ is a weights of a representation $(E|_{p_i}, \tilde{\sigma}|_{p_i})$.

By "obstruction theoritical" argument, we can show following proposition about "uniqueness" of a lift.

Proposition 2.2.2. *Let $(E_1, \tilde{\sigma}_1), (E_2, \tilde{\sigma}_2)$ be a G-equivariant vector bundles over (Σ, σ) then*

$$\exists gauge\ transformation\ \Phi : E_1 \to E_2\ s.t\ \tilde{\sigma}_2^{-1}\Phi\tilde{\sigma}_1$$

$$\Longleftrightarrow \mathrm{rank}E_1 = \mathrm{rank}E_2,\ \deg E_1 = \deg E_2,\ f_{(E_1,\tilde{\sigma}_1)} = f_{(E_2,\tilde{\sigma}_2)}.$$

This proposition means that a lift is determined (up to isomorphism via "Φ") uniquely by rank, deg and its fixed point date.

Remark 2.2. Similar results hold in the case $\Sigma^G = \emptyset$.

3 The group action on the moduli space of stable vector bundles over Riemann surfaces

3.1 The moduli space of stable vector bundles over Riemann surfaces

Let $E \to \Sigma$ be a Hermitian vector bundle of $\mathrm{rank}E = n$, $\deg E = d$ over Riemann surface Σ (here we consider a complex structure of Σ). We denote its connection space by $\mathcal{A} = \mathcal{A}(E)$ and the group of gauge transformations of E by $\mathcal{G} = \mathcal{G}(E)$. We can consider holomorphic structures on E and denote the space of holomorphic structures on E, the group of complexified ($\mathrm{GL}(n, \mathbb{C})$-valued) gauge transformations by $\mathcal{C} = \mathcal{C}(E)$, $\mathcal{G}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}(E)$ respectively. It is well known that in our case ($\dim_{\mathbb{C}} \Sigma = 1$) $\mathcal{A}$ and $\mathcal{C}$ can be identified by standard way (take $(0,1)$-part for a connection and take skew hermitian part for a holomorphic structure).

There exist $\mathcal{G}$ ($\mathcal{G}^{\mathbb{C}}$)-invariant subspaces of $\mathcal{A}$ and $\mathcal{C}$, $\mathcal{A}_c$ and $\mathcal{C}_s$ called "central Yang-Mills connections" and "stable holomorphic structures". See [2] for the definitions of these spaces. We denote the quatient space $\mathcal{A}_c/\mathcal{G}$ by $\mathcal{N}$. The following theorem was prooved by Narasimhan-Seshadri.

Theorem 3.1.1 (Narasimhan-Seshadri [7]). *If $(n, d) :=$ G.C.D$(n, d) = 1$, then under the identyfication $\mathcal{A} \cong \mathcal{C}$ as above, the quatient spaces $\mathcal{N}$ and $\mathcal{C}_s/\mathcal{G}^{\mathbb{C}}$ are homeomorphic. Moreover it carries a structure of smooth compact Kähler manifold of dimension (over $\mathbb{C}$) $4g - 3$ and its tangent space at $[A] \in \mathcal{N}$ is isomorphic to the vector space*

$$H^1(\Sigma; \mathrm{End}E) = \mathrm{coker}(d_A^{0,1} : \Omega^{0,0}(\Sigma; \mathrm{End}E) \to \Omega^{0,1}(\Sigma; \mathrm{End}E)).$$

We call this space $\mathcal{N}(= \mathcal{C}_s/\mathcal{G}^{\mathbb{C}})$ "the moduli space of stable vector bundles over Riemann surfce".

Remark 3.1. The condition "$(n, d) = 1$" is neccesaly for smoothness of $\mathcal{N}$, but in fact they showed similar result in the case $(n, d) \neq 1$.

3.2 The group action on $\mathcal{N}$

Here we assume G-action on Σ preserves its complex structure. Let $\pi : E \longrightarrow \Sigma$ be a Hermitian vector bundle of rank$E = n$, deg$E = d$. We define a "extended gauge transformation group" $\tilde{\mathcal{G}}$ by

$$\tilde{\mathcal{G}} := \{\tilde{f} : E \to E \text{ bundle iso} \mid \exists f \in G,\ \pi \circ \tilde{f} = f \circ \pi\}.$$

Then there exists natural exact sequence

$$1 \to \mathcal{G} \to \tilde{\mathcal{G}} \to G \to 1.$$

Remark 3.2. A splitting of this exact sequence coresponds to a lift of G-action on E and the set of equivalence classes of splittings by conjugation action of $\mathcal{G}$ corespond to the set of equivalence classes of lifts of G-action on E.

The group $\tilde{\mathcal{G}}$ acts on $\mathcal{A}$ by pullback so by the exact sequence above, quatient group $G = \tilde{\mathcal{G}}/\mathcal{G}$ acts on the quatient space $\mathcal{A}/\mathcal{G}$ and one can check this action preserves subspace $\mathcal{N}$. In this way we get a natural G-action on the moduli space of stable vector bundles $\mathcal{N}$.

Remark 3.3. By definition of the induced G-action, a equivalence class $[A] \in \mathcal{A}/\mathcal{G}$ is a fixed point of this action if and only if

$$\exists \tilde{\sigma} \in \tilde{\mathcal{G}} \text{ s.t } \pi \circ \tilde{\sigma} = \sigma \circ \pi,\ \tilde{\sigma}^* A = A.$$

Hereafter, we consider the case $\Sigma^G \neq \emptyset$. Then lift of G-actions on E does exists by Prop 2.2.1.

Fix a lift of G-action $\tilde{\sigma} : E \to E$ and define following notations:

$$\begin{aligned}
&\mathcal{A}^{\tilde{\sigma}} := \{A \in \mathcal{A} \mid \tilde{\sigma}^* A = A\} \\
&\mathcal{A}_c^{\tilde{\sigma}} := \mathcal{A}_c \cap \mathcal{A}^{\tilde{\sigma}} \\
&\mathcal{G}^{\tilde{\sigma}} := \{\phi \in \mathcal{G} \mid \phi \circ \tilde{\sigma} = \tilde{\sigma} \circ \phi\} \\
&\iota_{\tilde{\sigma}} : \mathcal{A}^{\tilde{\sigma}} \to \mathcal{A} \quad \text{(inclusion map)} \\
&\mathcal{N}_{\tilde{\sigma}} := \mathcal{A}_c^{\tilde{\sigma}}/\mathcal{G}^{\tilde{\sigma}}.
\end{aligned}$$

We also denote the induced map by

$$\iota_{\tilde{\sigma}} : \mathcal{N}_{\tilde{\sigma}} \to \mathcal{N}.$$

The map $\iota_{\tilde{\sigma}} : \mathcal{N}_{\tilde{\sigma}} \to \mathcal{N}$ is not injective apriori, but we can show the next propsition by using the irreducibility of central Yang-Mills connections in co-prime case.

Proposition 3.2.1.

$$If\ (n, d) = 1,\ then\ \iota_{\tilde{\sigma}}\ is\ injective.$$

By definition of the G-action on $\mathcal{N}$, we see that $\iota_{\tilde{\sigma}}(\mathcal{N}_{\tilde{\sigma}}) \subset \mathcal{N}^G$ (fixed point set of G-action). Moreover we can show the following proposition.

Proposition 3.2.2. *If $(n,d)=1$, then for any connected component Z of $\mathcal{N}^G$, there exists a lift of G-action $\tilde{\sigma}$ s.t $\iota_{\tilde{\sigma}}(\mathcal{N}_{\tilde{\sigma}}) = Z$ and such a lift is unique up to multiplication of gauge transformation*

$$\begin{aligned} \zeta : E &\rightarrow E \\ u &\rightarrow \zeta u. \end{aligned}$$

Where ζ is a complex number s.t $\zeta^p = 1$ and we regard $\zeta \in U(1)$ as a center elemant of $\mathcal{G}$. In paticular there exist $p =^\# G$ lifts satisfing that condition.

By these two propositions we can determine the fixed point set $\mathcal{N}^G$ and hereafter we will denote a connected component of fixed point set by $\mathcal{N}_{\tilde{\sigma}}$ for some lift of G-action $\tilde{\sigma}$.

Next, we consider about $\dim \mathcal{N}_{\tilde{\sigma}}$. We restrict to the simplest nontrivial case $n=2, d=1$.

Definition 3.2.3. For a lift of G-action $\tilde{\sigma}$ of the type

$$f_{(E,\tilde{\sigma})}(p_i) = \xi^{\varepsilon_i} + \xi^{\varepsilon_i'} \in R_2(G) \quad (p_i \in \Sigma^G)$$

(note that a lift of G-action is determined by $f_{(E,\tilde{\sigma})}$ from Prop 2.2.2.) we define

$$\Sigma^{\tilde{\sigma}} := \{p_i \in \Sigma^G \mid \varepsilon_i \neq \varepsilon_i'\}, \quad k_{\tilde{\sigma}} :=^\# \Sigma^{\tilde{\sigma}}.$$

Then we can write down the dimension formula as follows.

Proposition 3.2.4.

$$\dim_{\mathbb{C}} \mathcal{N}_{\tilde{\sigma}} = k_{\tilde{\sigma}} + 4g' - 3$$

where g' is a genus of quatient surface Σ/G.

To show this formula, note that for $[A] \in \mathcal{N}_{\tilde{\sigma}} \subset \mathcal{N}$, $T_{[A]}\mathcal{N} = H^1(\Sigma; \mathrm{End}E)$ so $T_{[A]}\mathcal{N}_{\tilde{\sigma}} = H^1(\Sigma; \mathrm{End}E)^{\tilde{\sigma}}$. One can compute $\dim H^1(\Sigma; \mathrm{End}E)^{\tilde{\sigma}}$ by Riemann-Roch formula, localization formula for equivariant K-theory and orthogonality of irreducible representations.

Natural projection $E/\tilde{\sigma} \rightarrow \Sigma/G$ does not define a vector bundle structure in general, but this projection defines a "V-bundle structure". So we can regard $\mathcal{A}^{\tilde{\sigma}}$ and $\mathcal{G}^{\tilde{\sigma}}$ as connections and gauge transfomation group for V-bundle $E/\tilde{\sigma}$. Mehta-Seshadri showed that V-bundles corespond to a vector bundles with additional structure called "palabolic bundles" and the space $\mathcal{N}_{\tilde{\sigma}}$ can be identified with the moduli space of "stable parabolic bundles". Moreover Nitsure gave a method to compute the Betti numbers of the moduli space of stable parabolic vector bundles and improving his method, we can compute the Betti numbers of $\mathcal{N}_{\tilde{\sigma}}$. In paticular we can check that whether $\mathcal{N}_{\tilde{\sigma}} \neq \emptyset$ or not for a given lift $\tilde{\sigma}$. See [3][6][8] for details.

Remark 3.4. We can write down the characteristic classes (Chern character etc) of the normal bundle $\nu_{\tilde{\sigma}} \to \mathcal{N}_{\tilde{\sigma}}$ by similar argument and using index for family. Here "write down" means the following:

In [2], Atiyah-Bott showed that there exists a "universal bundle" $\mathcal{U} \to \mathcal{N} \times \Sigma$ and cohomology classes of $\mathcal{N}$ obtained from characteristic clases of $\mathcal{U}$ generate cohomology ring of $\mathcal{N}$. We can express $ch(\nu_{\tilde{\sigma}})$ as combinations of these classes and natural cohomology classes arising from the representation of the fiber of E over fixed points.

By these arguments we could determine the fixed point set.

4 An application of localization formula

4.1 General setting for an apaplication

As we said in 3.1, $\mathcal{N}$ carries a Kähler structure so there exists a symplectic form ω. We can construct a G-equivariant Hermitian line bundle $\mathcal{L} \to \mathcal{N}$ with Hermitian connection ∇ s.t $c_1(\nabla) = \omega$ and cosider its Borel construction $\mathcal{L}_G \to \mathcal{N}_G$. Put $\omega_G := c_1(\mathcal{L}) \in H^2(\mathcal{N}_G;\mathbb{Z}) = H^2_G(\mathcal{N};\mathbb{Z})$ and denote its mod p reduced class by $\omega_G(p) \in H^2_G(\mathcal{N};\mathbb{Z}/p)$.

Note that equivariant cohomology class $\omega_G \in H^2_G(\mathcal{N};\mathbb{Z})$ is a lift of the symplectic class $[\omega] \in H^2(\mathcal{N};\mathbb{Z})$ with respect to the map $H^*_G(\mathcal{N};\mathbb{Z}) \to H^*(\mathcal{N};\mathbb{Z})$ arising from fiber bundle

$$\mathcal{N} \to \mathcal{N}_G \to BG.$$

Denote the constant map $f_{\mathcal{N}} : \mathcal{N} \to pt$ and induced map $f_{\mathcal{N}} : \mathcal{N}_G \to BG$ etc, consider the Gysin homomorphism for Borel cohomology

$$f_{\mathcal{N}*} : H^*_G(\mathcal{N}) \to H^*(BG).$$

Appling the localization formula for Borel cohomology to $\omega_G(p)^j \in H^{2j}(\mathcal{N};\mathbb{Z}/p)$ ($j = 0, 1, \cdots$), we obtain the following formula:

$$f_{\mathcal{N}*}\left(\omega_G(p)^j\right) = \sum_{\tilde{\sigma}} f_{\mathcal{N}_{\tilde{\sigma}}*}\left(\frac{\iota^*_{\tilde{\sigma}}(\omega_G(p)^j)}{e_G(\nu_{\tilde{\sigma}})}\right) \in H^*(BG;\mathbb{Z}/p)_* \qquad (\#)$$

where $e_G(\nu_{\tilde{\sigma}})$ is the (mod p reduced) equivariant Euler class of the normal bundle of the fixed point set $\mathcal{N}_{\tilde{\sigma}}$ and $H^*(BG;\mathbb{Z}/p)_*$ is a localized ring of the ring $H^*(BG;\mathbb{Z}/p)$ by appropriate multiplicatively closed subset.

Because the Gysin homomorphism associated to a fiber bundle is equal to integration along the fiber, its left hand side for $j = 4g - 3$ can be written

$$f_{\mathcal{N}*}\left(\omega_G(p)^{4g-3}\right) \equiv \mathrm{Vol}(\mathcal{N}) \bmod p.$$

Similar holds for a fixed point component. Where we put

$$\mathrm{Vol}M := \int_M \omega_M^{\frac{\dim_{\mathbb{R}} M}{2}}$$

for a symplectic manifold (M, ω_M).

From this formula, we can obtain following relations:

·For $0 \le j \le 4g-4 (= \dim \mathcal{N} - 1)$,

$$\sum_{\bar\sigma} f_{\mathcal{N}_{\bar\sigma}*}\left(\frac{\iota_{\bar\sigma}^*(\omega_G(p)^j)}{e_G(\nu_{\bar\sigma})}\right) = 0 \bmod p.$$

These formula contain relations between symplectic volume of fixed point set.

·For $j = 4g-3 (= \dim \mathcal{N})$,

$$f_{\mathcal{N}*}\left(\omega_G(p)^{4g-3}\right) = \sum_{\bar\sigma} f_{\mathcal{N}_{\bar\sigma}*}\left(\frac{\iota_{\bar\sigma}^*(\omega_G(p)^{4g-3})}{e_G(\nu_{\bar\sigma})}\right) \bmod p.$$

Remark 4.1. Formally, this method is analogy with "Duistemaat-Heckman's formula" for Hamiltonian S^1-action. Key points to prove "D-H's formula" are:
· Construct a equivariant cohomology class that is a lift of the symplectic class by using moment map.
· Consider the image of a localized Gysin map of the powers of that equivariant cohomology class.

In the case of a finite group action, moment map is trivial so we consider a equivariant prequantum line bundle $\mathcal{L}$ to lift the symplectic class.

4.2 Example

Consider genus 2 hyperellptic curve (Σ, σ), and Hermitian vector bundle E of rank$=2$, deg$=1$ over Σ. By classification of lifts, it turns out that there exist 192 lifts s.t $k_{\bar\sigma} = 1$, 160 lifts s.t $k_{\bar\sigma} = 3$ and 12 lifts s.t $k_{\bar\sigma} = 5$. Then $\dim \mathcal{N}_{\bar\sigma} = k_{\bar\sigma} - 3 = -2, 0, 2$ and we can check there does exist 80 fixed point components of $\dim \mathcal{N}_{\bar\sigma} = 0$ and 6 components of $\dim \mathcal{N}_{\bar\sigma} = 2$.

Let $\mathcal{N}_k$ $(k = 1, \cdots, 6)$ be the components of dim $= 2$ and put

$$\iota_k^*(\omega_G(2)) := \iota_k^*[\omega] + \kappa_k u \in H_G^2(\mathcal{N}_k; \mathbb{Z}/2).$$

Where $u \in H^2(BG; \mathbb{Z}/2) (\cong \mathbb{Z}/2)$ is a generater and $\kappa_k \in \mathbb{Z}/2$.

Remark 4.2. We can not have determined prameters κ_k yet. These numbers correspond to the weights of the representation of the fibers of the line bundle $\mathcal{L}|_{\mathcal{N}_k}$ over trivial G-space $\mathcal{N}_k$.

Computing (#) for $j = 1, \cdots, 5$ and combining them, we obtain the following relations:

Theorem 4.2.1.

$$\sum_{k=1}^{6} \mathrm{Vol}(\mathcal{N}_k) \equiv 0 \bmod 2$$

and

$$\sum_{k=1}^{6} \kappa_k \mathrm{Vol}(\mathcal{N}_k) \equiv \mathrm{Vol}(\mathcal{N}) \bmod 2.$$

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