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ABELIAN GROUP THEORY IN JAPAN

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1. Introduction

Honda began to study Abelian Group Theory in Japan at the middle of 20th century. He specially studied primary abelian groups at that time. So he introduced a lot of tools to study this field in the future. Later, Koyama also began to study primary abelian groups. In this article, we introduce their work.

All groups considered are abelian groups. Throughout this article, Q denotes the field of rational numbers, Z the ring of integers, N the set of all positive integers, and p a prime integer.

2. Notation

Definition 2.1. *Let G be a group. The subgroup*

$$G[p] = \{g \in G \mid pg = 0\}$$

is called the p-socle of G.

Definition 2.2. *Let G be a group and σ an ordinal. Then $p^\sigma G$ is defined as follows: $p^0G = G$,*

$$p^{\sigma+1}G = p(p^\sigma G),$$

and $p^\sigma G = \cap_{\rho<\sigma} p^\rho G$ whatever σ is a limit ordinal. The minimal ordinal τ such that $p^{\tau+1}G = p^\tau G$ will be called the p-length of G.

Definition 2.3. *A group G is said to be torsion if all elements of G are of finite order, while the group G is said to be torsion-free if all elements of G, except 0, are of infinite order.*

The ordinal ω is the minimal infinite ordinal.

Proposition 2.4. *Let G be a torsion-free group. Then the p-length of A is at most ω for every prime p.*

Proof. Suppose that $p^\omega G = \cap_{n\in\mathbf{N}} p^n G \neq 0$. Let $0 \neq z \in p^\omega G$. Then, for every $n \in \mathbf{N}$, there exists $x_n \in G$ such that $z = p^n x_n$. Since $p^n(px_{n+1} - x_n) = 0$ and G is torsion–free, we have $px_{n+1} = x_n$. Hence $z = px_1$ and $x_1 \in p^\omega G$. Thus $p(p^\omega G) = p^\omega G$ and so the p–length of the group G is ω. Hence the assertion is clear. □

Proposition 2.5. *The set T of all elements of finite order in a group G is a subgroup of G.*

The subgroup T in Proposition 2.5 is called the *maximal torsion subgroup* of G.

3. Summable p–Groups

Summable p–groups were discovered by Honda.

Definition 3.1. *A p–group G of length τ is said to be reduced if $p^\tau G =$ 0.*

Let G be a p–group of length τ. Define S_σ by

$$p^\sigma G[p] = p^{\sigma+1} G[p] \oplus S_\sigma \quad \text{for} \quad \sigma < \tau. \tag{3.2}$$

The direct sum $\oplus_{\sigma<\tau} S_\sigma$ fails, in general, to exhaust $G[p]$, i.e. $G[p] \neq \oplus_{\sigma<\tau} S_\sigma$. A simple counterexample is the following.

Example 3.3. *Let G be the maximal torsion subgroup of*

$$\prod_{n=1}^{\infty} \langle x_n \rangle, \quad o(x_n) = p^n.$$

Then G is not summable.

Let G be in as Example 3.3. For every nonnegative integer n, we can write

$$p^n G[p] = S_n \oplus p^{n+1} G[p], \quad S_n < G[p]. \tag{3.4}$$

Put $S_n = \langle p^n x_{n+1} \rangle$. Then $G[p] \neq \oplus_{n=0}^{\infty} S_n$, because $(x_1, px_2, \ldots, p^n x_{n+1}) \notin \oplus_{n=0}^{\infty} S_n$. For any other choice of S_n in (3.4), $G[p] \neq \oplus_{n=0}^{\infty} S_n$.

Definition 3.5. *Let G be a p–group of length τ. The p–group G is said to be summable if, for suitable choice of S_σ in* (3.2), *$G[p] = \oplus_{\sigma<\tau} S_\sigma$.*

From this definition, it is clear that:

(1) A direct sum of cyclic p–groups is summable.
(2) A direct sum of summable groups is again summable.

Theorem 3.6. *[Honda] Countable reduced p–groups are summable.*

Theorem 3.7. *[Honda] A summable p–group G satisfies* $p^{\omega_1}G = 0$. *The ordinal* ω_1 *is the minimal uncountable ordinal.*

By Ulm Theorem, countable p–groups are already characterized.

4. N–High Subgroups

Definition 4.1. *Let* G *be a group and let* A *and* N *be subgroups of* G. *If* A *is maximal with respect to the property of being disjoint from* N, *then* A *is called an* N*-high subgroup of* G.

The existence of N–high subgroups is guaranteed by Zorn's lemma.

Honda seemed to consider the following problem. Let G be a group and A a subgroup of G. Then is there a subgroup B of G such that $G = A \oplus B$?

Definition 4.2. *A subgroup* A *of a group* G *is said to be an absolute direct summand of* G *if, for every* A*-high subgroup* B *of* G,

$$G = A \oplus B.$$

The following is an example of an absolute direct summand.

Proposition 4.3. *Let* G *be a group and* A *a subgroup of* G. *Suppose that* $nA = A$ *for all* $n \in \mathbf{N}$. *Then* A *is an absolute direct summand of* G.

The study of N–high subgroups in arbitrary groups seems to have started with Honda and Irwin and Walker independently. Irwin posed the following problems.

For what subgroups N *of a group* G *is it true that*

(1) *all* N*–high subgroups are pure (see Definition 5.6)*
(2) *all* N*–high subgroups are isomorphic*
(3) *all* N*–high subgroups are endomorphic images of* G *?*

Though Irwin originally posed these problems for a primary group, we may consider them in arbitrary groups. The first one was completely solved by Pierce. This author partially solved the second one. However, the others are not yet completely solved, though numerous authors have studied these problems.

5. Essential Subgroups

Honda also studied essential subgroups and used this concept to define neat subgroups.

Definition 5.1. *Let* G *be a group. A subgroup* A *of* G *is said to be essential in* G *if* $A \cap B \neq 0$ *for every nonzero subgroup* B *of* G. *In this case,* G *is called an essential extension of* A.

For example, the p–socle is an essential subgroup of a p–group.

We occasionally use the expression "a maximal essential extension subgroup E of a subgroup A in a group G" meaning implicitly that the subgroup E is maximal among the essential extension subgroups of A.

Proposition 5.2. *Let G be a group and A a subgroup of G. Then there exists a maximal essential extension subgroup of G containing A.*

Definition 5.3. *A subgroup A of a group G is said to be neat in G if, for every prime p,*

$$A \cap pG = pA.$$

Definition 5.4. *Let G be a group and A subgroup of G. A subgroup N of G containing A is a neat hull of A in G if, among the neat subgroups of G containing A, there exists a minimal one.*

He proved in his book written in Japanese that the maximal essential subgroups of a given subgroup are neat hulls of the given subgroup. Hence he originally proved the following important result.

Theorem 5.5. *[Honda] Let G be a group and A subgroup of G. Then there exists a neat hull of A in G.*

Definition 5.6. *A subgroup A of a group G is said to be pure in G if, for all $n \in \mathbf{N}$,*

$$A \cap nG = nA.$$

Theorem 5.5 is also useful. Then we considered that for every subgroup A of a group G, is there a minimal pure subgroup H of G containing A ? However, there exists a subgroup B of some group G that there is no minimal pure subgroup of G containing B.

Definition 5.7. *A subgroup A of a group G is said to be purifiable in G if, among the pure subgroups of G containing A, there exists a minimal one.*

Koyama also presented in her paper a good example not to be purifiable in a given group. This author got interested in purifiable subgroups at middle of 1980s and has been studied these subgroups since then.

6. Torsion–Complete Groups

Definition 6.1. *A group G is said to be a torsion–complete p–group if G is the maximal torsion subgroup of the p–adic completion of a direct sum of cyclic p–groups.*

The group G in Example 3.3 is a torsion–complete p–group.

Koyama studied torsion–complete p–groups. Before her study, we already knew that if G is a torsion–complete p–groups, then the closure of a pure subgroup of G is a direct summand of G. She obtained the following useful result.

Theorem 6.2 (Koyama). *A reduced p–group G is torsion–complete if and only if, for every pure subgroup A of G, the closure of A is a direct summand of G.*