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Approximate Computation of Pseudovarieties

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1 Introduction

This paper is concerned with a generalization of the notion of a variety of an ideal I, that we call a *pseudovariety* in analogy with the standard terminologies "pseudozero"[6, 7, 8, 9] and "pseuodspectrum"[9, 10]. Other works use "pseudozero set"[3] or "root neighborhoods"[4] to describe this concept.

"To state the main idea as directly as possible, the pseudozero set of f is the union of the zero sets of all systems $\hat{f}$ that are "acceptable approximations" of f in the sense that they come from f by small changes in the coefficients."[3]

The first thing to note about pseudovarieties (a technical definition will come later) is that they depend on the specified generators (basis) of the ideal I. They are not, therefore, true algebraic objects like the variety $V(I)$, which is independent of the basis for the ideal I. However, we prefer to retain the term "variety" as a subword of the definition because other features of varieties are preserved, and association of the words may help us to think about these things.

Let us first consider the simplest kind of pseudovariety, as follows. For multivariate polynomials $p_i \in \mathbb{C}[x_1, \cdots, x_s]$, define:

Definition 1 (Straightforward Pseudovariety)

$$PV\left(I_{\epsilon;p}\right) = \bigcup_{||\Delta p_i|| \leq \epsilon} V\left(\langle p_1 + \Delta p_1, \cdots, p_m + \Delta p_m\rangle\right)$$

where $\Delta p_i \in \mathbb{C}[x_1, \cdots, x_s]$ and the norm $||\cdot||$ means the 2-norm of the vector of polynomial coefficients.

One of the goals of this paper is to explore methods for visualization of the pseudovariety $PV(I_{\epsilon;p})$. We will use this notation to describe the stability of roots and a monodromy condition for systems.

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2 Properties for pseudovarieties

Some basic properties, but helpful also for visualization, are shown for the affine varieties $V = V(f_1, \cdots, f_m)$ and $W = V(g_1, \cdots, g_n)$ in the text book [2] as follows.

Lemma 2
If $V, W \subset \mathbb{C}^s$ are affine varieties, then so are $V \cap W$ and $V \cup W$.

We would like to show these properties will be preserved for pseudovarieties $V = PV(I_{\epsilon;f_1,\cdots,f_m})$ and $W = PV(I_{\epsilon;g_1,\cdots,g_n})$ in some sense.

Lemma 3

- *If $V, W \subset \mathbb{C}^s$ are pseudovarieties w.r.t. ϵ, then so is $V \cap W$.*
- *If unit polynomials $||f_i|| = 1, ||g_j|| = 1$ are assumed, then $V \cup W$ is a subset of a pseudovariety w.r.t. $\sqrt{2\pi}\epsilon$.*

Proof We claim that

$$V \cap W = PV(I_{\epsilon;f_1,\cdots,f_m,g_1,\cdots,g_n})$$

and

$$V \cup W \subset PV(I_{\sqrt{2\pi}\epsilon;f_ig_j:1\leq i\leq m,1\leq j\leq n}).$$

The first equality is trivial to prove, since the coefficients of each polynomials are perturbating independently. The second one is proved as follows.

If $(a_1, \cdots, a_s) \in V$, then, at least, one of the neighborhoods $\tilde{f}_i, 1 \leq i \leq m$ must vanish at this point, which implies that all of the $\tilde{f}_i g_j$ for $1 \leq j \leq n$ also vanish at $(a_1, \cdots, a_s)$. By the way,

$$\begin{aligned} ||fg||_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} f(z)\overline{f(z)} \cdot g(z)\overline{g(z)}dz \\ &\leq \left[\frac{1}{2\pi}\int_0^{2\pi} f(z)\overline{f(z)}dz\right] \cdot \left[\int_0^{2\pi} g(z)\overline{g(z)}dz\right] \\ &= 2\pi \cdot ||f||^2 \cdot ||g||^2. \end{aligned}$$

Therefore,

$$||\tilde{f}_i g_j - f_i g_j|| \leq \sqrt{2\pi} \cdot ||\tilde{f}_i - f_i|| \cdot ||g_j|| = \sqrt{2\pi}\epsilon.$$

Thus, $V \subset PV(I_{\sqrt{2\pi}\epsilon;f_ig_j:1\leq i\leq m,1\leq j\leq n})$, and $W \subset PV(I_{\sqrt{2\pi}\epsilon;f_ig_j:1\leq i\leq m,1\leq j\leq n})$ follows similarly. This proves that $V \cup W \subset PV(I_{\sqrt{2\pi}\epsilon;f_ig_j:1\leq i\leq m,1\leq j\leq n})$. ∎

The first property about $V \cap W$ may give a solution for visualization of pseudovarieties of overdetermined system, i.e. $m > s$.

3 Pseudozero

A useful theorem for the problem is known as a pseudozero criterion for systems of multivariate polynomials. This has been described by Stetter in [8]. We would like to give a brief explanation for the 2-norm case, and then we will show later how it works for visualization of pseudovarieties.

A polynomial in s variables with support $J \subset \mathbb{N}_0^s$ may be written as $p(x) = \sum_{j\in J} a_j x^j \in \mathbb{C}[x_1, \cdots, x_s]$. The tolerance associated with p is defined by a nonnegative vector $e \in \mathbb{R}^{|J|}$ whose components $e_j \geq 0$ correspond to the coefficients a_j, $j \in J$. Let

$$J' := \{j \in J : e_j > 0\} \neq \emptyset, \text{ and } |J'| \leq |J|,$$

then the following defines the concept of e-neighborhood.

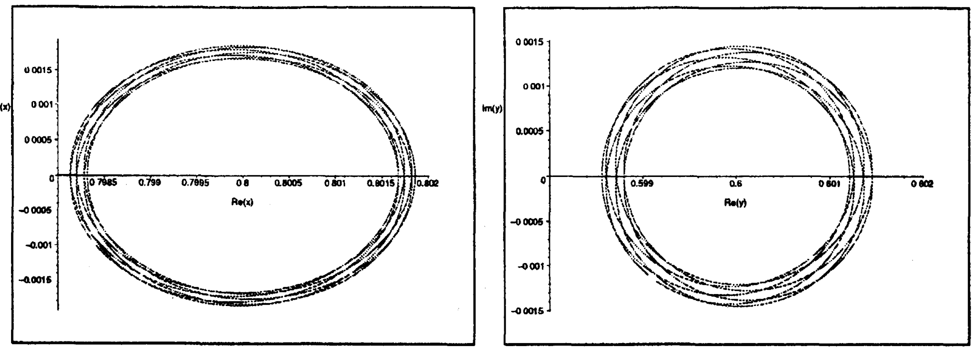

Figure 1: $\epsilon = 0.001$

Definition 4 (e-neighborhood)
The e-neighborhood $N(p, e)$ of the polynomial p with tolerance e consists of those polynomials $\tilde{p} \in \mathbb{C}[x_1, \cdots, x_s]$, $\tilde{p}(x) = \sum \tilde{a}_j x^j$, which satisfy

$$||(\cdots, \frac{|\tilde{a}_j - a_j|}{e_j}, \cdots)|| \leq 1, \tag{1}$$

for $\tilde{a}_j = 0,\ j \notin J$ and $\tilde{a}_j = a_j, j \in J \backslash J'$.

Using the definition, the pseudovariety may be described as follows.

$$PV\left(I_{e;p}\right) = \bigcup_{p_i + \Delta p_i \in N(p_i, e_i)} V\left(\langle p_1 + \Delta p_1, \cdots, p_m + \Delta p_m \rangle\right) \tag{2}$$

If $z \in PV(I_{e;p})$, then z is called a *pseudozero* of a system. A pseudozero criterion is described as follows.

Theorem 5 (Pseudozero criterion)
$z \in \mathbb{C}^s$ is in $PV(I_{e;p})$ if and only if

$$|p_i(z)| \leq \left\| \begin{pmatrix} \vdots \\ e_j |z|^j \\ \vdots \end{pmatrix} \right\|, \quad \textit{for } i = 1, \cdots, m.$$

The theorem can be proved by using the Hölder inequality.

4 Method for visualization

The method we use to visualize a pseudovariety for the case $m = s$ is to find an algebraic characterization of the boundary of the pseudovariety, that we then display by a numerical parameterization. That is, we find some points on the pseudovariety by a method such as Newton's method, and then use numerical parameterization (i.e. solving a differential equation numerically) with these as initial points. Figures 1, 2, and 3 show projections of these parameterizations for the example

$$x^2 + y^2 - 1 = 0, \quad 25xy - 12 = 0\,.$$

We see from these figures that the isolated roots of the original system can merge to a double root if the perturbation is as large as $\epsilon = 0.05$.

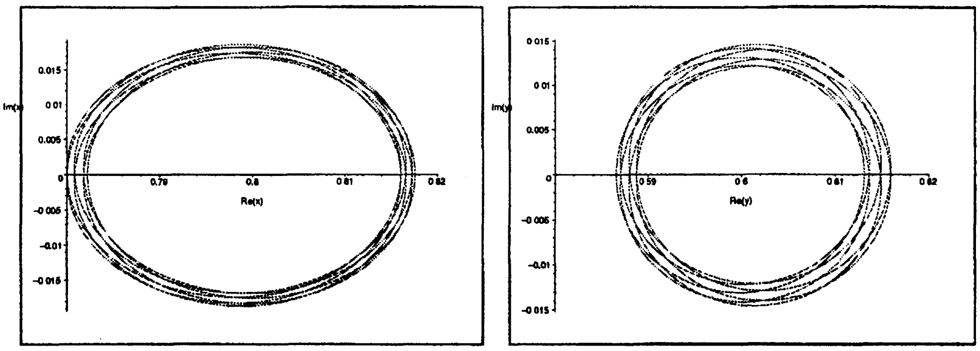

Figure 2: $\epsilon = 0.01$

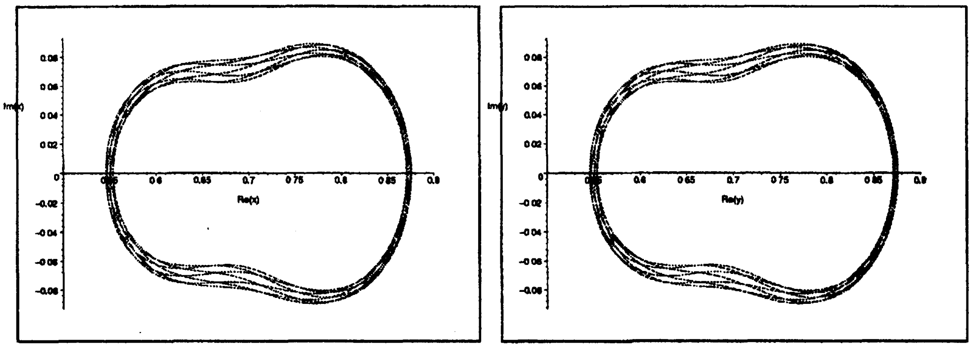

Figure 3: $\epsilon = 0.05$

5 Stability of roots

The condition number for a zero of systems of polynomials was defined in [5] by using linearization. It should be a useful tool for small enough perturbations. Here we would like to show stability of roots from our point of view by using our visualization tools.

If all pseudozeros in $PV(I_{e;p})$ are isolated, e.g. Figure 1 and Figure 2, the conditioning of roots may be defined as

$$\max_{t_1,\cdots,t_s} \{|x_1(t_1,\cdots,t_s) - x_1^0|,\cdots,|x_s(t_1,\cdots,t_s) - x_s^0|\} \tag{3}$$

where $(x_1^0,\cdots,x_s^0)$ is a root of original system $p_1,\cdots,p_s$. (min also taken, to ensure roots correspond) This may be directly obtained from the solution of the boundaries $x_1(t_1,\cdots,t_s),\cdots,x_s(t_1,\cdots,t_s)$.

For example, about the example in the section **3**, the maximum perturbation of roots is bounded by

$$\max\{|x - x^0|, |y - y^0|\} = 0.001881 \quad \text{for } \epsilon = 0.001$$
$$\max\{|x - x^0|, |y - y^0|\} = 0.020089 \quad \text{for } \epsilon = 0.01$$

6 Monodromy

We observed, e.g. in Figure 3, that ϵ may change the monodromy group[1]. This asks for the sizes and values of ϵ and coefficients of Δp_i such that roots interchange. We may find these values as follows.

$$\text{minimize } ||\Delta p_1||^2 + \cdots + ||\Delta p_s||^2$$

subject to

$$p_i + \Delta p_i = 0, \ i = 1,\cdots,s,$$

$$\det(J(p_1 + \Delta p_1, \cdots, p_s + \Delta p_s)) = 0,$$

where J is the Jacobian matrix w.r.t. polynomials $p_1 + \Delta p_1, \cdots, p_s + \Delta p_s$. We may solve this problem by practical methods of optimization, e.g. Lagrange multipliers.

7 Conclusion

We described about pseudovarieties for systems of polynomials on the following issues :

- A method for visualizing low-dimensional projections of pseudovarieties,
- Understanding stability of roots of nearby systems of polynomials,
- Decide if nearby systems have multiple roots with respect to ϵ.

There are several remaining works concerning about the visualization method :

- Verify the backward error of the solutions,
- Compare visualization methods with the one given in [3].

Furthermore, related works may arise about finding nearest singularities of polynomials :

- Find nearest singularities of algebraic curves of bivariate polynomials,
- Find nearest positive-dimensional system of a zero-dimensional system.

For univariate polynomials, there are some works to compute nearest singular polynomials, e.g. [11].

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