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Generalization of operator type Shannon inequality and its reverse one

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Abstract. We shall state the following generalization of operator type Shannon inequality and its reverse one as a simple corollary of parametric extensions of Shannon inequality in Hilbert space operators.

Let $\{A_1, A_2, ..., A_n\}$ *and* $\{B_1, B_2, ..., B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H. *If* $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, *then*

$$\begin{aligned}\sum_{j=1}^{n} S_2(A_j|B_j) &\geq \Big[\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big] \log\Big[\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big] \geq \log\Big[\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big] \\ &\geq \sum_{j=1}^{n} S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^{n} S(A_j|B_j) \\ &\geq -\log\Big[\sum_{j=1}^{n} A_jB_j^{-1}A_j\Big] \geq -\Big[\sum_{j=1}^{n} A_jB_j^{-1}A_j\Big] \log\Big[\sum_{j=1}^{n} A_jB_j^{-1}A_j\Big] \\ &\geq \sum_{j=1}^{n} S_{-1}(A_j|B_j)\end{aligned}$$

where $S_q(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ *for* $A > 0$, $B > 0$ *and any real number* q *and* $S(A|B) = S_0(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ *which is the relative operator entropy of* $A > 0$ *and* $B > 0$.

Our results can be considered as parametric extensions of the following celebrated Shannon inequality ([3],[5] and [233 p ,1]) which is very useful and so famous in information theory. *Let* $\{a_1, a_2, ..., a_n\}$ *and* $\{b_1, b_2, ..., b_n\}$ *be two probability vectors. Then*

$$0 \geq \sum_{j=1}^{n} a_j \log b_j - \sum_{j=1}^{n} a_j \log a_j \text{ (see inequalities (2.4) of Corollary 2.4).}$$

§1 Introduction

First the Shannon inequality asserts: *Let* $\{a_1, a_2, ..., a_n\}$ *and* $\{b_1, b_2, ..., b_n\}$ *be two probability vectors. Then*

$$(1.1)\qquad 0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j}.$$

We remark that $0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j}$ in (1.1) is equivalent to $D = \sum_{j=1}^{n} a_j \log \frac{a_j}{b_j} \geq 0$ which is the original number type Shannon inequality and this D is called "divergence" in [3] and [5].

In this paper we shall state parametric extensions of Shannon inequality and its reverse one in Hilbert space operators.

A bounded linear operator T on a Hilbert space H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Definition 1.1. $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q is defined by

$$S_q(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}.$$

We recall that $S_0(A|B) = A^{\frac{1}{2}}(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} = S(A|B)$ is the relative operator entropy in [2] and $S(A|I) = -A\log A$ is the usual operator entropy in [4].

Definition 1.2. *$A\natural_q B$ for $A > 0$ and $B >$ and any real number q is defined by*

$$A\natural_q B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q A^{\frac{1}{2}}$$

and $A\natural_p B$ for $p \in [0,1]$ just coincides with $A\sharp_p B$ which is well known as p-power mean.

We remark that $S_1(A|B) = -S(B|A)$ and moreover $S_q(A|B) = -S_{1-q}(B|A)$ for any q.

Following after Definition 1.1, The original Shannon inequality can be expressed as follows:

$$0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} = \sum_{j=1}^{n} a_j^{\frac{1}{2}}(\log a_j^{\frac{-1}{2}} b_j a_j^{\frac{-1}{2}})a_j^{\frac{1}{2}} = \sum_{j=1}^{n} S(a_j|b_j).$$

Consequently $0 \geq \sum_{j=1}^{n} S(a_j|b_j)$ in the original Shannon inequality can be extented to $0 \geq \sum_{j=1}^{n} S(A_j|B_j)$ in operator version case (2.4) of Corollary 2.4, so that the form of (1.1) is convenient for operator type extension. We can summarize the following contrast:

The original Shannon inequality and its reverse one	*The operator version Shannon inequality and its reverse one*
$0 \geq \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \geq -\log \sum_{j=1}^{n} \frac{a_j^2}{b_j}.$	$0 \geq \sum_{j=1}^{n} S(A_j\vert B_j) \geq -\log \sum_{j=1}^{n} A_j B_j^{-1} A_j.$
for $a_j, b_j > 0$ *with* $1 = \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j.$	for $A_j, B_j > 0$ with $I = \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j.$

§2 Parametric extensions of operator reverse type Shannon inequality derived from two operator concave functions $f_1(t) = \log t$ and $f_2(t) = -t \log t$

Firstly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = \log t$.

Theorem 2.1. *Let $p \in [0,1]$ and also let $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^{n} A_j \sharp_p B_j \leq I$, where I means the identity operator on H. Then*

$$\log\Big[\sum_{j=1}^{n}(A_j \natural_{p+1} B_j) + t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] - \log t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j) \tag{2.1}$$

$$\geq \sum_{j=1}^{n} S_p(A_j\vert B_j)$$

$$\geq -\log\Big[\sum_{j=1}^{n}(A_j \natural_{p-1} B_j) + t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j)\Big] + \log t_0(I - \sum_{j=1}^{n} A_j \sharp_p B_j)$$

for fixed real number $t_0 > 0$, where $S_p(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

Secondly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = -t \log t$.

Theorem 2.2. *Let $p \in [0,1]$ and also let $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^{n} A_j \sharp_p B_j \leq I$, where I means the identity operator on H. Then*

$$(2.2) \qquad \sum_{j=1}^{n} S_{p+1}(A_j|B_j)$$

$$\geq \Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$

$$-t_0 \log t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j) \quad \textit{for fixed real number } t_0 > 0,$$

and

$$(2.2') \qquad \sum_{j=1}^{n} S_{p-1}(A_j|B_j)$$

$$\leq -\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$

$$+t_0 \log t_0(I - \sum_{j=1}^{n} A_j\sharp_p B_j) \quad \textit{for fixed real number } t_0 > 0,$$

where $S_q(A|B)$ is defined in Definition 1.1 and $A\natural_q B$ is defined in Definition 1.2.

We shall state the following result which can be shown by combining Theorem 2.1 with Theorem 2.2.

Corollary 2.3. *Let $p \in [0,1]$ and also let $\{A_1, A_2, ..., A_n\}$ and $\{B_1, B_2, ..., B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^{n} A_j\sharp_p B_j \leq I$, where I means the identity operator on H. Then*

$$(2.3) \quad \sum_{j=1}^{n} S_{p+1}(A_j|B_j)$$

$$\geq \Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big] \log\Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$

$$\geq \log\Big[\sum_{j=1}^{n}(A_j\natural_{p+1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$

$$\geq \sum_{j=1}^{n} S_p(A_j|B_j)$$

$$\geq -\log\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j) + (I - \sum_{j=1}^{n} A_j\sharp_p B_j)\Big]$$

$$\geq -\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j)+(I-\sum_{j=1}^{n}A_j\sharp_p B_j)\Big]\log\Big[\sum_{j=1}^{n}(A_j\natural_{p-1}B_j)+(I-\sum_{j=1}^{n}A_j\sharp_p B_j)\Big]$$

$$\geq \sum_{j=1}^{n} S_{p-1}(A_j|B_j)$$

where $S_q(A|B)$ *is defined in Definition 1.1 and* $A\natural_q B$ *is defined in Definition 1.2.*

Corollary 2.3 easily implies the following result which can be considered as *operator version of Shannon inequality and its reverse one.*

Corollary 2.4. *Let* $\{A_1, A_2, ..., A_n\}$ *and* $\{B_1, B_2, ..., B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* H. *If* $\sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j = I$, *then*

$$(2.4)\quad \sum_{j=1}^{n} S_2(A_j|B_j) \geq \Big[\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big]\log\Big[\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big] \geq \log\Big[\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big]$$
$$\geq \sum_{j=1}^{n} S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^{n} S(A_j|B_j)$$
$$\geq -\log\Big[\sum_{j=1}^{n} A_jB_j^{-1}A_j\Big] \geq -\Big[\sum_{j=1}^{n} A_jB_j^{-1}A_j\Big]\log\Big[\sum_{j=1}^{n} A_jB_j^{-1}A_j\Big]$$
$$\geq \sum_{j=1}^{n} S_{-1}(A_j|B_j).$$

Remark 2.1. We recall $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q as follows:

$$S_q(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q(\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}.$$

By an easy calculation we have

$$\tfrac{d}{dq}[S_q(A|B)] = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^q[\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}]^2A^{\frac{1}{2}} \geq 0,$$

so that $S_q(A|B)$ is an increasing function of q, and it is interesting to point out that the decreasing order of the positions of $\sum_{j=1}^{n} S_2(A_j|B_j)$, $\sum_{j=1}^{n} S_1(A_j|B_j)$, $\sum_{j=1}^{n} S(A_j|B_j)$, and $\sum_{j=1}^{n} S_{-1}(A_j|B_j)$ in (2.4) of Corollary 2.4 is quite reasonable since $\sum_{j=1}^{n} S(A_j|B_j) = \sum_{j=1}^{n} S_0(A_j|B_j)$.

This paper will appear elsewhere with complete proofs.

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