数理解析研究所講究録 1396 巻 2004 年 112-123

The Choquet integral as a piecewise linear function

櫻井 智章, 室伏 俊明
Tomoaki Sakurai, Toshiaki Murofushi
東京工業大学 大学院総合理工学研究科 知能システム科学専攻
Department of Computational Intelligence and Systems Science,
Tokyo Institute of Technology

Abstract

In this paper, the Choquet integral is considered as one of the representations of piecewise linear functions, and the mutual transformation with Chua's canonical form, which is another representation of piecewise linear functions, is given. Moreover, the transformation method into state-variable representation, which is also a representation of piecewise linear functions, from Choquet integral is given. Finally, the relationship with an existing generalization of Chua's canonical form is also investigated.

1 Introduction

Recently, a lot of studies have been made on the application of Choquet integral to model non-dynamical multi-input/one-output systems such as multi-attribute evaluation classification and information fusion [6, 16]. If the number of inputs is n, then the number of the parameters of Choquet integral model is $2^n - 1$ while that of the linear model is n. This brings a great power of description to a Choquet integral model, however, it also brings a problem of complex structure. Therefore, in order to overcome this problem, several studies have been made on the multi-level Choquet integral [4, 10], which in general can reduce the number of the parameters hence the complexity. So far, Murofushi and Narukawa [12] have shown that every piecewise linear function is representable as a multi-level non-monotonic Choquet integral with constant terms. On the other hand, there are other well-known representations of piecewise linear functions such as Chua's canonical form [2], state-variable representation [8] and max-min representation [5, 13], and so far various researches on piecewise linear functions have been done, from practical and theoretical points of view [7, 8]. Until now, however, there have been no unified researches. The objectives of this research are: (i) to built a unified theory about piecewise linear functions, (ii) to establish the efficient analysis technique for piecewise linear models, with the consideration that the multi-level Choquet integral is one of the representations of piecewise linear functions.

This paper gives the mutual transformation between the Choquet integral and Chua's canonical form, and the transformation from Choquet integral to state-variable representation. Moreover, the relationship of the Choquet integral to a generalization of Chua's canonical form [9] is also considered.

Throughout this paper, n is assumed to be a positive integer, and $X = \{1, 2, \dots, n\}$. 2^X denotes the power set of X. The cardinality of a set B is denoted by $|B|$. Moreover, $\mathbb{R}$ denotes the set of real numbers, max and min operators are denoted by $\vee$ and $\wedge$, respectively, and for

$x \in \mathbb{R}$ we write $x^+ = x \vee 0$ and $x^- = (-x)^+$. Unless otherwise noted, all vectors are column vectors, and the inner product of two vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ is denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. The transposition of a matrix (or a vector) A is denoted by A^T.

2 Preliminaries

In this section, existing results relevant to the Choquet integral and a piecewise linear function are introduced. Moreover, Chua's canonical form is also introduced.

2.1 Fuzzy measure and Choquet integral

This subsection describes existing results in fuzzy measure theory. The following is the definition of fuzzy measure in this paper.

Definition 2.1. [11, 4] A set function $\mu : 2^X \to \mathbb{R}$ is called a *fuzzy measure* if
(1) $\mu(\emptyset) = 0$.
μ is called a *monotone* fuzzy measure if it fulfills (1) and the following:
(2) $\mu(A) \leqq \mu(B)$ whenever $A \subset B$.

Remark 1. Usually, a set function fulfilling (1) is called a *non-monotonic* fuzzy measure, and a set function fulfilling (1) and (2) is called a fuzzy measure [11, 4]. We, however, adopt the above nonstandard terminology so that we deal mainly with set functions fulfilling (1) in this paper.

Definition 2.2. [4] The *Choquet integral* of a function $f : X \to \mathbb{R}$ with respect to a fuzzy measure μ is defined by

$$\text{(C)} \int_X f(j)\, d\mu(j) = \sum_{k=1}^{n} f(j_k)[\mu(A_k) - \mu(A_{k+1})] \tag{2.1}$$

where $\{j_1, j_2, \dots, j_n\} = X$, $f(j_1) \leqq f(j_2) \leqq \dots \leqq f(j_n)$, $A_k = \{j_k, j_{k+1}, \dots, j_n\}$ for $k = 1, 2, \dots, n$ and $A_{n+1} = \emptyset$.

Definition 2.3. [4] Let μ be a fuzzy measure. The *Möbius inverse* of μ is the set function $\mu^{\mathrm{m}} : 2^X \to \mathbb{R}$ defined as

$$\mu^{\mathrm{m}}(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \quad \forall A \subset X.$$

Definition 2.4. [4] For a positive integer k, a fuzzy measure μ is called *k-additive* if $\mu^{\mathrm{m}}(A) = 0$ whenever $|A| > k$, and there exists at least one subset $A \subset X$ such that $|A| = k$ and $\mu^{\mathrm{m}}(A) \neq 0$. In this case, we say that the *order of additivity* of μ is k.

Proposition 2.1. [4] *Let μ be a fuzzy measure on X, then the following holds.*

$$\mu(A) = \sum_{B \subset A} \mu^{\mathrm{m}}(B), \quad \forall A \subset X.$$

Proposition 2.2. [4] *The Choquet integral of a function $f : X \to \mathbb{R}$ with respect to a fuzzy measure μ is given by*

$$\text{(C)} \int_X f(j)\, d\mu(j) = \sum_{\substack{A \subset X \\ A \neq \emptyset}} \bigwedge_{j \in A} f(j) \mu^{\mathrm{m}}(A).$$

2.2 Piecewise linear functions

In this subsection, the fundamental definitions relevant to a piecewise linear function are introduced.

Definition 2.5. [17] A finite collection $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ of pairs of a vector $\boldsymbol{\alpha}_i \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and a scalar $\beta_i \in \mathbb{R}$ is called a *linear partition* of $\mathbb{R}^n$ if it fulfills the following condition:

(lp) if $i \neq j$, there is no $\lambda \in \mathbb{R}$ such that $\lambda\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_j$ and $\lambda\beta_i = \beta_j$.

Each $(\boldsymbol{\alpha}_i, \beta_i)$ is called a *boundary hyperplane.*

The *family of region* generated by a linear partition $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ of $\mathbb{R}^n$ is the family $\mathcal{R}$ of subsets of $\mathbb{R}^n$ defined as $\mathcal{R} = \{R_I \mid I \subset \{1, 2, \dots, l\}, \dim(R_I) = n\}$, where

$$R_I = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \langle \boldsymbol{\alpha}_i, \boldsymbol{x} \rangle \geqq \beta_i \text{ for all } i \in I, \\ \langle \boldsymbol{\alpha}_i, \boldsymbol{x} \rangle \leqq \beta_i \text{ for all } i \notin I \end{array} \right\}.$$

Remark 2. $\mathbb{R}^n = \bigcup \mathcal{R}$ holds.

Definition 2.6. [17] Let $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ be a linear partition of $\mathbb{R}^n$ and $\mathcal{R}$ be the family of regions generated by $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$. Two regions $R_I, R_J \in \mathcal{R}$ are called *(i-)neighbors* if $I \triangle J = \{i\}$.

Definition 2.7. [2] A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called a *piecewise-linear function* if there exists a linear partition $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ of $\mathbb{R}^n$ which fulfills the following condition:

(pwl) For every $R \in \mathcal{R}$, there exist a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$ such that

$$f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}, \quad \forall \boldsymbol{x} \in R.$$

The right-hand side above $A\boldsymbol{x} + \boldsymbol{b}$ is called the *linear component* of f on R. Moreover, A is called the *Jacobian* of f on R.

Remark 3. Every piecewise linear function is continuous. Moreover, every piecewise linear function f has infinitely many linear partition of $\mathbb{R}^n$ which fulfills the condition **(pwl)**.

The following proposition indicates that Choquet integral is a piecewise linear function.

Proposition 2.3. [12][14] *Let μ be a fuzzy measure on X, then the following function $\varphi_\mu : \mathbb{R}^n \to \mathbb{R}$ is a piecewise linear function*

$$\varphi_\mu(x_1, x_2, \dots, x_n) = (\mathrm{C}) \int_X x_j \, d\mu(j), \tag{2.2}$$

where the integrand in the right hand side is $j \mapsto x_j$. Moreover, the piecewise linear function φ_μ has a linear partition $\{(e_{ij}, 0)\}_{1 \leqq i < j \leqq n}$ of $\mathbb{R}^n$, where $e_{ij} = (e_{ij1}, e_{ij2}, \dots, e_{ijn})$ is defined as

$$e_{ijk} = \begin{cases} 1 & \textit{if } k = i, \\ -1 & \textit{if } k = j, \\ 0 & \textit{otherwise.} \end{cases}$$

The family of regions generated by $\{(e_{ij}, 0)\}_{1 \leqq i < j \leqq n}$ is $\mathcal{R} = \{R_\sigma\}_{\sigma \in S}$, where S is the set of permutations on X and for $\sigma \in S$

$$R_\sigma = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leqq \cdots \leqq x_{\sigma(n)}\}.$$

Furthermore, the linear component of φ_μ on R_σ is given by the right-hand side of (2.1) with the substitution of $f(j_k) = x_{\sigma(k)}$ and $j_k = \sigma(k)$, $(k = 1, 2, \dots, n)$.

Henceforth, the Choquet integral of a function $j \mapsto x_j$ with respect to a fuzzy measure μ will be denoted by $\varphi_\mu(\boldsymbol{x})$ like as in (2.2).

2.3 Chua's canonical form

The expression form based on the definition of a piecewise linear function (Definition 2.7) has problems such as a lot of parameters, the difficulty of analysis, and the immense cost of calculation. Because of these problems, Chua introduced the following expression form.

Definition 2.8. [2] A piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ possesses a *Chua's canonical form* if f is expressed as

$$f(\boldsymbol{x}) = \boldsymbol{a} + B\boldsymbol{x} + \frac{1}{2}\sum_{i=1}^{l} \boldsymbol{c}_i |\langle \boldsymbol{\alpha}_i, \boldsymbol{x}\rangle - \beta_i|, \tag{2.3}$$

where l is a nonnegative integer, $B \in \mathbb{R}^{m\times n}$, $\boldsymbol{\alpha}_i \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, $\boldsymbol{a} \in \mathbb{R}^m$, $\boldsymbol{c}_i \in \mathbb{R}^m \setminus \{\boldsymbol{0}\}$, $\beta_i \in \mathbb{R}$ $(i = 1, 2, \ldots, l)$, and $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ fulfills **(lp)**.

Chua's canonical form is unique in the sense that, if a piecewise linear function (2.3) is represented as

$$f(\boldsymbol{x}) = \boldsymbol{a}' + B'\boldsymbol{x} + \frac{1}{2}\sum_{i=1}^{l'} \boldsymbol{c}'_i |\langle \boldsymbol{\alpha}'_i, \boldsymbol{x}\rangle - \beta'_i|,$$

then $\boldsymbol{a} = \boldsymbol{a}'$, $B = B'$, $l = l'$ and there exist a bijection $\pi : \{1, 2, \ldots, l\} \to \{1, 2, \ldots, l\}$ and positive numbers $\gamma_1, \gamma_2, \ldots, \gamma_l$ such that for every $i \in \{1, 2, \ldots, l\}$

$$\boldsymbol{c}_i = \gamma_i \boldsymbol{c}'_{\pi(i)}, \qquad \boldsymbol{\alpha}_i = \gamma_i^{-1}\boldsymbol{\alpha}'_{\pi(i)}, \qquad \beta_i = \gamma_i^{-1}\beta'_{\pi(i)}.$$

Based on the observation above, throughout the paper we put on Chua's canonical form (2.3) the constraint that $||\boldsymbol{\alpha}_i||_\infty = ||(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})||_\infty = \sup_{j=1,2,\ldots,n} |\alpha_{ij}| = 1$ for $i = 1, 2, \ldots, l$.

Besides the uniqueness, Chua's canonical form has merits such as a concise expression, a small number of parameters, and the explicit information on a linear partition of f, which is given as $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ by $\boldsymbol{\alpha}_i$'s and β_i's in (2.3). However, as shown in Remark 5 below, there is a demerit that a piecewise linear function does not necessarily have a Chua's canonical form; in other words, the class of piecewise linear functions possessing Chua's canonical form is very small.

Definition 2.9. [2] Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a piecewise linear function. f is said to possess the *consistent variation property* if there exists a linear partition $\{(\boldsymbol{\alpha}_i, \beta_i)\}_{i=1}^{l}$ of $\mathbb{R}^n$ fulfilling the following condition **(cv)**:

(cv) For every boundary hyperplane $(\boldsymbol{\alpha}_i, \beta_i)$, there exists a matrix $C_i \in \mathbb{R}^{m\times n}$ such that, for every pair of i-neighboring regions (R_{iI}^+, R_{iI}^-), it holds that

$$A_{iI}^+ - A_{iI}^- = C_i,$$

where A_{iI}^+ and A_{iI}^- are the Jacobians on R_{iI}^+ and R_{iI}^-, respectively,

$$R_{iI}^+ = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \langle \boldsymbol{\alpha}_l, \boldsymbol{x}\rangle \geqq \beta_l \text{ for all } l \in I \cup \{i\}, \\ \langle \boldsymbol{\alpha}_l, \boldsymbol{x}\rangle \leqq \beta_l \text{ for all } l \notin I \cup \{i\} \end{array} \right\},$$

$$R_{iI}^- = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \langle \boldsymbol{\alpha}_l, \boldsymbol{x}\rangle \geqq \beta_l \text{ for all } l \in I, \\ \langle \boldsymbol{\alpha}_l, \boldsymbol{x}\rangle \leqq \beta_l \text{ for all } l \notin I \end{array} \right\},$$

and $I \subset \{1, 2, \ldots, l\} \setminus \{i\}$.

Remark 4. When (**cv**) is fulfilled, there exists a unique vector $\boldsymbol{c}_i \in \mathbb{R}^m$ such that $C_i = \boldsymbol{c}_i \boldsymbol{\alpha}_i^T$. Moreover, this $\boldsymbol{c}_i$ coincides with the constant vector $\boldsymbol{c}_i$ in the right-hand side of (2.3).

Proposition 2.4. [2] *A piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ possesses a Chua's canonical form if and only if f possesses the consistent variation property.*

Remark 5. A piecewise linear function does not necessarily possess a Chua's canonical form [2]. $\boldsymbol{c}_i|\langle \boldsymbol{\alpha}_i, \boldsymbol{x}\rangle - \beta_i|$ in (2.3) expresses the variation of the linear component of f when crossing over the boundary hyperplane $(\boldsymbol{\alpha}_i, \beta_i)$. Chua's canonical form, equivalently the condition (**cv**), requires that this variation is consistent independent of the crossing point over $(\boldsymbol{\alpha}_i, \beta_i)$. Clearly, it is a very strong condition.

2.4 State-variable representation

van Bokhoven has introduced a new expression form of a piecewise linear function from a view-point of nonlinear circuit theory.

Definition 2.10. [8] The correspondence f from $\boldsymbol{x} \in \mathbb{R}^n$ to $\boldsymbol{y} \in \mathbb{R}^m$ is called a *complementarity correspondence* if there exist a nonnegative integer k and matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{k\times n}$, $D \in \mathbb{R}^{k\times k}$, and vectors $\boldsymbol{g} \in \mathbb{R}^m$, $\boldsymbol{h} \in \mathbb{R}^k$ such that

$$\begin{aligned} \boldsymbol{y} &= A\boldsymbol{x} + B\boldsymbol{u} + \boldsymbol{g}, \\ \boldsymbol{j} &= C\boldsymbol{x} + D\boldsymbol{u} + \boldsymbol{h}, \\ \boldsymbol{u}, \boldsymbol{j} &\geqq \boldsymbol{0}, \quad \langle \boldsymbol{u}, \boldsymbol{j}\rangle = 0. \end{aligned}$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{j}$ are called the *state-variables*, and the expression using these state-variables is called a *state variable representation*.

The state-variable representation has advantages such as the ability to express all the piece-wise linear functions and, moreover, the ability to express correspondences, or multivalued functions, which map each value of $\boldsymbol{x}$ to one or more values for $\boldsymbol{y}$. On the other hand, it has at least two disadvantages. One is that function values cannot be calculated easily. The problem to find $\boldsymbol{y}$ for each $\boldsymbol{x}$ results in a linear complementarity problem (Definition 2.11 below) by substituting $\boldsymbol{q} = C\boldsymbol{x} + \boldsymbol{h}$; that is, in order to calculate a function value, we must solve the linear complementarity problem each time. Another disadvantage is the lack of methods to find a representation with minimum dimensional state-variables. A piecewise linear function has infinitely many state-variable representations, and the dimensions of their state-variables are generally different; a method to find a minimum dimensional representation (the minimum realization problem of state-variable representation) has not been clarified.

Definition 2.11. [3][8] For a given matrix $D \in \mathbb{R}^{k\times k}$ and vector $\boldsymbol{q} \in \mathbb{R}^k$, the problem to find a pair of vectors $\boldsymbol{j}$ and $\boldsymbol{u}$ that fulfill the following conditions is called a *linear complementarity problem*.

$$\boldsymbol{j} = D\boldsymbol{u} + \boldsymbol{q}, \tag{2.4a}$$
$$\boldsymbol{u}, \boldsymbol{j} \geqq \boldsymbol{0}, \quad \langle \boldsymbol{u}, \boldsymbol{j}\rangle = 0. \tag{2.4b}$$

The constraint (2.4b) is called *complementarity condition*.

The coefficient matrix D in (2.4a) is classified according to properties of the linear comple-mentarity problem [3][8]. Two matrix classes related to piecewise linear functions are introduced especially here.

Definition 2.12. **(P)** A matrix $D \in \mathbb{R}^{k\times k}$ is said to be a P-matrix if all its principal minors are positive. The class of such matrices is called Class P.
(ULT) A matrix $D \in \mathbb{R}^{k\times k}$ is said to be a unit lower triangular matrix, a ULT-matrix for short, if it is a lower triangular matrix and all its diagonal elements are 1. The class of such matrices is called Class ULT.

Clearly, Class ULT is a proper subset of Class P. The classes of complementarity correspondences associated with them, however, coincide with each other (Proposition 2.5 below).

Definition 2.13. The class of complementarity correspondences possessing a state-variable representation with a coefficient matrix D in Class P [resp. ULT] is called Class P [resp. ULT].

Proposition 2.5. [8] Classes P and ULT of complementarity correspondences coincide with each other.

The following proposition is important for the existence and uniqueness of solutions to the linear complementarity problem.

Proposition 2.6. [3][8] The linear complementarity problem (2.4) has a unique solution for all $q \in \mathbb{R}^k$ if and only if D belongs to Class P.

Furthermore, the following holds.

Theorem 2.1. [15] The class of all piecewise linear functions coincides with Class ULT, or equivalently Class P, of complementarity correspondence.

2.5 max-min representation

It is has shown that every piecewise linear function is representable as a max-min combination of its linear components [5, 13].

Proposition 2.7. [5, 13] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a piecewise linear function and $\{g_1, g_2, \cdots, g_l\}$ be the set of all its distinct linear components, then there exists an incomparable (with respect to $\subset$) family $\{S_j\}_{j\in J}$ of subsets of $\{1, 2, \ldots, l\}$ such that

$$f(\boldsymbol{x}) = \bigvee_{j\in J} \bigwedge_{i\in S_j} g_i(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{2.5}$$

The right-hand side of (2.5) is called a disjunctive normal form of a max-min polynomial in the variables g_k, or simply called a max-min representation.

The above assertion is valid for vector-valued piecewise linear functions also.

Corollary 2.1. [13] Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a piecewise linear function and $\{g_1, g_2, \cdots, g_l\}$ be the set of all its distinct linear components, then there exists a family $\{S_j^k\}_{j\in J,\ 1\leqq k\leqq m}$ of subsets of $\{1, 2, \ldots, l\}$ such that

$$f_k(\boldsymbol{x}) = \bigvee_{j\in J} \bigwedge_{i\in S_j^k} g_k^{(i)}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^n,\ 1 \leqq k \leqq m,$$

where

$$f = (f_1, f_2, \cdots, f_m), \qquad g_i = (g_1^{(i)}, g_2^{(i)}, \cdots, g_m^{(i)}) \text{ for } 1 \leqq i \leqq l.$$

3 Mutual transformation

This section gives the mutual transformation between Choquet integral and Chua's canonical form. Moreover, the transformation method from Choquet integral to state-variable representation is also given.

3.1 From Choquet integral to Chua's canonical form

In the case of Choquet integral, the consistent variation property is expressed as follows.

Lemma 3.1. *Let μ be a fuzzy measure. Then the following three conditions are equivalent to each other.*

(i) *The Choquet integral $\varphi_\mu(\boldsymbol{x})$ possesses the consistent variation property.*

(ii) *For every pair $i,\ j \in X$ with $i < j$, there exists $c_{ij} \in \mathbb{R}$ such that for all $A \subset X \setminus \{i, j\}$*

$$\mu(\{i,j\} \cup A) - \mu(\{j\} \cup A) - \mu(\{i\} \cup A) + \mu(A) = c_{ij}.$$

(iii) $\mu^{\mathrm{m}}(A) = 0$ *for all $A \subset X$ with $|A| > 2$.*

The following theorem follows from Definition 2.4, Proposition 2.4, and Lemma 3.1.

Theorem 3.1. *Let μ be a fuzzy measure. Then the Choquet integral $\varphi_\mu(\boldsymbol{x})$ possesses Chua's canonical form if and only if μ is at most 2-additive. Moreover, Chua's canonical form of the Choquet integral is given as*

$$\varphi_\mu(\boldsymbol{x}) = \sum_{i \in X} \left(\mu^{\mathrm{m}}(\{i\}) + \frac{1}{2} \sum_{i \neq j} \mu^{\mathrm{m}}(\{i,j\}) \right) x_i - \frac{1}{2} \sum_{i<j} \mu^{\mathrm{m}}(\{i,j\}) \cdot |\langle \boldsymbol{e}_{ij}, \boldsymbol{x} \rangle|.$$

Example 3.1. Let $X = \{1,2,3,4\}$, and consider the following fuzzy measure μ:

$$\begin{aligned}
&\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 1,\ \mu(\{4\}) = 3,\\
&\mu(\{1,2\}) = \mu(\{2,3\}) = 2,\ \mu(\{1,4\}) = 4,\\
&\mu(\{1,3\}) = \mu(\{2,4\}) = \mu(\{3,4\}) = 3,\\
&\mu(\{1,2,3\}) = 4,\ \mu(\{2,3,4\}) = 3,\\
&\mu(\{1,2,4\}) = 4,\ \mu(\{1,3,4\}) = 5,\ \mu(X) = 5.
\end{aligned}$$

Obviously, μ is 2-additive and Chua's canonical form of $\varphi_\mu(\boldsymbol{x})$ is given as

$$\varphi_\mu(\boldsymbol{x}) = 1.5x_1 + 0.5x_2 + x_3 + 2x_4 - 0.5|x_1 - x_3| + 0.5|x_2 - x_4| + 0.5|x_3 - x_4|.$$

3.2 From Chua's canonical form to Choquet integral

The following theorem gives the transformation from Chua's canonical form to the Choquet integral.

Theorem 3.2. *Let a piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}$ possess a Chua's canonical form (2.3). Then f is representable as a Choquet integral φ_μ if and only if the parameters $\boldsymbol{a}$, $\boldsymbol{\alpha}_k$'s, and β_k's of (2.3) fulfill the following three conditions:*

- $a = 0$.
- *there exists an injection* $\rho : \{1, 2, \ldots, l\} \to \binom{X}{2}$ *such that* $\boldsymbol{\alpha}_k = \mathbf{e}_{ij}$ *for* $\rho(k) = \{i, j\}$ *and* $i < j$, *where* $\binom{X}{2}$ *denotes the family of all two-element subsets of* X.
- $\beta_k = 0$ *for all* $k \in \{1, 2, \ldots, l\}$.

In this case, the Möbius inverse of the fuzzy measure μ *is given as follows:*

$$\mu^{\mathrm{m}}(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ b_i + \dfrac{1}{2}\displaystyle\sum_{j \neq i} c(\{i, j\}) & \text{if } A = \{i\}, \\ -c(\{i, j\}) & \text{if } A = \{i, j\} \\ & \quad \text{and } i \neq j, \\ 0 & \text{if } |A| > 2, \end{cases} \tag{3.1}$$

where b_i *is the i-th component of* $B \in \mathbb{R}^{1 \times n}$ *in* (2.3),

$$c(\{i, j\}) = \begin{cases} c_k & \text{if } \rho(k) = \{i, j\}, \\ 0 & \text{otherwise}, \end{cases}$$

and $c_k \in \mathbb{R}^1 \setminus \{0\}$ *is the k-th coefficient in* (2.3). *Moreover,* μ *is monotone if and only if for every* $A \subset X$ *and for every* $i \in A$

$$b_i - \frac{1}{2}\sum_{j \in A \setminus \{i\}} c(\{i, j\}) + \frac{1}{2}\sum_{j \notin A} c(\{i, j\}) \geqq 0. \tag{3.2}$$

proof. The first assertion and Eq. (3.1) follow from Theorem 3.1 and the uniqueness of Chua's canonical form. Hence it is sufficient to show the monotonicity condition (3.2). In order to show this, we use the following equivalence [1]:

$$\mu \text{ is monotone} \iff \forall A \subset X, \forall i \in A; \sum_{B: i \in B \subset A} \mu^{\mathrm{m}}(B) \geqq 0.$$

Then we obtain the monotonicity condition (3.2) by using (3.1) as follows:

$$\begin{aligned} \sum_{B: i \in B \subset A} \mu^{\mathrm{m}}(B) &= \sum_{\substack{B: i \in B \subset A \\ |B| \leqq 2}} \mu^{\mathrm{m}}(B) \\ &= b_i + \frac{1}{2}\sum_{j \neq i} c(\{i, j\}) - \sum_{\substack{j \in A \\ j \neq i}} c(\{i, j\}) \\ &= b_i - \frac{1}{2}\sum_{j \in A \setminus \{i\}} c(\{i, j\}) + \frac{1}{2}\sum_{j \notin A} c(\{i, j\}). \end{aligned}$$

□

Remark 6. The fuzzy measure μ in Theorem 3.2 is obtained by an application of Proposition 2.1 to (3.1) as follows:

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ b_i + \frac{1}{2}\sum_{j \neq i} c(\{i,j\}) & \text{if } A = \{i\}, \\ b_i + b_j + \frac{1}{2}\sum_{k \neq i,j} c(\{i,k\}) + \frac{1}{2}\sum_{k \neq i,j} c(\{j,k\}) & \text{if } A = \{i,j\} \text{ and } i \neq j, \\ \sum_{\substack{B \subset A \\ |B|=2}} \mu(B) - (|A|-2)\sum_{i \in A} \mu(\{i\}) & \text{if } |A| > 2. \end{cases}$$

3.3 State-variable representation of Choquet integral

In this section, state-variable representation of Choquet integral is given. Notice that, since a piecewise linear function has infinitely many state-variable representations as mentioned in Subsection 2.4, the representation given here is one of them.

We use the following binary relation $\preceq$ on 2^X. For $E, F \subset X$,

$$E \preceq F \overset{\text{def}}{\iff} E = \{f \in F \mid f \leqq e\} \text{ for some } e \in F,$$

where $\leqq$ in the right-hand side above is the ordinary order $\leqq$ on $\{1, 2, \ldots, n\} = X$.

Example 3.2. $E \preceq \{2,3,5\}$ iff $E = \{2\}$, $\{2,3\}$, or $\{2,3,5\}$.

Next, one of the state variable representations of Choquet integral is given.

Theorem 3.3. *Every Choquet integral $\varphi_\mu(\boldsymbol{x})$ possesses the following state-variable representation:*

$$\begin{aligned} \varphi_\mu(\boldsymbol{x}) &= A\boldsymbol{x} + B\boldsymbol{u}, \\ \boldsymbol{j} &= C\boldsymbol{x} + D\boldsymbol{u}, \\ \boldsymbol{u}, \boldsymbol{j} &\geqq \boldsymbol{0}, \quad \langle \boldsymbol{u}, \boldsymbol{j} \rangle = 0, \end{aligned} \tag{3.3}$$

where if we write $\mathfrak{X} = \{E \subset X \mid |E| \geqq 2\}$,

$$\begin{aligned} \boldsymbol{j} &= (j_E)_{E \in \mathfrak{X}} \in \mathbb{R}^{|\mathfrak{X}|}, \\ \boldsymbol{u} &= (u_E)_{E \in \mathfrak{X}} \in \mathbb{R}^{|\mathfrak{X}|}, \\ A &= (a_i)_{i \in X} \in \mathbb{R}^{1 \times n} : \quad a_i = \sum_{F \succeq \{i\}} \mu^{\mathrm{m}}(F), \\ B &= (b_E)_{E \in \mathfrak{X}} \in \mathbb{R}^{1 \times |\mathfrak{X}|} : \quad b_E = -\sum_{F \succeq E} \mu^{\mathrm{m}}(F), \\ C &= (c_{E,j})_{E \in \mathfrak{X}, j \in X} \in \mathbb{R}^{|\mathfrak{X}| \times n} : \quad c_{E,j} = \begin{cases} 1 & \textit{if } j = \max E, \\ -1 & \textit{if } j = \min E, \\ 0 & \textit{otherwise}, \end{cases} \\ D &= (d_{E,F})_{E,F \in \mathfrak{X}} \in \mathbb{R}^{|\mathfrak{X}| \times |\mathfrak{X}|} : \quad d_{E,F} = \begin{cases} 1 & \textit{if } F \preceq E, \\ 0 & \textit{otherwise}. \end{cases} \end{aligned}$$

The coefficient matrix D above is a P-matrix. Moreover, if $\boldsymbol{u} = (u_{E_1}, u_{E_2}, \ldots, u_{E_N})^T$, where $N = |\mathfrak{X}|$, and if $i \leqq j$ whenever $E_i \preceq E_j$, then D is a ULT-matrix; for example, an arrangement of the members of $\mathfrak{X}$ in cardinality-ascending order, i.e., $i \leqq j$ whenever $|E_i| \leqq |E_j|$, makes D a ULT-matrix (See the following example).

Example 3.3. Let $X = \{1,2,3\}$, and consider the following fuzzy measure μ:

$$\mu(\{1\}) = \mu(\{2\}) = 1,\ \mu(\{3\}) = 2,\ \mu(\{1,2\}) = \mu(\{2,3\}) = 2,\ \mu(\{1,3\}) = 3,\ \mu(X) = 4.$$

The coefficients in the state-variable representation (3.3) of the Choquet integral with respect to μ are given as follows:

$$A = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} -1 & 0 & 0 & -1 \end{pmatrix},\ C = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

where $\boldsymbol{u} = (u_{\{1,2\}}, u_{\{2,3\}}, u_{\{1,3\}}, u_X)^T$.

4 Generalization of Chua's canonical form

In Section 3, it was shown that at most 2-additivity of fuzzy measure is a necessary and sufficient condition for Choquet integral to possess Chua's canonical form. In this section, we investigate the relationship between k-additivity of fuzzy measure and a high-level canonical form of Choquet integral. The high-level canonical form of piecewise linear functions is a generalization of Chua's canonical form introduced by Lin et al. [9]. Our observation shows that the high-level canonical form is not suitable as a "canonical form."

4.1 High-level canonical form

In this subsection, existing results about the generalized Chua's canonical form are introduced, and several properties are described. Note that the following definition is essentially same as that in [9].

Definition 4.1. An affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called 0th-*level canonical.* For a positive integer K, a piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called Kth-*level canonical* if there exist a nonnegative integer l, a matrix $C \in \mathbb{R}^{m \times l}$, and $(K-1)$th-level canonical piecewise linear functions $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^l$ such that

$$f(\boldsymbol{x}) = g(\boldsymbol{x}) + C|h(\boldsymbol{x})| \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \tag{4.1}$$

where $|h| = (|h_1|, |h_2|, \dots, |h_l|)^T$ for $h = (h_1, h_2, \dots, h_l)^T$.

Remark 7. By the definition above, obviously every Kth-level canonical piecewise linear function is $(K+1)$th-level canonical.

By definition, a piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ possesses a Chua's canonical form if and only if f is first-level canonical.

The following proposition shows that every piecewise linear function can be expressed as (4.1).

Proposition 4.1. [9] *For every piecewise linear function f there exists a nonnegative integer K such that f is Kth-level canonical.*

The following properties (i) – (iii) can be easily seen from the definition of high-level canonical form by induction. Note that the proof of (iii) uses the following well-known formulae:

$$x \wedge y = \frac{1}{2}(x + y - |x - y|), \qquad x \vee y = \frac{1}{2}(x + y + |x - y|).$$

Proposition 4.2. **(i)** *If $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ are K_1th- and K_2th-level canonical, respectively, then the product $f_1 \times f_2 : \mathbb{R}^n \to \mathbb{R}^{m_1+m_2}$ is $\max\{K_1, K_2\}$th-level canonical.*
(ii) *If $f_1 : \mathbb{R}^n \to \mathbb{R}^m$, $f_2 : \mathbb{R}^n \to \mathbb{R}^m$ are K_1th- and K_2th-level canonical, respectively, then their linear combination $f = \lambda f_1 + \nu f_2 : \mathbb{R}^n \to \mathbb{R}^m$, where $\lambda, \nu \in \mathbb{R}$, is $\max\{K_1, K_2\}$th-level canonical.*
(iii) *If $f_1 : \mathbb{R}^n \to \mathbb{R}$, $f_2 : \mathbb{R}^n \to \mathbb{R}$ are K_1th- and K_2th-level canonical, respectively, then $f_1 \wedge f_2 : \mathbb{R}^n \to \mathbb{R}$ and $f_1 \vee f_2 : \mathbb{R}^n \to \mathbb{R}$ are $(\max\{K_1, K_2\} + 1)$th-level canonical.*

4.2 Relation with the Choquet integral

This subsection clarifies the relationship between the level of canonical form of Choquet integral φ_μ and the order of additivity of the fuzzy measure μ.

Theorem 4.1. *The Choquet integral φ_μ with respect to a k-additive fuzzy measure μ is a $\lceil \log_2 k \rceil$th-level canonical piecewise linear function, where $\lceil x \rceil$ expresses the smallest integer greater than or equal to x.*

proof. For each $A (\neq \emptyset) \subset X$, we write $f_A(\boldsymbol{x}) = \bigwedge_{i \in A} p_i(\boldsymbol{x})$, where p_i is the projection onto the i-th coordinate, i.e., $p_i(\boldsymbol{x}) = p_i(x_1, x_2, \ldots, x_n) = x_i$. Obviously, p_i is a 0th-level canonical piecewise linear function. It can be shown from Proposition 4.2 (iii) by induction that f_A is a $\lceil \log_2 |A| \rceil$th-level canonical piecewise linear function. By Proposition 2.2, the Choquet integral $\varphi_\mu(\boldsymbol{x})$ with respect to a k-additive fuzzy measure μ is expressed as

$$\varphi_\mu(\boldsymbol{x}) = \sum_{\substack{A \subset X \\ 0 < |A| \leqq k}} f_A(\boldsymbol{x}) \mu^{\mathrm{m}}(A).$$

Therefore, by Proposition 4.2 (ii), $\varphi_\mu(\boldsymbol{x})$ is a $\lceil \log_2 k \rceil$th-level canonical piecewise linear function. $\square$

By Theorem 4.1 the Choquet integrals with respect to 3- and 4-additive fuzzy measures are both second-level canonical piecewise linear functions, and by Theorem 3.1 neither is first-level canonical. Despite the mathematical difference between 3- and 4-additivities, the canonicity level cannot differentiate them. Generally, for $2^{K-1} < k < k' \leqq 2^K$, the Choquet integrals with respect to

k- and k'-additive fuzzy measures are both Kth-level canonical. The order of additivity of a fuzzy measure cannot be identified from the canonicity level of the Choquet integral.

5 Concluding remarks

This paper has given a necessary and sufficient condition for the Choquet integral to possess Chua's canonical form, a necessary and sufficient condition for Chua's canonical form to be representable as a Choquet integral, and the mutual transformation between them.

Moreover, the relationship between the generalized Chua's canonical form and the Choquet integral was investigated. From the observations in Section 4, we can conclude that the representation using absolute value signs such as (4.1) is not suitable as "canonical form" for the Choquet integral. In addition, the canonicity level is too coarse as a scale of complexity of piecewise linear functions. A further direction of this study will be to find another canonical form of piecewise linear functions which can characterize the order of additivity of fuzzy measures.

References

[1] A. Chateauneuf and J.-Y. Jaffray, Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion, *Mathematical Social Sciences*, **17** (1989) 263–283.

[2] L.O. Chua and A.-C. Deng, Canonical piecewise-linear representation, *IEEE Trans. Circuits and Systems*, **35** (1988) 101–111.

[3] R.W. Cottle, J.-S. Pang, and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, San Diego (1992).

[4] K. Fujimoto and T. Murofushi, Hierarchical decomposition of the Choquet integral, *Fuzzy measures and integrals: Theory and Applications*, Physica-Verlag, Heidelberg (2000) 94–103.

[5] V.V. Gorokhovik and O.I. Zorko, Piecewise affine functions and polyhedral sets, *Optimization*, **31** (1994) 209–221.

[6] M. Grabisch, T. Murofushi, and M. Sugeno (eds.), *Fuzzy Measure and Integrals: Theory and Applications*, Physica-Verlag, Heidelberg (2000).

[7] M. Johansson, *Piecewise Linear Control Systems*, Springer (2003).

[8] D.M.W. Leenaerts and W.M.G. van Bokhoven, *Piecewise Linear Modeling and Analysis*, Kluwer, Boston (1998).

[9] J.-N. Lin, H.-Q. Xu, and R. Unbehauen, A generalization of canonical piecewise-linear functions, *IEEE Trans. Circuits and Systems*, **41** (1994) 345–347.

[10] T. Murofushi, M. Sugeno, Separated hierarchical decomposition of the Choquet integral, *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, **5** (1997) 563–583.

[11] T. Murofushi, M. Sugeno, and M. Machida, Non-monotonic fuzzy measures and the Choquet integral, *Fuzzy Sets and Systems*, **64** (1994) 73–86.

[12] T. Murofushi and Y. Narukawa, A characterization of multi-step discrete Choquet integral, *Sixth International Conference on Fuzzy Sets Theory and Its Applications, Abstracts*, (2002) p. 94.

[13] S. Ovchinnikov, Max-min representation of piecewise linear functions, *Contributions to Algebra and Geometry*, **43** (2002) 297–302.

[14] S. Ovchinnikov, Piecewise linear aggregation functions, *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems*, **10** (2002) 17–24.

[15] T. Sakurai, *Representations of piecewise linear functions and Choquet integral over a finite set*, Master's Thesis, Tokyo Institute of Technology, 2004 (in Japanese).

[16] K. Tanaka and M. Sugeno, A study on subjective evaluations of printed color images, *Int. J. of Approximate Reasoning*, **5** (1991) 213–222.

[17] P.B. Zwart, Multivariate splines with nondegenerate partitions, *SIAM J. Numer. Anal.*, **10** (1973) 665–673.