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CP-CONVEXITY AND ENTROPY OF CP-MAPS

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ABSTRACT. The purpose of this note is to review some basic results on CP-convexity and introduce new entropy of CP-maps in the context of CP-convexity theory, and discuss the relations with other entropies of CP-maps defined in quantum physics and entanglements in quantum information theory.

1. Introduction.

The notion of *completely positive map* between C*-algebras, abbreviated by *CP-map* in this note, was mathematically initiated by W.F. Stinespring [17], and was first introduced into physics by K. Kraus [14]. He showed that, assuming that observables of a physical system are described by bounded operators $a \in B(H)$ on a Hilbert space H, after an interaction with an exterior, they are changed by a normal contractive completely positive map (so called *operation*) ψ such that

$$\psi(a) = \sum_i S_i^* a S_i \quad \text{with} \quad S_i \in B(H) \quad \text{such that} \quad \sum_i S_i^* S_i \leq I_H.$$

Note that, using the polar decomposition $S_i = u_i|S_i|$, this can be rewritten as

$$\psi(a) = \sum_i |S_i^*|\varphi_i(a)|S_i| \quad \text{with } |S_i| \in B(H)^+ \quad \text{such that } \sum_i |S_i|^2 \leq I_H,$$

where $\varphi_i(a) = u_i^* a u_i$ is a *conditional transform* with a partial isometry u_i on H. We can also show that it can be decomposed as

$$\psi(a) = \sum_i V_i^* \phi_i(a) V_i \quad \text{with } V_i \in B(H) \quad \text{such that } \sum_i V_i^* V_i \leq I_H,$$

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where $\phi_i(a) = U_i^* a U_i$ is a *unitary transform* with a unitary U_i on H, but this time the operator coefficients V_i cannot be positive in general (cf. [11]). Our motivation was to define an operator convexity for operations where unitary transforms and conditional transforms are characterized as extreme elements which would represent minimal interactions in physics.

For more general setting, we shall consider the set of all CP-maps from a C*-algebra A to $B(H)$, and denote it by $CP(A, B(H))$. Then $\psi \in CP(A, B(H))$ is said to be a *CP-convex combination* of $\psi_i \in CP(A, B(H))$ if it can be decomposed as

$$\psi = \sum_i S_i^* \psi_i S_i \quad \text{with} \quad S_i \in B(H) \quad \text{such that} \quad \sum_i S_i^* S_i \leq I_H,$$

which will be abbreviated as

$$\psi = CP\text{-}\sum_i S_i^* \psi_i S_i.$$

For example, we have seen above that every operation can be decomposed into a CP-convex combination of unitary transforms, and it is also a CP-convex combination of conditional transforms with positive operator coefficients. In a series of works [5-11], we developed *CP-convexity theory* which "quantizes" scalar convexity theory for state spaces for C*-algebras. In this note, we shall define new entropy of CP-maps using the CP-coefficients, which vanishes at the extreme elements in the sense of the above operator convexity.

2. Notations.

We prepare some basic results and notations for CP-maps. Recall that, by the Stinespring representation theorem [17], every CP-map $\psi \in CP(A, B(H))$ can be represented as $\psi = V^* \pi V$ where π is a representation of A, and V is a bounded linear operator from H to H_π. We denote by p_ψ the support projection of ψ (i.e., the support projection $s(V^*V)$ of V^*V), and then $H_\psi := p_\psi H$ is the support of ψ.

A CP-map $\psi \in CP(A, B(H))$ is called a *CP-state* if it is contractive, and we denote by $Q_H(A)$ the set of all CP-states, i.e.,

$$Q_H(A) = \{\psi = V^* \pi V \in CP(A; B(H)); \|V\| \leq 1\}.$$

In particular, the set of all *unital* CP-states will be denoted by $S_H(A)$, i.e.,

$$S_H(A) = \{\psi = V^*\pi V; V^*V = I_H\}.$$

$\psi \in CP(A : B(H))$ is *pure* in the cone $CP(A, B(H))$ iff π is irreducible, and we denote by $P_H(A)$ the set of all pure elements in $CP(A, B(H))$, and by $PS_H(A) = P_H(A) \cap S_H(A)$ the set of all unital pure CP-states.

Recall also that $Rep(A : H)$ [resp. $Rep_c(A : H)$, $Irr(A : H)$] represents the set of all [resp. cyclic, irreducible] representations of A on H (i.e., whose representation spaces are subspaces of H). Since every representation can be decomposed into a direct sum of cyclic representations, we can show that

$$Q_H(A) = CP\text{-conv}\, Rep_c(A : H).$$

3. CP-extreme states.

The natural questions would be as follows. "What are the extreme CP-maps in CP-convexity?" "How can a CP-map be decomposed into the extreme elements?" The answer should generalizes the pure states $P(A)$ and Choquet theory on the state space $S(A)$.

Recall that, in the example of the decomposition of operation, we saw two types of CP-decompositions. Naturally, the definition of CP-extreme elements is not unique, and this would be a difficult part of operator convexity, but it will turn out to be an advantage of CP-convexity which allows us to deal with both algebraic decomposition and statistical decomposition. (cf. [11])

Definition 1. A CP-state is defined to be *CP-extreme* if $\psi = CP\text{-}\sum_i v_i^* \psi_i v_i$, then ψ_i is unitarily equivalent to ψ, i.e., there exists a partial isometry u_i with $u_i^* u_i = p_\psi$ and $u_i u_i^* = p_{\psi_i}$ such that $\psi_i = u_i \psi u_i^*$ and $v_i = c_i u_i$ with $c_i \in \mathbb{C}$ for all i. We denote by $D_H(A)$ the set of all CP-extreme states.

Theorem 1.
(i) If $\dim H = \infty$, *then* $D_H(A) = Irr(A : H)$.
(ii) If $1 < \dim H < \infty$, *the* $D_H(A) = Irr(A : H) \cup PS_H(A)$.
(iii) If $\dim H = 1$, *then* $D_H(A) = P(A)$.

In the case of operations, the extreme elements in the above sense are irreducible representations of $B(H)$ on H, which are exactly unitary transforms.

This definition of CP-extreme elements are useful for algebraic arguments (cf. [10]). On the other hand, we have another definition of CP-extreme states, restricting the CP-coefficients to positive operators, which we shall call *positive* CP-convex combination.

Definition 2. A CP-state is defined to be *positively CP-extreme* if $\psi = CP\text{-}\sum_i v_i \psi_i v_i$ with $v_i \geq 0$, then $\psi_i = \psi$. We denote by $E_H(A)$ the set of all positively CP-extreme states.

Theorem 2.

$$\begin{aligned} E_H(A) &= Irr(A:H) \cup PS_H(A) \\ &= \{\psi = u^* \pi u \in P_H(A);\ u^*u = I_H \text{ or } uu^* = p_\pi\}. \end{aligned}$$

Though the above definition and result are simple, in the case of operation it cannot single out the conditional transforms as extreme elements. For this, we shall give a weaker version of Definition 2.

Definition 2'. A CP-state is defined to be *conditionally CP-extreme* if $\psi = CP - \sum_i v_i \psi_i v_i$ with $v_i \geq 0$, then $s(v_i)\psi_i s(v_i) = \psi$. We denote by $E_H^c(A)$ the set of all conditionally CP-extreme states.

Theorem 2'. $E_H^c(A) = \{\psi = u^* \pi u \in P_H(A); u^*u = p_\psi\}$.

Thus conditional transforms are conditionally CP-extreme states, and they are important in applications. For example, annihilation and creation in Fock Hilbert space are conditionally CP-extreme, so minimal interactions as expected, where CP-coefficients include the information of correlation in quantum information theory.

We note that the non-commutative version of Gelfand-Naimark theorem was realized on the CP-extreme elements $Irr(A:H)$ for a sufficient large H [10]. Also, Choquet's representation theorem was generalized for CP-convexity by introducing *CP-measure* (operation valued measure) and its integration theory (see [9] for details).

CP-Choquet theorem. *Let A and H be separable. Then, for any CP-state $\psi \in Q_H(A)$, there exists a CP-measure λ_ψ supported by $D_H(A)$ such that*

$$\psi(a) = \int_{Q_H(A)} \hat{a} d\lambda_\psi \quad \text{for all } a \in A.$$

4. Entropy of CP-states.

Assume that a CP-state $\psi \in Q_H(A)$ is atomic (i.e., it has an atomic representing CP-measure λ_ψ) and it is decomposed as

$$\psi = \sum_{i=1}^{\infty} V_i^* \pi_i V_i \quad \text{where } \pi_i \in Irr(A:H) \text{ and } V_i \in B(H),\ \sum_{i=1}^{\infty} V_i^* V_i \le I_H.$$

If we set $V_0 = (I_H - \sum_{i=1}^{\infty} V_i^* V_i)^{\frac{1}{2}}$, then

$$\psi = \sum_{i=0}^{\infty} V_i^* \pi_i V_i \quad \text{where } \pi_i \in Irr(A:H) \text{ and } V_i \in B(H),\ \sum_{i=0}^{\infty} V_i^* V_i = I_H.$$

Now, how can we define an "entropy" of ψ with the CP-coefficients V_i which vanishes at the extreme CP-maps?

As a first attempt, we can define an operator entropy $S(\psi)$ by

$$S(\psi) = -\sum_i V_i^* V_i \ln V_i^* V_i,$$

however, this vanishes at any atomic representation $\pi = \oplus_i \pi_i = \sum_i p_{\pi_i} \pi_i p_{\pi_i}$ ($\pi_i \in Irr(A:H)$) which is not CP-extreme, and also the above definition depends on the way of the decomposition. We also note that the scalar convexity does not work here for the CP-state space $Q_H(A)$, since any representation $\pi \in Rep(A, H)$ of A on H is an extreme point of $Q_H(A)$ (cf. [9; Appendix]), which may not be CP-extreme if it is not irreducible.

Let $\rho \in T(H_\psi)_1^+$ be a normal state of $B(H_\psi)$, and consider an affine set $S_H^\psi(A)_\rho$ in the cone $CP(A : B(H_\psi))$ (which may not be included in the CP-state space $Q_{H_\psi}(A)$ defined by

$$S_{H_\psi}^\psi(A)_\rho = \{\varphi \in CP(A : B(H_\psi))\,;\, \|\varphi\|_\rho = \|\psi\|_\rho\}.$$

where $\|\varphi\|_\rho := \rho(\varphi(1))$. Then, $S_{H_\psi}^\psi(A)_\rho$ is a BW-compact convex set which includes ψ, so that ψ can be decomposed into a scalar convex combination of pure CP-maps

$$\psi = \sum_{i=0}^{\infty} \lambda_i \psi_i \quad \text{with } \psi_i \in P_{H_\psi}(A) \cap S_{H_\psi}^\psi(A)_\rho \ (i \ge 1) \text{ and } \lambda_i > 0,\ \sum_{i=0}^{\infty} \lambda_i = 1,$$

where $\lambda_i = \omega(V_i^* V_i)$ for $i \geq 1$ and $\lambda_0 = 1 - \sum_{i=1}^{\infty} \lambda_i$, and $\psi_0 = 0$, $\psi_i = \lambda_i^{-1} V_i^* \pi_i V_i$ for $i \geq 1$. We can then define

$$S_\rho^1(\psi) := \inf\{-\sum_{i=0}^{\infty} \lambda_i \ln \lambda_i\},$$

where inf is taken over all possible atomic CP-extreme decomposition of ψ.

In particular, let ψ be a unital operation, and suppose

$$\psi = \sum_i V_i^* \cdot V_i \quad \text{with } V_i \in B(H),\ \sum_i V_i^* V_i = I_H,$$

and let $\rho \in T(H)_1^+$ be a faithful normal state. Then, Lindblad [15] defined an entropy of ψ with respect to ρ by

$$S_\rho^2(\psi) := S(M_\rho) \quad \text{where } [M_\rho]_{i,j} = \mathrm{Tr}\, V_i \rho V_j^*.$$

On the other hand, Alicki [1] showed that this is equivalent to

$$S_\rho^3(\psi) := S(\rho_\psi) \quad \text{where } \rho_\psi = \sum_i (\cdot, V_i)_\rho V_i \in T(H_\rho)_1^+$$

where H_ρ is the GNS-representation space of $B(H)$ with respect to $\mathrm{Tr}\rho(\cdot)$.

Lemma. *If ψ is a unital operation and $\rho \in T(H)_1^+$ is faithful, then*

$$S_\rho^1(\psi) = S_\rho^2(\psi) = S_\rho^3(\psi).$$

Thus our entropy $S_\rho^1(\psi)$ is a generalization of the Lindblad entropy to non-unital CP-states, and non-faithful $\rho \in T(H_\psi)_1^+$, so that we shall call it the *Lindblad entropy* of ψ with respect to $\rho \in T(H_\psi)_1^+$, and denote it by $S_\rho^L(\psi)$.

Now, one of our main theorem in [12] is stated as follows.

Theorem 3. *Let $\psi \in Q_H(A)$ be an atomic CP-state, then $S_\rho^L(\psi) = 0$ for all $\rho \in T(H_\psi)_1^+$ iff ψ is a conditionally CP-extreme state.*

5. Entropy of operations.

Let $\omega \in (B(H) \otimes B(H))_*$ be a normal atomic composite state of the tensor product $B(H) \otimes B(H)$, and so it is decomposed as

$$\omega = \sum_i \lambda_i \omega_i \quad \text{where } \omega_i \text{ is pure},\ \lambda_i > 0,\ \sum_i \lambda_i = 1.$$

Then the *entanglement of formation* $S^E(\omega)$ of ω is defined by

$$S^E(\omega) := \inf\{\sum_i \lambda_i \hat{S}(\omega_i)\} \quad \text{with } \hat{S}(\omega_i) = S(\omega_i(1 \otimes \cdot)),$$

where inf is taken over all possible pure decompositions (cf. [16], [12]). We note that $S^E(\omega) = 0$ if and only if ω is a separable state, i.e., ω is of the form $\omega = \sum_i \mu_i p_i \otimes q_i$ where p_i and q_i are pure states of $B(H)$, and $\mu_i > 0, \sum_i \mu_i = 1$.

On the other hand, we have one to one correspondence between $\omega \in (B(H) \otimes B(H))_*$ and $\varphi_\omega \in CP(B(H), T(H))_n$ (where n represents the normal part) such that

$$\varphi_\omega = \sum_i \lambda_i \varphi_{\omega_i} \quad \text{with } \varphi_{\omega_i} = V_i^*(\cdot)V_i \in P_H(B(H)) \cap CP(B(H), T(H))_n,$$

where the correspondence is given by $\omega(a \otimes b) = \operatorname{Tr}(\varphi_\omega(a)\,{}^t b)$ for $a, b \in B(H)$.

Definition. $\varphi \in CP(B(H), T(H))_n$ is called a *tracial operation,* and suppose that

$$\varphi = \sum_i V_i^* \cdot V_i \quad \text{with } \varphi(1) = \sum_i V_i^* V_i \in T(H)_1^+.$$

We then define the *entanglement of formation* $S^E(\varphi)$ of φ by

$$S^E(\varphi) := \inf \sum_i \lambda_i S(\lambda_i^{-1} V_i^* V_i) \quad \text{with } \lambda_i = \operatorname{Tr} V_i^* V_i$$

and the *entropy* $S(\varphi)$ of φ by

$$S(\varphi) := \inf\{-\sum_i \operatorname{Tr}(V_i^* V_i \ln V_i^* V_i)\}$$

where inf is taken over all possible decompositions above.

We note here that

$$-\sum_i \operatorname{Tr}(V_i^* V_i \ln V_i^* V_i) = -\sum_i \lambda_i \ln \lambda_i + \sum_i \lambda_i S(\lambda_i^{-1} V_i^* V_i),$$

so if we define $\omega_\varphi(a \otimes b) = \operatorname{Tr}(\varphi(a)\,{}^t b)$ for $a, b \in B(H)$, then the above defined entropy of a tracial operation φ measures how far the corresponding composite state ω_φ is from pure and separable states. We can then show the following results.

Theorem 4. *Let $\psi \in CP(B(H), T(H))_n$ be a tracial operation. Then we have*

(i) $S(\varphi) \geq S(\omega_\varphi) + S^E(\varphi) \geq 0$.

(ii) $S(\varphi) = 0$ *iff* ω_φ *is a separable pure state.*

Let ψ be an operation and let $\rho \in T(H)_1^+$, then $\rho^{\frac{1}{2}}\psi\rho^{\frac{1}{2}} \in CP(B(H), T(H))_n$ defines a tracial operation, so the above defined entropy of tracial operations can be applied to define an entropy of operations.

Definition. We define the entropy $S_\rho(\psi)$ of an operation ψ with respect to $\rho \in T(H_\psi)_1^+$ by

$$S_\rho(\psi) := S(\rho^{\frac{1}{2}}\psi\rho^{\frac{1}{2}}) - S(\rho)$$

Our another main result can be stated as follows.

Theorem 5. *Let ψ be an operation. Then we have*

(i) $S_\rho(\psi) \geq S_\rho^L(\psi) + S_\rho^E(\psi) - S(\rho) \geq 0$ *for all* $\rho \in T(H_\psi)_1^+$.

(ii) $S_\rho(\psi) = 0$ *for all* $\rho \in T(H_\psi)_1^+$ *iff* $\psi = \omega I_\psi$ *with* $\omega \in P(B(H))$ *or* ψ *is a conditional transform.*

In [12], more detailed properties of the above entropy will be discussed. We hope that the entropies of CP-maps will be useful for quantum dynamical entropy, which was initiated by G.G. Emch [3, 4] and A. Connes and E. Stϕrmer [2], and the Lindblad entropy was defined for this purpose ([15], see also [1]). There are many references on this subject, so we just mention the most recent work by A. Kossakowski, M Ohya and N. Watanabe [13]

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