数理解析研究所講究録 1396 巻 2004 年 42-48

Strong Convergence Theorem by the Hybrid and Extragradient Method for Nonexpansive Mappings and Monotone Mappings

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Abstract

In this paper we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping. The iterative process is based on two well known methods - hybrid and extragradient. We obtain a strong convergence theorem for three sequences generated by this process.

1 Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geqslant 0$$

for all $u, v \in C$. The *variational inequality problem* is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geqslant 0$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. A mapping A of C into H is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geqslant \alpha \|Au - Av\|^2$$

for all $u, v \in C$; see [1], [4]. It is obvious that an α-inverse-strongly-monotone mapping A is monotone and Lipschitz-continuous. A mapping S of C into itself is called *nonexpansive* if

$$\|Su - Sv\| \leqslant \|u - v\|$$

for all $u, v \in C$; see [8]. We denote by $F(S)$ the set of fixed points of S. For finding an element of $VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex and a mapping A of C into H is α-inverse-strongly-monotone, Iiduka, Takahashi and Toyoda [2] introduced the following iterative scheme by a hybrid method:

$$\begin{cases} x_0 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n) \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda_n \subset [a, b]$ for some $a, b \in (0, 2\alpha)$. They showed that if $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$, generated by this iterative process, converges strongly to $P_{VI(C,A)}x$. On the other hand, for solving the variational inequality problem in a finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex and a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz-continuous, Korpelevich [3] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in R^n \\ \bar{x}_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n) \end{cases} \tag{1}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, 1/k)$. He showed that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1), converge to the same point $z \in VI(C, A)$.

In this paper, by an idea of combining hybrid and extragradient methods, we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. Then we obtain a strong convergence theorem for three sequences generated by this process.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leqslant \|x - y\|$ for all $y \in C$. P_C is called *the metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geqslant 0; \tag{2}$$

$$\|x - y\|^2 \geqslant \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{3}$$

for all $x \in H$, $y \in C$; see [8] for more details. Let A be a monotone mapping of C into H. In the context of variational inequality problem this implies

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u,) \quad \forall \lambda > 0.$$

It is also known that H satisfies Opial's condition [6], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geqslant 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geqslant 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k–Lipschitz-continuous mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e. $N_C v = \{w \in H : \langle v - u, w \rangle \geqslant 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [7].

3 Strong Convergence Theorem

In this section we prove a strong convergence theorem by a combined hybrid-extragradient method for nonexpansive mappings and monotone, k-Lipshitz-continuous mappings.

Theorem 3.1 *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n) \\ z_n = SP_C(x_n - \lambda_n A y_n) \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0,1,2,\dots$, where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(S)\cap VI(C,A)}x$.

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n = 0,1,2,\dots$. As $C_n = \left\{z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0\right\}$, we also have C_n is convex for every $n = 0,1,2,\dots$. Put $t_n = P_C(x_n - \lambda_n A y_n)$ for every $n = 0,1,2,\dots$. Let $u \in F(S) \cap VI(C,A)$. From (3), monotonicity of A and $u \in VI(C,A)$, we have

$$\begin{aligned} \|t_n - u\|^2 &\leqslant \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n (\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle) \\ &\leqslant \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle. \end{aligned}$$

Further, since $y_n = P_C(x_n - \lambda_n A x_n)$ and A is k-Lipschitz-continuous, we have

$$\begin{aligned} &\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\ &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\ &\leqslant \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\ &\leqslant \lambda_n k \|x_n - y_n\| \|t_n - y_n\|. \end{aligned}$$

So, we have

$$\begin{aligned} \|t_n - u\|^2 &\leqslant \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\ &\leqslant \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\ &\leqslant \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\ &\leqslant \|x_n - u\|^2. \end{aligned} \tag{4}$$

Therefore from $z_n = St_n$ and $u = Su$, we have

$$\|z_n - u\| = \|St_n - Su\| \leq \|t_n - u\| \leq \|x_n - u\| \tag{5}$$

for every $n = 0,1,2,\dots$ and hence $u \in C_n$. So, $F(S) \cap VI(C,A) \subset C_n$ for every $n = 0,1,2,\dots$. Next, let us show by mathematical induction that $\{x_n\}$ is well-defined and $F(S) \cap VI(C,A) \subset C_n \cap Q_n$ for every $n = 0,1,2,\dots$. For $n = 0$ we have $Q_0 = C$. Hence we obtain $F(S) \cap VI(C,A) \subset C_0 \cap Q_0$. Suppose that x_k is given and $F(S) \cap VI(C,A) \subset C_k \cap Q_k$ for some $k \in N$. Since $F(S) \cap VI(C,A)$ is nonempty,

$C_k \cap Q_k$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for every $z \in C_k \cap Q_k$. Since $F(S) \cap VI(C, A) \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0$ for $z \in F(S) \cap VI(C, A)$ and hence $F(S) \cap VI(C, A) \subset Q_{k+1}$. Therefore, we obtain $F(S) \cap VI(C, A) \subset C_{k+1} \cap Q_{k+1}$.

Let $t_0 = P_{F(S) \cap VI(C,A)} x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and $t_0 \in F(S) \cap VI(C, A) \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|t_0 - x\| \tag{6}$$

for every $n = 0, 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. We also have

$$\|z_n - u\| = \|St_n - Su\| \leq \|t_n - u\| \leq \|x_n - u\|$$

for some $u \in F(S) \cap VI(C, A)$. So, $\{z_n\}$ and $\{t_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 0, 1, 2, \ldots$. Therefore, there exists $c = \lim_{n \to \infty} \|x_n - x\|$. Since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x\|^2 + \|x_n - x\|^2 + 2 \langle x_{n+1} - x, x - x_n \rangle \\ &= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2 \langle x_n - x_{n+1}, x - x_n \rangle \\ &\leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2 \end{aligned}$$

for every $n = 0, 1, 2, \ldots$. This implies that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \leq 2 \|x_{n+1} - x_n\|$$

for every $n = 0, 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in F(S) \cap VI(C, A)$, from (4) and (5) we obtain

$$\|z_n - u\|^2 \leq \|t_n - u\|^2 \leq \|x_n - u\|^2 + \left(\lambda_n^2 k^2 - 1\right) \|x_n - y_n\|^2.$$

Therefore, we have

$$\begin{aligned} \|x_n - y_n\|^2 &\leq \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\|^2 - \|z_n - u\|^2 \right) \\ &= \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| - \|z_n - u\| \right) \left(\|x_n - u\| + \|z_n - u\| \right) \\ &\leq \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| + \|z_n - u\| \right) \|x_n - z_n\|. \end{aligned}$$

Since $\|x_n - z_n\| \to 0$, we obtain $x_n - y_n \to 0$. From (4) and (5) we also have

$$\begin{aligned} \|z_n - u\|^2 &\leq \|t_n - u\|^2 \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\ &\leq \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \|x_n - y_n\|^2 + \lambda_n^2 k^2 \|y_n - t_n\|^2 \\ &\leq \|x_n - u\|^2 + \left(\lambda_n^2 k^2 - 1\right) \|y_n - t_n\|^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} \|t_n - y_n\|^2 &\leq \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\|^2 - \|z_n - u\|^2 \right) \\ &= \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| - \|z_n - u\| \right) \left(\|x_n - u\| + \|z_n - u\| \right) \\ &\leq \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| + \|z_n - u\| \right) \|x_n - z_n\|. \end{aligned}$$

Since $\|x_n - z_n\| \to 0$, we obtain $t_n - y_n \to 0$. Since A is k-Lipschitz-continuous, we have $Ay_n - At_n \to 0$. From $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$ we also have $x_n - t_n \to 0$. Since

$$\|t_n - St_n\| = \|t_n - z_n\| \le \|t_n - x_n\| + \|x_n - z_n\|,$$

we have $\|t_n - St_n\| \to 0$.

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some u. We can obtain that $u \in F(S) \cap VI(C, A)$. First, we show $u \in VI(C, A)$. Since $x_n - t_n \to 0$ and $x_n - y_n \to 0$, we have $\{t_{n_i}\} \rightharpoonup u$ and $\{y_{n_i}\} \rightharpoonup u$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [7]. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av \rangle \ge 0$ for all $t \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n Ay_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - v \rangle \ge 0$$

and hence

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n \right\rangle \ge 0.$$

Therefore from $w - Av \in N_C v$ and $t_{n_i} \in C$, we have

$$\begin{aligned} \langle v - t_{n_i}, w \rangle &\ge \langle v - t_{n_i}, Av \rangle \\ &\ge \langle v - t_{n_i}, Av \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \right\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\ge \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned}$$

Hence, we obtain $\langle v - u, w \rangle \ge 0$ as $i \to \infty$. Since T is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in VI(C, A)$.

Let us show $u \in F(S)$. Assume $u \notin F(S)$. From Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty} \|t_{n_i} - u\| &< \liminf_{i\to\infty} \|t_{n_i} - Su\| \\ &= \liminf_{i\to\infty} \|t_{n_i} - St_{n_i} + St_{n_i} - Su\| \\ &\le \liminf_{i\to\infty} \|St_{n_i} - Su\| \\ &\le \liminf_{i\to\infty} \|t_{n_i} - u\|. \end{aligned}$$

This is a contradiction. So, we obtain $u \in F(S)$. This implies $u \in F(S) \cap VI(C, A)$.

From $t_0 = P_{F(S)\cap VI(C,A)}x$, $u \in F(S) \cap VI(C, A)$ and (6), we have

$$\|t_0 - x\| \le \|u - x\| \le \liminf_{i\to\infty} \|x_{n_i} - x\| \le \limsup_{i\to\infty} \|x_{n_i} - x\| \le \|t_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|u - x\|.$$

From $x_{n_i} - x \rightharpoonup u - x$ we have $x_{n_i} - x \to u - x$ and hence $x_{n_i} \to u$. Since $x_n \in P_{Q_n}x$ and $t_0 \in F(S) \cap VI(C, A) \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|t_0 - x_{n_i}\|^2 = \langle t_0 - x_{n_i}, x_{n_i} - x \rangle + \langle t_0 - x_{n_i}, x - t_0 \rangle \ge \langle t_0 - x_{n_i}, x - t_0 \rangle.$$

As $i \to \infty$, we obtain $-\|t_0 - u\|^2 \ge \langle t_0 - u, x - t_0 \rangle \ge 0$ by $t_0 = P_{F(S)\cap VI(C,A)}x$ and $u \in F(S) \cap VI(C, A)$. Hence we have $u = t_0$. This implies that $x_n \to t_0$. It is easy to see $y_n \to t_0$, $z_n \to t_0$.

4 Applications.

Using Theorem 3.1, we prove some theorems in a real Hilbert space.

Theorem 4.1 *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H such that $VI(C,A)$ is nonempty. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n) \\ z_n = P_C(x_n - \lambda_n A y_n) \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0,1,2,...$, where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{VI(C,A)}x$.

Proof. Putting $S = I$, by Theorem 3.1, we obtain the desired result.
Remark. See Iiduka, Takahashi and Toyoda [2] for the case when A is α-inverse-strongly-monotone.

Theorem 4.2 *Let C be a closed convex subset of a real Hilbert space H and S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = S x_n \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0,1,2,....$ Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(S)}x$.

Proof. Putting $A = 0$, by Theorem 3.1, we obtain the desired result.
Remark. See also Nakajo and Takahashi [5] for more general result.

Theorem 4.3 *Let H be a real Hilbert space. Let A be a monotone, k-Lipschitz-continuous mapping of H into itself and S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = S(x_n - \lambda_n A(x_n - \lambda_n A x_n)) \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0,1,2,...$, where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,1/k)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(S)\cap A^{-1}0}x$.

Proof. We have $A^{-1}0 = VI(H,A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result.
Remark. Notice that $F(S) \cap A^{-1}0 \subset VI(F(S),A)$. See also Yamada [9] for the case when A is a strongly monotone and Lipschitz continuous mapping of a real Hilbert space H into itself and S is a nonexpansive mapping of H into itself.

Theorem 4.4 *Let H be a real Hilbert space. Let A be a monotone, k-Lipschitz-continuous mapping of H into itself and $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = J_r^B(x_n - \lambda_n A(x_n - \lambda_n A x_n)) \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{A^{-1}0 \cap B^{-1}0} x$.

Proof. We have $A^{-1}0 = VI(H, A)$ and $F\left(J_r^B\right) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result.

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