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Asymptotic Analysis of Confluent Hypergeometric Partial Differential Equations in Many Variables

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1 Introduction

The confluent differential equation in one variable, known as the Kummer differential equation

$$x\frac{d^2}{dx^2}w+(\gamma-x)\frac{d}{dx}w-\beta w=0,$$

is studied by several authors in various ways. Among those, the so-called Borel-Laplace-Ecalle method is a powerful one, which is explained for example in [1]. This method is applicable to an analysis of the Humbert confluent hypergeometric differential equations Φ_2 in 2 variables

$$\begin{aligned} x\frac{\partial^2}{\partial x^2}w+y\frac{\partial}{\partial y}\frac{\partial}{\partial x}w+(\gamma-x)\frac{\partial}{\partial x}w-\beta w &= 0,\\ y\frac{\partial^2}{\partial y^2}w+x\frac{\partial}{\partial x}\frac{\partial}{\partial y}w+(\gamma-y)\frac{\partial}{\partial y}w-\beta' w &= 0, \end{aligned}$$

and we can obtain formal solutions, asymptotic solutions and so-called Stokes multipliers (see [2], [3]).

It is also applicable to an asymptotic analysis of the Humbert confluent hypergeometric partial differential equations in $m(>2)$ variables. Here, the author gives an overview of it.

2 Humbert confluent hypergeometric partial differential equations Φ_D

The system of Humbert confluent hypergeometric partial differential equations Φ_D is as follows:

$$x_k\frac{\partial^2 u}{\partial x_k^2}+\sum_{l\neq k}x_l\frac{\partial^2 u}{\partial x_k\partial x_l}+(\gamma-x_k)\frac{\partial u}{\partial x_k}-\beta_k u=0,$$

where $\beta_k(k=1,\cdots,m)$ and γ are not non-negative integers.

We consider this system in $M=(P^1(\mathbf{C}))^m$. The system has irregular singularities on $H=\bigcup_{k=1}^m H_k$, where $H_k=P^1(\mathbf{C})\times\cdots\times\{\infty\}\times\cdots\times P^1(\mathbf{C})$.

For simplicity, let p be a point in $H\setminus\bigcup_{k\neq l}(H_k\cap H_l)$, we consider the formal solutions and asymptotic solutions to Φ_D near the point.

Proposition 1. We have $(m+1)$ linearly independent formal solutions. Among them, $(m-1)$ formal solutions are convergent and 2 formal solutions are divergent.

Near a point $(\infty,x_2,\ldots,x_m)$ with bounded $x_2,\ldots,x_m$, we have divergent solutions of the following forms

$$e^{x_1}x_1^{\beta_1-\gamma}\hat{V}(\beta_1,\beta_2,\ldots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1}),$$

and

$$x_1^{-\beta_1}\hat{U}(\beta_1,\beta_2,\ldots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1}).$$

Here, we put

$$\begin{aligned} &\hat{V}(\beta_1,\beta_2,\ldots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1})\\ =&\sum_{n=0}^{\infty}P_n(\beta_1,\beta_2,\ldots,\beta_m,\gamma,x_2,\cdots,x_m)x_1^{-n}, \end{aligned}$$

with the polynomials

$$\begin{aligned} &P_n(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m)\\ = &\sum_{\ell=0}^{n}\frac{(\gamma-\beta_1+\ell)_{n-\ell}(1-\beta_1)_{n-\ell}}{(n-\ell)!\ell!}\sum_{j_2+\dots+j_m=\ell}\frac{(\beta_2)_{j_2}\dots(\beta_m)_{j_m}\ell!}{j_2!\dots j_m!}x_2^{j_2}\dots x_m^{j_m}\end{aligned}$$

and

$$\begin{aligned} &\hat{U}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1})\\ = &\sum_{n=0}^{\infty}\frac{(\beta_1)_n(\beta_1-\gamma+1)_n}{n!}\Phi_D^{m-1}(\beta_2,\dots,\beta_m;\gamma-\beta_1-n;x_2,\cdots,x_m)(-x_1)^{-n},\end{aligned}$$

where $\Phi_D^{m-1}(\beta_2,\dots,\beta_m;\gamma-\beta_1-n;x_2,\cdots,x_m)$ is the Humbert confluent hypergeoetric function in $(m-1)$ variables with the parameter $(\beta_2,\dots,\beta_m;\gamma-\beta_1-n)$,

$$\begin{aligned} &\Phi_D^{m-1}(\beta_2,\dots,\beta_m;\gamma-\beta_1-n;x_2,\cdots,x_m)\\ = &\sum_{j_2=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\frac{(b_2)_{j_2}\cdots(b_m)_{j_m}x_2^{j_2}\cdots x_m^{j_m}}{(\gamma-\beta_1-n)_{j_2+\dots+j_m}j_2!\cdots j_m!}.\end{aligned}$$

In the above, we use the Pochhammer symbol $(b)_s=(b+1)\cdots(b+s-1)$.

Proposition 2. The divergent formal series

$$\hat{V}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1})$$

and

$$\hat{U}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1})$$

are of Gevrey order 1 as $x_1\to\infty$ uniformly on a bounded domain D in the $(x_2,\dots,x_m)$-space.

Definition. For a formal expression $e^{\rho x_1}x_1^{-\lambda}\hat{p}(x)$ with a complex number ρ, a non-negative integer λ and a formal series $\hat{p}(x)=\sum_{n=0}^{\infty}c_n(x_2,\dots,x_m)x_1^{-n}$, we define the Borel transform as follows,

$$\hat{B}_1(e^{\rho x_1}\sum_{n=0}^{\infty}c_nx_1^{-\lambda-n})(\xi_1)=\sum_{n=0}\frac{c_n}{\Gamma(n+\lambda)}(\rho+\xi_1)^{n+\lambda-1}.$$

Proposition 3. The Borel transforms of divergent solutions are holomorphic functions in a domain in $\mathbf{C}^m$, which are analytically prolongeable.

In fact,

$$\begin{aligned} &\hat{B}_1(e^{x_1}x_1^{\beta_1-\gamma}\hat{V}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1}))(\xi_1)\\ = &\frac{1}{\Gamma(\gamma-\beta_1)}(-\xi_1)^{\beta_1-1}(1+\xi_1)^{\gamma-\beta_1-1}\\ \times &\Phi_D^{m-1}(\beta_2,\dots,\beta_m;\gamma-\beta_1;(1+\xi_1)x_2,\cdots,(1+\xi_1)x_m),\end{aligned}$$

$$\begin{aligned} &\hat{B}_1(x_1^{-\beta_1}\hat{U}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1}))(\xi_1)\\ = &\frac{1}{\Gamma(\beta_1)}\xi_1^{\beta_1-1}(1+\xi_1)^{\gamma-\beta_1-1}\\ \times &\Phi_D^{m-1}(\beta_2,\dots,\beta_m;\gamma-\beta_1;(1+\xi_1)x_2,\cdots,(1+\xi_1)x_m).\end{aligned}$$

Between

$$B_1(\hat{v})(\xi_1)=\hat{B}_1(e^{x_1}x_1^{\beta_1-\gamma}\hat{V}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1}))(\xi_1)$$

and

$$B_1(\hat{u})(\xi_1)=\hat{B}_1(x_1^{-\beta_1}\hat{U}(\beta_1,\beta_2,\dots,\beta_m,\gamma,x_2,\cdots,x_m,x_1^{-1}))(\xi_1),$$

we have a relation

$$\Gamma(\beta_1)B_1(\hat{u})(\xi_1) = \Gamma(\gamma-\beta_1)(-1)^{-\beta_1+1}B_1(\hat{v})(\xi_1),$$

Definition. Consider a function $f(\xi_1, x_2, \dots, x_m)$ which is holomorphic and exponentially small in a tubular neighborhood in the first variable and a bounded domain in the other variables. We define the generalized Laplace transforms of $f(\xi_1, x_2, \dots, x_m)$, as follows

$$\int_{C(q,\theta)} \exp(-x_1\xi_1) f(\xi_1, x_2, \dots, x_m) d\xi_1,$$

where $C(q,\theta)$ is a following path of integral. For a point q in the tubular neighborhood, $C(q,\theta)$ is a path on which $\arg(\xi_1 - q)$ is taken to be initially θ and finally $\theta + 2\pi$.

Proposition 4. The Laplace transforms of Borel transforms of divergent solutions are holomorphic functions in a suitable angular domain with the summit p in $P^1(\mathbf{C}) \times \mathbf{C}^{m-1}$, where they are actual solutions to the sysytem Φ_D with asymptotic expansions of Gevrey order 1. Here, the asymptotic expansions coincide with the divergent solutions, respectively.

In fact, for

$$-2\pi < \theta < 0,$$

the Laplace integral

$$\frac{1}{\Gamma(\gamma-\beta_1)} \int_{C(-1,\theta)} \exp(-x_1\xi_1)(-\xi_1)^{\beta_1-1}(1+\xi_1)^{\gamma-\beta_1-1}$$
$$\times \quad \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma-\beta_1; (1+\xi_1)x_2, \cdots, (1+\xi_1)x_m) d\xi_1$$

is defined and represents a holomorphic function in the first variable x_1 in the angular domain (mod. 2π)

$$\frac{\pi}{2} < \arg(-\xi_1 x_1) < \frac{3\pi}{2},$$

namely,

$$-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,$$

because $\exp(-x_1\xi_1)$ tends to 0 as ξ_1 tends to the infinity. By considering the analytic prolongation, we obtain an actual solution v in the angular domain

$$-\frac{5\pi}{2} < \arg x_1 < \frac{\pi}{2}.$$

For

$$-\pi < \theta < \pi,$$

the Laplace integral

$$\frac{1}{\Gamma(\beta_1)} \int_{C(0,\theta)} \exp(-x_1\xi_1)\xi_1^{\beta_1-1}(1+\xi_1)^{\gamma-\beta_1-1}$$
$$\times \quad \Phi_D^{m-1}(\beta_2, \dots, \beta_m; \gamma-\beta_1; (1+\xi_1)x_2, \cdots, (1+\xi_1)x_m) d\xi_1$$

is defined and represents a holomorphic function in the first variable x_1 in the angular domain (mod. 2π)

$$\frac{\pi}{2} < \arg(-\xi_1 x_1) < \frac{3\pi}{2},$$

namely,

$$-\frac{\pi}{2} - \theta < \arg x_1 < \frac{\pi}{2} - \theta,$$

because $\exp(-x_1\xi_1)$ tends to 0 as ξ_1 tends to the infinity. By considering the analytic prolongation, we obtain an actual solution u in the angular domain

$$-\frac{3\pi}{2} < \arg x_1 < \frac{3\pi}{2}.$$

Proposition 5 We have fundamental systems of solutions

$$(e_1u, e_2v, w_2, \dots, w_m)$$

in the angular domain

$$-\frac{3\pi}{2} < \arg x_1 < \frac{\pi}{2}.$$

and

$$(e_1u, e_2v(x_1e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)}, w_2, \dots, w_m)$$

in the angular domain

$$-\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}.$$

where

$$w_2 = x_1^{-\beta_1}x_2^{\beta_1-\gamma+1}h_2, \dots, w_m = x_1^{-\beta_1}x_m^{\beta_1-\gamma+1}h_m$$

with holomorphic functions $h_2, \dots, h_m$ at the point p.

Then, we have the relations

$$\begin{aligned} & (e_1u, e_2v(x_1e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)}) \\ = \ & (e_1u, e_2v)\begin{pmatrix} 1 & c_{12} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

in the angular domain

$$-\frac{\pi}{2} < \arg x_1 < \frac{\pi}{2}.$$

and

$$\begin{aligned} & (e_1u(x_1e^{-2i\pi})e^{2i\pi(-\beta_1)}, e_2v(x_1e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)}) \\ = \ & (e_1u, e_2v(x_1e^{-2i\pi})e^{2i\pi(-\gamma+\beta_1)})\begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix} \end{aligned}$$

in the angular domain

$$\frac{\pi}{2} < \arg x_1 < \frac{3\pi}{2}.$$

In the above, we use the following constants

$$\begin{aligned} e_1 &= (e^{2i\pi\beta_1} - 1)^{-1}, \\ e_2 &= (e^{2i\pi(\gamma-\beta_1)} - 1)^{-1}, \\ c_{12} &= \frac{-2i\pi}{\Gamma(1-\beta_1)\Gamma(\gamma-\beta_1)}, \\ c_{21} &= \frac{-2i\pi e^{i\pi(\gamma-2\beta_1)}}{\Gamma(\beta_1)\Gamma(1-\gamma+\beta_1)}. \end{aligned}$$

References

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[2] Majima, H., Stokes structure of the confluent hypergeometric differential equations in two variables and resurgent equations, in 'Collection Travaux en cours 48'(1994), pp31–34.

[3] Obayashi, A., Asymptotic Analysis of the Confluent Hypergeometric Differential Equations in Two Variables, Master Thesis, Ochanomizu University (2004)