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Announcement of the Toulouse Project Part 2

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Last September (September, 2003) Koike, Nishikawa and we reported results on the Stokes geometry of higher order Painlevé equations at a conference held in Toulouse ([KKNT1]). At that occasion we emphasized that our report is the first step in our trial to understand the analytic structure of solutions of higher order Painlevé equations; without presenting any detailed program, we then named the trial "Toulouse Project" after Toulouse, with which Painlevé is closely tied.

We have recently been able to make one step further in Toulouse Project, and we announce the result here:

Any 0*-parameter solution of a higher order Painlevé equation* $(P_J)_m$ $(J = \mathrm{I}, \mathrm{II}\text{-}1, \mathrm{II}\text{-}2; m = 1, 2, \cdots)$ *can be formally reduced to a* 0*-parameter solution of* $(P_{\mathrm{I}})_1$*, i.e., the traditional Painlevé equation* (P_{I}) *with a large parameter, near its turning point of the first kind (in the sense of [KKNT1]).*

It is clear that this result is a substantial generalization of our earlier result ([KT1]) on the reduction of 0-parameter solutions of the second order Painlevé equations; there are only six equations covered by our earlier result, but now infinitely many equations are covered by the above stated result "Toulouse Project Part 2".

Using this opportunity we present our current dream of "Toulouse Project" with some comments.

Part 1: Stokes geometry of higher order Painlevé equations.

See [KKNT1] and [KKNT2] for the details.

Part 2: Reduction of a 0-parameter solution of a higher order Painlevé equation to a 0-parameter solution of the first Painlevé equation near its turning point of the first kind.

This is what we announced above. See [KT2] for the details.

Part 3: Study of the structure of a 0-parameter solution of a higher order Painlevé equation near its turning point of the second kind.

Part 4: Construction of $(2m)$-parameter solutions of $(P_J)_m$ ($J = \mathrm{I}, \mathrm{II}\text{-}1, \mathrm{II}\text{-}2; m = 1, 2, \cdots$).

We plan to make use of Hamiltonian form of a higher order Painlevé equation in the construction. (See [T] for the prototype of such construction.)

Part 5: Study of the structure of a $(2m)$-parameter solution of $(P_J)_m$ ($J = \mathrm{I}, \mathrm{II}\text{-}1, \mathrm{II}\text{-}2; m = 1, 2, \cdots$) near its turning point.

Part 6: Connection formula for a solution of a higher order Painlevé equation near its turning point.

We believe that the result in Part 2 and the expected results in Part 3 and Part 5 will be basic tools in completing this part.

Part 7: Study of the structure (of solutions) of a higher order Painlevé equation near a crossing point of its Stokes curves.

As Nishikawa first observed by a computer-assisted study of Stokes geometry of a higher order Painlevé equation ([N]), some unexpected degeneracy of the Stokes geometry of the underlying Lax pair often occurs in a neighborhood of a crossing point of Stokes curves of the Painlevé equation. Here we use the wording "some unexpected degeneracy" to mean that two turning points of (one of) the Lax pair are connected by a Stokes curve despite the fact that the parameter t does not lie on any (ordinary) Stokes curves of the Painlevé equation. Although it is not always the case that this phenomenon, the so-called Nishikawa phenomenon, is observed near a crossing point of Stokes curves of the Painlevé equation, the totality of points where the Nishikawa phenomenon is observed, if any, forms a curved ray emanating from the crossing point; we call such a ray a new Stokes curve ([KKNT1], [KKNT2]). Study of the structure of solutions of the Painlevé equation in a neighborhood of a new Stokes curve is a challenging problem. We surmise that the study of Stokes geometry of the Lax pair in the large will give us a clue to this issue.

References

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[KKNT2] ———: On the complete description of the Stokes geometry for the first Painlevé hierarchy. This proceedings.

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