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On the exact WKB analysis of microdifferential operators of WKB type

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One of the reporters, T. Kawai, was supposed to give a talk at the parent conference of this symposium, Colloque en l'honneur de Louis Boutet de Monvel (Paris, juin 2003), but several troubles blocked him to do so. Then Kawai asked T. Koike to report on behalf of the four reporters on the subject described in the title, trusting him with the following recorded message:

I am very sorry for being unable to attend this conference because of several personal issues. I sincerely apologize to Professor Boutet de Monvel and the organizing committee for my absence. Now, looking over the list of speakers, I feel quite nostalgic. So please allow me to indulge in some personal recollections.

There was a conference at Orsay in September 1972, where I was, so to speak, a debutante in France. After the end of my talk, I noticed a majestic guy was approaching me. Probably because of his hairstyle, I felt as if I had been approached by a lion. Fortunately the lion was a good and kind lion, and he enthusiastically encouraged me concerning the theory of pseudo-differential operators of infinite order, which both of us were interested in then. This is how Professor Boutet and I first met. Needless to say, I had known him by name through the renowned paper of Boutet de Monvel and Krée titled "Pseudo-differential operators and Gevrey classes". Actually the formal norm introduced there played a critically important role in the paper by Sato, Kashiwara and myself. As a matter of fact, however, I had known his name before I encountered the paper.

In 1967 Professor Reiji Takahashi came back to the University of Tokyo after his extended stay in France. One day he was chatting with several students in the common room. Probably to spur us, he told us the following story. For a while, "I" stands for Professor Takahashi and "you" stands for the students sitting around him then.

"You may regard yourselves brilliant, and it might be reasonable in Japan. But you should not forget that there are many really ingenious students in the world. For example, I met a terribly ingenious young guy when I was an assistant of Professor (Henri) Cartan. He always solved all the problems in the best way I could imagine. I was overwhelmed by him. His name was Boutet de Monvel. You should work hard as you are to compete with such ingenious guys."

So, to work hard, let me stop indulging in personal recollections and start scientific discussions. Here I pass the baton to one of my collaborators, Dr. Koike, and he presents our recent results together with some open problems in the exact WKB analysis.

What we reported this time (on March 8, 2004) is essentially the same as that reported by Koike on June 26, 2003 with one exception: the open problems mentioned in the above message of Kawai have been first numerically and then analytically resolved. In a word, the notorious point x_B that appears in the Berk-Book equation ([BB])

$$(1)\quad \gamma^2(\exp x^2)\psi = \Big[-2(\eta^{-1}\frac{d}{dx})^{-2} + 4(\eta^{-1}\frac{d}{dx})^{-3}\exp(-(\eta^{-1}\frac{d}{dx})^{-2})\int_0^{(\eta^{-1}d/dx)^{-1}}\exp t^2 dt\Big]\psi,$$

that is, the point x_B satisfying

$$(2)\qquad \gamma^2 \exp x_B^2 = 1,$$

is an accumulation point of simple turning points of (1). Although the distribution of virtual turning points of (1) and their effects on the connection formulas for WKB solutions of (1) have not been analyzed yet, we hope that the course is now set by the above result ([AKKT2, Appendix]).

Besides this characterization of the point x_B, the title of this symposium tempted us to lay particular emphasis this time on the close relationship between microlocal analysis and exact WKB analysis of microdifferential operators of WKB type. (See also [DP].) As a generalization of differential operators of WKB type ([AKKT1]), microdifferential operators of WKB type admit, in general, infinitely many phases in their WKB solutions. At the same time, they contain negative order differentiation as is observed in the Berk-Book equation (1). Otherwise stated, we have to find out WKB solutions, in particular their phase functions, for an operator which is not defined on the 0-section (i.e., the part where $\xi = 0$ with ξ denoting the symbol of d/dx). This might sound impossible at first, but intertwining the Borel transform P_B of a microdifferential operator P of WKB type by $\exp(T(x, \partial_y)\partial_y)$ with an operator T of order 0 results in a quantized contact transformation of P_B determined by T. Hence, barring some degenerate situation (that is, at the point x_0 where the top order part of $\partial_x T$ vanishes), the concrete computation of the WKB solutions can be done at the zero section for the quantized contact transformed operator. Thus the genuine problem is how to write down explicitly the quantized contact transformed operator. Here we employ an ingenious idea of Malgrange ([M]) in writing down the composition of microdifferential operators. (See also [A].) By making use of several techniques in microlocal analysis, we arrive at the following Riccati-type equation (5) for the η^{-1} multiple of the logarithmic derivative $S = S_0(x) + \eta^{-1}S_1(x) + \cdots$ of a WKB solution ψ of the equation

$$P\psi = 0 \tag{3}$$

near a point x_0 that satisfies

$$S_0(x_0) \neq 0. \tag{4}$$

$$\exp(\eta^{-1}\partial_\zeta\partial_z)\sigma(P_B)(x, \zeta + \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}\frac{\partial^k}{\partial x^k}S(x, \eta), \eta)\Big|_{z=\zeta=0} = 0. \tag{5}$$

See [AKKT2] for the precise definition of the operator P and the detailed derivation of (5).

Although the equation (5) looks quite scaring, it can be recursively solved if

$$\partial_\zeta P_0(x, S_0(x)) \neq 0. \tag{6}$$

Note that S_0 is determined by the equation

$$P_0(x, S_0(x)) = 0, \tag{7}$$

and hence we encounter infinitely many phase functions S_0, in general. Once S_0 is fixed, we find

$$S_1 = -(\partial_\zeta P_0(x, S_0))^{-1}(\partial_\zeta^2 P_0(x, S_0)\frac{1}{2!}S_0' + P_1(x, S_0)), \tag{8}$$

etc. in a recursive manner ([AKKT2, Theorem 2.1]).

At the end of this report, we note our expectation that the analysis in [AKKT2, Appendix] of the function $P(x, z)$ given by

$$-2z^2(1 - 2z\exp(-z^2)\int_0^z \exp t^2 dt) - 1 - 2x \tag{9}$$

will turn out to be useful in application (e.g. [L]).

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