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Symbolic Structures on Corner Manifolds

D. Calvo, C.-I. Martin, and B.-W. Schulze

Abstract

Differential operators on a manifold M with singularities of order m are degenerate in a natural way (in corresponding 'stretched' coordinates). We establish natural scales of weighted cone and edge Sobolev spaces (with multiple weights) on such manifolds and formulate principal symbolic hierarchies, consisting of $m+1$ components. Moreover, we illustrate the iterative way to pass from the singularity order m to $m+1$.

Contents

Introduction

Operators on a manifolds with higher (regular) corners, have a principal symbolic hierarchy which is responsible for ellipticity and parametrices. As is known from the case of a manifold with smooth edges, cf. [22] or [25], there is an edge symbolic structure which consists of families of operators on an infinite model cone parametrised by the cotangent bundle (minus the zero section) of the edge. For smooth edges the model cone $X^{\Delta} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ of local wedges has a smooth base X, and $r \to \infty$ can be interpreted as a conical exit to infinity (here $r \in \mathbb{R}_+$ is the axial variable). For higher singularities the base X is not smooth. In such a case X^{Δ} has edges and corners up to infinity.

The program of the calculus for smooth edges as well as for corners of different kind, cf. [24], [27], [15], [17], shows that specific structures have to be developed for making the approach iterative, cf. [26]. One of the main issues is to understand the higher analogues of the principal edge symbolic structure, represented by operators in weighted Sobolev spaces on X^{Δ}. In the present paper we give a new definition of the higher spaces (elementary compared with the one in [26]) which points out the aspect of manifolds with exits to infinity and non-smooth cross section. The case of cross sections of singularity order 2 is treated in [3], while elements for the higher case may be found in [18]. The present note gives an overview of a part of these results.

Our considerations are embedded into the general program to establish a satisfying analysis on manifolds with singularities (stratified spaces). There is a vast variety of investigations in the literature, devoted to the index of elliptic operators, cf. Teleman [30], [31], and Nistor [21], Nazaikinskij, Savin, Schulze and Sternin [19], [20], Fedosov, Tarkhanov and Schulze [9], [8], Loya

[16], to the nature of appropriate weighted function spaces, cf. Schulze [23], Hirschmann [11], Brasselet and Teleman [2], or to other specific problems, cf. Seiler [28], Gil and Mendoza [10], Dines, Harutjunjan, and Schulze [4]. Concerning more references, also with respect to models with singularities in the applied sciences, cf. Kapanadze and Schulze [12].

Higher corner spaces are also of interest in anisotropic form in connection with long-time asymptotics of solutions to parabolic equations on a spatial configuration with singularities, cf. Krainer and Schulze [14] and Krainer [13].

1 Manifolds with higher corners

Definition 1.1 *By a manifold with corners of order m we understand a topological space M which is equipped with a chain of subspaces*

$$M \supset M' \supset M'' \supset \ldots \supset M^{(m)} \tag{1}$$

(where $M^{(0)} := M, M^{(m+1)} = \emptyset$) such that

(i) *$M^{(j)} \setminus M^{(j+1)}$ is a C^∞ manifold for $j = 0, \ldots, m$;*

(ii) *$M^{(j)}$ is of order $m - j$ (order 0 means C^∞) for $j = 1, \ldots, m$,*

(iii) *every $y \in M^{(j)} \setminus M^{(j+1)}$ has a neighbourhood V modelled on a wedge*

$$X_{j-1}^{\Delta} \times \Omega \tag{2}$$

where X_{j-1} is a manifold of order $j-1$, $j = 1, \ldots, m$, and $\Omega \subseteq \mathbb{R}^{q_j}$ open.

In addition we require some regularity of the transition maps between the local wedges, inductively defined in terms of isomorphisms of such singular manifolds, cf. the constructions below.

The homeomorphisms $\alpha : V \to X_{j-1}^{\Delta} \times \Omega$ will also be referred to as singular charts on M.

Note that $M^{(m)}$ is a C^∞ manifold, and $M \setminus M^{(m)}$ is of order $m-1$. In the singular case the notation 'manifold' is to be understood in a generalised sense. In fact, we are speaking about a special category of stratified spaces. In future such spaces are assumed to be a countable union of compact subsets.

Let $\mathfrak{M}_m$ denote the category of manifolds of singularity order m.

Because of the iterative process we mainly look at singular charts for the case $j = m$

$$\alpha : V \to X_{m-1}^{\Delta} \times \Omega \tag{3}$$

for a neighbourhood V of $y \in Y := M^{(m)}$ in M (of course, the following observations are true in analogous form for all j).

Every such α restricts to isomorphisms

$$\alpha_{\mathrm{reg}} : V \setminus Y \to X_{m-1}^{\wedge} \times \Omega \tag{4}$$

for $X_{m-1}^{\wedge} := \mathbb{R}_+ \times X_{m-1}$, $\Omega \subseteq \mathbb{R}^{q_m}$ open, and

$$\alpha' : V \cap Y \to \Omega; \tag{5}$$

α' is then a diffeomorphism. From (4) we obtain a splitting of variables

$$(r, x, y) \in \mathbb{R}_+ \times X_{m-1} \times \Omega. \tag{6}$$

Example 1.2 (i) *Let X be a C^∞ manifold, and set $M = X^\Delta$ which is the infinite cone with base X and vertex $\{v\}$ (represented by $\{0\} \times X$ in the corresponding quotient space, cf. the notation in the beginning). In this case we have $m = 1$ and $M' = \{v\}$.*

(ii) *Let M be a C^∞ manifold with boundary. We then have $m = 1$ and $M' = \partial M$. The local model wedge in this case is the half-space with $\overline{\mathbb{R}}_+$ (the inner normal to the boundary) as the model cone.*

(iii) *Let $M = \{x \in \mathbb{R}^m : 0 \leq x_j \leq 1$ for $j = 1, \ldots, m\}$. Then M is of singularity order m. To save space we only describe the singular subspaces for $m = 3$. In this case M' is the surface of the cube, M'' consists of the edges including corner points, and M''' are the corner points.*

Remark 1.3 *For convenience, in the constructions below we make some simplifying assumptions that are not really necessary. In general the manifolds $X_{j-1} \in \mathfrak{M}_{j-1}$ in (2) may depend on $y \in M^{(j)} \setminus M^{(j+1)}$. We will assume that $X_{j-1}(y)$ is $\mathfrak{M}_{j-1}$-isomorphic to $X_{j-1}(\tilde{y})$ for all $y, \tilde{y} \in M^{(j)} \setminus M^{(j+1)}$ and for all j. This is the case, for instance, in Example* 1.2 (i), (ii), (iii).

For $M \in \mathfrak{M}_m$ we set

$$\dim M = 1 + \dim X_{(m-1)} + q_m$$

for $q_m = \dim M^{(m)}$, assuming that the dimension is already defined on $\mathfrak{M}_{m-1}$. It follows that

$$\dim M = 1 + \dim X_{(j-1)} + q_j$$

for $q_j = \dim(M^{(i)} \setminus M^{(j+1)})$, $j = 1, \ldots, m$, and $\dim M = q_0 = \dim(M \setminus M')$.

To a manifold with singularities we can form the so called stretched manifold. For instance, the stretched manifold $\mathbb{M}$ to the cone $M = X^\Delta$ of Example 1.2 (i) is defined by $\mathbb{M} = \overline{\mathbb{R}}_+ \times X$.

An interesting category are manifolds W with smooth edges Y. It this case we have $m = 1$ and $W' = Y$. Apart from the general construction at the beginning they can alternatively be introduced by first defining their stretched manifolds $\mathbf{W}$.

$\mathbf{W}$ is given as a C^∞ manifold with boundary $\partial\mathbf{W}$, and $\partial\mathbf{W}$ is a bundle over Y the fibre of which is a C^∞ manifold X. In simplest cases X is closed and compact. If $\pi : \partial\mathbf{W} \to Y$ denotes the bundle projection we can pass to the quotient space $W := \mathbf{W}/\sim$ with respect to the equivalence relation $w \sim w \iff \{\pi w = \pi w'$ when $w, w' \in \partial\mathbf{W}$ or $w = w'$ when $w, w' \notin \partial\mathbf{W}\}$.

From the definition we obtain a continuous map

$$\pi : \mathbf{W} \to W$$

(for simplicity, again denoted by π) such that $\pi|_{\partial\mathbf{W}}$ is just the bundle projection mentioned before and $\pi|_{\mathrm{int}\,\mathbf{W}}$ the identity map on $\mathrm{int}\,\mathbf{W}$. We also set

$$\mathbf{W}_{\mathrm{reg}} := \mathbf{W} \setminus \partial\mathbf{W}, \quad \mathbf{W}_{\mathrm{sing}} := \partial\mathbf{W}.$$

An isomorphism $\mathbf{W} \to \widetilde{\mathbf{W}}$ between two stretched manifolds with edge is defined as a diffeomorphism between the respective C^∞ manifolds with boundary which restrict to bundle isomorphisms $\mathbf{W}_{\mathrm{sing}} \to \widetilde{\mathbf{W}}_{\mathrm{sing}}$. If $W := \mathbf{W}/\sim$ and $\widetilde{W} := \widetilde{\mathbf{W}}/\sim$ are the associated manifolds with edges, a homeomorphism $W \to \widetilde{W}$ is said to be an isomorphism if it is induced by an isomorphism $\mathbf{W} \to \widetilde{\mathbf{W}}$ between the associated stretched manifolds.

It is often convenient to interpret $\mathbf{W}$ as a submanifold of its double $2\mathbf{W}$ (which is a C^∞ manifold) obtained by gluing together two copies $\mathbf{W}_\pm$ of $\mathbf{W}$ along their common boundary (we then identify $\mathbf{W}_+$ with $\mathbf{W}$).

In a similar manner we can proceed with an arbitrary manifold M with singularities of order m. We interpret the C^∞ manifold $Y := M^{(m)}$ as a 'higher' edge. The transition maps of the local wedges (2) will be defined in such a way that they generate the structure of an X_{m-1} bundle $\mathbb{M}_{\mathrm{sing}}$ over Y with the projection $\pi : \mathbb{M}_{\mathrm{sing}} \to Y$ which belongs to $\mathfrak{M}_{m-1}$. By induction we assume that isomorphisms are already defined up to the order $m-1$. Also $\mathbb{R} \times \mathbb{M}_{\mathrm{sing}}$ as well as $\mathbb{R}_+ \times \mathbb{M}_{\mathrm{sing}}$ belong to $\mathfrak{M}_{m-1}$. Observe that $\overline{\mathbb{R}}_+ \times \mathbb{M}_{\mathrm{sing}}$ can be regarded as an $\overline{\mathbb{R}}_+ \times X_{m-1}$-bundle over Y, and there is then a quotient map $\overline{\mathbb{R}}_+ \times \mathbb{M}_{\mathrm{sing}} \to (\overline{\mathbb{R}}_+ \times \mathbb{M}_{\mathrm{sing}})/\sim$ to an X^Δ_{m-1}-bundle over Y induced by the fibrewise maps $\overline{\mathbb{R}}_+ \times X_{m-1} \to X^\Delta_{m-1}$.

In order to specify the above requirement (iii) on the local wedges we now assume (for the case $j = m$) that $Y = M^{(m)}$ has a neighbourhood U in M such that there is a homeomorphism

$$U \to (\overline{\mathbb{R}}_+ \times \mathbb{M}_{\mathrm{sing}})/\sim \tag{7}$$

which restricts to an $\mathfrak{M}_{m-1}$-isomorphism (i.e., in the sense of the category $\mathfrak{M}_{m-1}$)

$$U \setminus Y \to \mathbb{R}_+ \times \mathbb{M}_{\mathrm{sing}} \tag{8}$$

and a diffeomorphism $U \cap Y \to Y$. Two homeomorphisms (7) are called equivalent if the transition map $\mathbb{R}_+ \times \mathbb{M}_{\mathrm{sing}} \to \mathbb{R}_+ \times \mathbb{M}_{\mathrm{sing}}$ is the restriction of an $\mathfrak{M}_{m-1}$-isomorphism $\chi : \mathbb{R} \times \mathbb{M}_{\mathrm{sing}} \to \mathbb{R} \times \mathbb{M}_{\mathrm{sing}}$ to $\mathbb{R}_+ \times \mathbb{M}_{\mathrm{sing}}$ such that χ restricts to an isomorphism $\{0\} \times \mathbb{M}_{\mathrm{sing}} \to \{0\} \times \mathbb{M}_{\mathrm{sing}}$ of X_{m-1}-bundles.

This allows us to attach $\mathbb{M}_{\mathrm{sing}}$ to $M \setminus Y$ in an invariant manner, and we obtain in this way the stretched manifold $\mathbb{M} := (M \setminus Y) \cup \mathbb{M}_{\mathrm{sing}}$ associated with M. In this connection we set

$$\mathbb{M}_{\mathrm{reg}} := \mathbb{M} \setminus \mathbb{M}_{\mathrm{sing}}$$

which is $\mathfrak{M}_{m-1}$-isomorphic to $M \setminus Y$. From the definition we immediately obtain a map

$$\pi : \mathbb{M} \to M$$

which restricts to the bundle projection $\mathbb{M}_{\mathrm{sing}} \to Y$ and to an $\mathfrak{M}_{m-1}$-isomorphism $\mathbb{M}_{\mathrm{reg}} \to M \setminus Y$.

Remark 1.4 *For technical reasons we content ourselves with isomorphisms $\chi : \mathbb{R} \times \mathbb{M}_{\mathrm{sing}} \to \mathbb{R} \times \mathbb{M}_{\mathrm{sing}}$ in the above description of transition maps such that there is an $\varepsilon > 0$ with $\chi(r, \cdot) = \chi(0, \cdot)$ for all $|r| < \varepsilon$.*

The double $2\mathbb{M}$ of $\mathbb{M}$ can be obtained by gluing together two copies $\mathbb{M}_\pm$ of $\mathbb{M}$ along the common subset $\mathbb{M}_{\mathrm{sing}}$. There is then a neighbourhood $2\mathbb{U}$ of $\mathbb{M}_{\mathrm{sing}}$ in $2\mathbb{M}$ which is $\mathfrak{M}_{m-1}$-isomorphic to $\mathbb{R} \times \mathbb{M}_{\mathrm{sing}}$ and such that this restricts to an isomorphism $\mathbb{M}_{\mathrm{sing}} \to \{0\} \times \mathbb{M}_{\mathrm{sing}}$ in the sense of X_{m-1}-bundles. In particular, this isomorphism restricts to a map $\mathbb{U}_+ := (2\mathbb{U}) \cap \mathbb{M}_+ \to \overline{\mathbb{R}}_+ \times \mathbb{M}_{\mathrm{sing}}$ and to an isomorphism of $\mathbb{M}_{\mathrm{sing}}$ to itself and factorises to (7).

An isomorphism $\mathbb{M} \to \widetilde{\mathbb{M}}$ between two stretched manifolds belonging to objects $M, \widetilde{M} \in \mathfrak{M}_m$ is defined as the restriction of an $\mathfrak{M}_{m-1}$-isomorphism $\chi : 2\mathbb{M} \to 2\widetilde{\mathbb{M}}$ to a map $\mathbb{M} \to \widetilde{\mathbb{M}}$ such that $\chi|_{\mathbb{M}_{\mathrm{sing}}} : \mathbb{M}_{\mathrm{sing}} \to \widetilde{\mathbb{M}}_{\mathrm{sing}}$ is an isomorphism between corresponding X_{m-1}- and $\widetilde{X}_{m-1}$-bundles. By passing to the spaces $M, \widetilde{M}$ themselves we obtain the notion of an $\mathfrak{M}_m$-isomorphism $M \to \widetilde{M}$. In this way we have the category $\mathfrak{M}_m$ including isomorphisms, and we can start the procedure again.

By a similar scheme we can inductively define morphisms in the category $\mathfrak{M}_m$.

2 Operators with symbolic hierarchies

If M is a manifold of singularity order $m \in \mathbb{N}$ there is a subspace $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$ of differential operators $A \in \mathrm{Diff}^{\mu}(M \setminus M')$ of order μ defined as follows. By hypotheses we already have $\mathrm{Diff}^{\mu}_{\mathrm{deg}}(M \setminus Y)$ on $M \setminus Y$ which is of singularity order $m-1$. Then

$$A \in \mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$$

is characterised by the conditions

$$A|_{M \setminus Y} \in \mathrm{Diff}^{\mu}_{\mathrm{deg}}(M \setminus Y),$$

and, in the splitting of variables $(r_m, x, y_m) \in \mathbb{R}_+ \times X_{m-1} \times \Omega$ near Y, $\Omega_m \subseteq \mathbb{R}^{q_m}$, (coming from a localisation of (8) for a chart on Y) the operator A takes the form

$$A = r_m^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r_m, y_m)\Big(-r_m \frac{\partial}{\partial r_m}\Big)^j \tag{9}$$

with coefficients $a_{j\alpha}(r_m, y_m) \in C^{\infty}\big(\overline{\mathbb{R}}_+ \times \Omega_m, \mathrm{Diff}^{\mu-(j+|\alpha|)}_{\mathrm{deg}}(X_{m-1})\big)$. One of the assumptions in the iterative process to organise a calculus of operators on M is that up to the singularity order $m-1$ there is a principal symbol

$$\sigma(A|_{M\setminus Y}) := \big(\sigma_j(A|_{M\setminus Y})\big)_{j=0,\dots,m-1}.$$

$\sigma_0(A|_{M\setminus Y})$ is nothing other than the standard homogeneous principal symbol of $A|_{M\setminus M'}$. For $A \in \mathrm{Diff}^{\mu}_{\mathrm{deg}}(M)$ itself we define

$$\sigma(A) := \big(\sigma(A|_{M\setminus Y}), \sigma_{\wedge}(A)\big)$$

where the extra component $\sigma_{\wedge}(A)$ is a family of operators

$$\sigma_{\wedge}(A)(y_m, \eta_m) := r_m^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(0, y_m)\Big(-r_m \frac{\partial}{\partial r_m}\Big)^j (r_m \eta_m)^{\alpha}$$

acting in a scale of weighted Sobolev spaces on $X^{\wedge}_{m-1} = \mathbb{R}_+ \times X_{m-1}$ denoted by

$$\mathcal{K}^{s,\gamma}(X^{\wedge}_{m-1}), \quad \gamma = (\gamma', \gamma_m) \in \mathbb{R}^m, \tag{10}$$

for $\gamma' \in \mathbb{R}^{m-1}$, $\gamma_m \in \mathbb{R}$. One of the main aspects of this article is to give an impression on the nature of these spaces and their iterative definition. As a result we then obtain a family of continuous operators

$$\sigma_{\wedge}(A)(y_m, \eta_m) : \mathcal{K}^{s,\gamma}(X^{\wedge}_{m-1}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}_{m-1}),$$

$\gamma - \mu := (\gamma' - \mu, \gamma_m - \mu)$, $(y_m, \eta_m) \in T^*\Omega_m \setminus 0$, with many natural properties. The experience with the calculus of (pseudo-) differential operators on (say, compact) manifolds M with corner singularities up to order 2 (cf. [27]) is that A should be the upper left corner of an $(m+1) \times (m+1)$-block matrix operator

$$\mathcal{A} : \bigoplus_{k=0}^{m} H^s_k(M^{(k)}) \to \bigoplus_{j=0}^{m} \widetilde{H}^{s-\mu}_j(M^{(j)})$$

with specific weighted Sobolev spaces of smoothness s on the submanifolds $M^{(j)}$ of $M = M^{(0)}$, cf. (1).

We do not develop the full story here; more details in that sense may be found in [26]. Let us only note that there is a generalisation of $\sigma_\wedge(A)$ to a principal symbol $\sigma_\wedge(\mathcal{A})$ for the block matrix $\mathcal{A}$. In the elliptic case $\sigma_\wedge(\mathcal{A})(y_m, \eta_m)$, $(y_m, \eta_m) \in T^*Y \setminus 0$, has to be a family of isomorphisms which is just an analogue of the Shapiro-Lopatinskij condition.

Note that the idea to associate block matrix operators with an elliptic operator A in the upper left corner such that the resulting operator is Fredholm has a long history and is realised in many specific theories, e.g., for Sobolev problems, cf. Sternin [29] (with the terminology boundary and coboundary operators), 'standard' boundary value problems with the transmission property at the boundary, cf. Boutet de Monvel [1] (with the terminology trace and Poisson operators), pseudo-differential boundary value problems without the transmission property, cf. Vishik and Eskin [32], [7], edge and corner operators [24], cf. Egorov and Schulze [6] (with the terminology trace and potential operators), and in other contexts [5], [4].

Observe that Laplace-Beltrami operators belonging to specific Riemannian metrics are of the form (9) for $\mu = 2$. For instance, consider (for the case $m = 2$) a Riemannian metric of the form

$$dr_2^2 + r_2^2\{dr_1^2 + r_1^2 g_{X_1}(r_1, y_1, r_2, y_2) + dy_1^2\} + dy_2^2$$

for a C^∞ manifold X_1 and a family of Riemannian metrics g_{X_1} or X_1 smoothy depending on

$$(r_1, y_1, r_2, y_2) \in \overline{\mathbb{R}}_+ \times \Omega_1 \times \overline{\mathbb{R}}_+ \times \Omega_2$$

(smooth up to $r_1 = 0$, $r_2 = 0$), $\Omega_j \subseteq \mathbb{R}^{q_j}$ open, $j = 1, 2$. The space $M \in \mathfrak{M}_2$ in this case is given by

$$M = \left\{\overline{\mathbb{R}}_+ \times (X_1^\Delta \times \Omega_1)/(\{0\} \times (X_1^\Delta \times \Omega_1))\right\} \times \Omega_2,$$

$M^{(2)} = \Omega_2$, and $\mathbb{M} = \overline{\mathbb{R}}_+ \times (X_1^\Delta \times \Omega_1) \times \Omega_2$, $2\mathbb{M} = \mathbb{R} \times (X_1^\Delta \times \Omega_1) \times \Omega_2$.

3 Corner Sobolev spaces of second generation

We now give a definition of spaces $\mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge)$ for $(\gamma_1, \gamma_2) \in \mathbb{R}^2$, when W is a compact manifold with smooth edges Y, cf. [3], knowing a corresponding definition of $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ for a closed C^∞ manifold X. In order to motivative the construction we briefly recall the construction of $\mathcal{K}^{s,\gamma_1}(X^\wedge)$. First we have the scale of standard Sobolev spaces $H^s(X)$, $s \in \mathbb{R}$, on X. Let $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l)$ denote the space of all classical parameter-dependent pseudo-differential operators on X of order $\mu \in \mathbb{R}$, with parameters $\lambda \in \mathbb{R}^l$. For every $\mu \in \mathbb{R}$ there exists an element $R^\mu(\lambda) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l)$ that induces isomorphisms

$$R^\mu(\lambda) : H^s(X) \to H^{s-\mu}(X)$$

for all $\lambda \in \mathbb{R}^l$, $s \in \mathbb{R}$. Let $H^s(\mathbb{R} \times X)$ denote the completion of the space $C_0^\infty(\mathbb{R} \times X)$ with respect to the norm

$$\left\{\int \|R^s(v) F_{p\to v} u)v\|^2_{L^2(X)} dv\right\}^{1/2}.$$

Here $F_{p\to v}$ is the one-dimensional Fourier transform on $\mathbb{R}$ and $R^s(v) \in L^s_{\mathrm{cl}}(X; \mathbb{R}_v)$ is a corresponding order reducing family of order s in the above-mentioned sense.

For the constructions below we refer to another equivalent definition of the cylindrical Sobolev spaces $H^s(\mathbb{R} \times X)$, namely, as the space all $u(p, \cdot) \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)$ such that

$$(\varphi u) \circ (1 \times \alpha^{-1}) \in H^s(\mathbb{R}_p \times \mathbb{R}^n_x)$$

for every chart $\alpha : U \to \mathbb{R}^n_x$ on X and every $\varphi \in C_0^\infty(U)$.

Let us set $(S_\beta u)(p) := e^{-(\frac{1}{2}-\beta)p}u(e^{-p})$, $p \in \mathbb{R}$, and

$$\mathcal{H}^{s,\gamma_1}(X^\wedge) := (S_{\gamma_1-\frac{n}{2}})^{-1}H^s(\mathbb{R}\times X)$$

for $n = \dim X$. We now define $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ for $(r_1,\cdot) \in X^\wedge$ near $r_1 = 0$ by

$$\omega_1(r_1)\mathcal{K}^{s,\gamma_1}(X^\wedge) = \omega_1(r_1)\mathcal{H}^{s,\gamma_1}(X^\wedge)$$

where ω_1 is any cut-off function on the half-axis (i.e., $\omega_1 \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega_1 \equiv 1$ near $r_1 = 0$). In remains to explain $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ for large r_1.

Let us set $B := \{y_0 \in \mathbb{R}^n : |y_0| < 1\}$ and

$$\Gamma := \{(r_1, r_1y_0) \in \mathbb{R}^{1+n} : r_1 \in \mathbb{R}_+, y_0 \in B\}.$$

On X we consider a chart $U \to B$, $x \to y_0$, and form the map

$$\beta_U : (r_1,x) \to (r_1, r_1y_0) =: (r_1,\tilde{y}_0),$$

$$\beta_U : \mathbb{R}_+ \times U \to \Gamma \subset \mathbb{R}^{1+n}.$$

An element $u \in H^s_{\mathrm{loc}}(\mathbb{R}\times X)|_{\mathbb{R}_+\times X}$ is said to belong to $H^s_{\mathrm{cone}}(X^\wedge)$ if for every chart $U \to B$ with the associated map β_U we have

$$(1-\omega_1)\varphi u \circ \beta_U^{-1} \in H^s(\mathbb{R}^{1+n}_{r_1,\tilde{y}_0})$$

for every cut-off function $\omega_1(r_1)$ and every $\varphi \in C_0^\infty(U)$.

We now define

$$\mathcal{K}^{s,\gamma_1}(X^\wedge) = \omega_1\mathcal{H}^{s,\gamma_1}(X^\wedge) + (1-\omega_1)H^s_{\mathrm{cone}}(X^\wedge)$$

for any choice of a cut-off function $\omega_1(r_1)$.

Remark 3.1 *The spaces $\mathcal{K}^{s,\gamma_1}(X^\wedge)$ can be endowed with scalar products in which they are Hilbert spaces. Setting*

$$(\kappa_\lambda u)(r_1,x) = \lambda^{\frac{n+1}{2}}u(\lambda r_1, x)$$

for $\lambda \in \mathbb{R}_+$, we obtain a strongly continuous group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomophisms on the space $\mathcal{K}^{s,\gamma_1}(X^\wedge)$, for every $s, \gamma_1 \in \mathbb{R}$.

The $\mathcal{K}$-spaces of second generation on the infinite cone $W^\wedge \ni (r_2,\cdot)$ for a compact manifold W with edge Y refer again to a construction near the tip $r_2 \to 0$ and near the exit $r_2 \to \infty$.

For $r_2 \to 0$ we have a corner configuration, cf. [27], while for $r_2 \to \infty$ we have a manifold with edge that has a conical exit to infinity.

An important tool are the abstract edge Sobolev spaces from [22].

Definition 3.2 *Let H be a Hilbert space which is endowed with a strongly continuous group of isomophisms*

$$\kappa_\lambda : H \to H, \quad \lambda \in \mathbb{R}_+.$$

Then $\mathcal{W}^s(\mathbb{R}^q, H)$ for $s \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm

$$\left\{\int \langle\eta\rangle^{2s}\|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H d\eta\right\}^{1/2}.$$

Together with Remark 3.1 we obtain the spaces

$$\mathcal{W}^s(\mathbb{R}^{q_1}, \mathcal{K}^{s,\gamma_1}(X^\wedge))$$

for every $s, \gamma_1 \in \mathbb{R}$. Then, if W is a (say, compact) manifold with smooth edges, we obtain corresponding global spaces $\mathcal{W}^{s,\gamma}(W)$.

By corner Sobolev spaces of second generation we understand weighted spaces on a manifold M of singularity order 2. Locally such a manifold M is modelled on

$$W^\Delta \times \Omega_2$$

for an open set $\Omega_2 \subseteq \mathbb{R}^{q_2}$ and a manifold W of singularity order 1, locally modelled on

$$X^\Delta \times \Omega_1$$

for an open set $\Omega_1 \subseteq \mathbb{R}^{q_1}$ and a C^∞ manifold X. We assume here X to be closed compact and W compact.

Similarly as before, in order to define spaces of the kind $\mathcal{W}^{s,(\gamma_1,\gamma_2)}(M)$, we need (here weighted) cylindrical Sobolev spaces

$$\mathcal{W}^{s,\gamma_1}(\mathbb{R}_p \times W), \tag{11}$$

$(p, \cdot) \in \mathbb{R} \times W$, as well as a local analogue $\mathcal{W}^{s,\gamma_1}_{\text{loc}}(\mathbb{R} \times W)$ of (11) and weighted cone spaces of the type

$$\mathcal{W}^{s,\gamma_1}_{\text{cone}}(W^\wedge), \tag{12}$$

where $\gamma_1 \in \mathbb{R}$ denotes the weight that is connected with the axial variable $r_1 \in \mathbb{R}_+$ for the local model cone X^Δ.

To define (11) we first recall that we have the spaces $H^s(\mathbb{R} \times 2\mathbf{W})$ from the discussion in the beginning, using the fact that 2W is a closed compact C^∞ manifold. Then $\mathcal{W}^{s,\gamma_1}(\mathbb{R} \times W)$ is defined to be the space of all $u \in H^s_{\text{loc}}(\mathbb{R} \times (W \setminus Y))$ such that

(i)
$$(1 - \omega_1)u \in H^s(\mathbb{R} \times 2\mathbf{W})|_{\mathbb{R}\times\mathbf{W}_{\text{reg}}}$$

for every cut-off function ω_1 on $\mathbf{W}$ (that is equal to 1 near $\partial\mathbf{W}$ and 0 outside a collar neighbourhood of $\partial\mathbf{W}$);

(ii) for every singular chart $\alpha : V \to X^\Delta \times \mathbb{R}^{q_1}$ on W near a point $y \in Y$

(cf. the formula (3)) and the induced map

$$1 \times \alpha_{\text{reg}} : \mathbb{R} \times (V \setminus Y) \to \mathbb{R} \times \mathbb{R}_+ \times X \times \mathbb{R}^{q_1},$$

$$(1 \times \alpha_{\text{reg}}) : (p, \cdot) \to (p, r_1, x, y_1),$$

we have

$$\varphi(1 \times \alpha^{-1}_{\text{reg}})^* \omega_1 u \in \mathcal{W}^s(\mathbb{R}_p \times \mathbb{R}^{q_1}, \mathcal{K}^{s,\gamma_1}(X^\wedge))$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{q_1})$ and the cut-off function ω_1 from (i).

A slight modification of this definition gives us the space $\mathcal{W}^{s,\gamma_1}_{\text{loc}}(\mathbb{R} \times W)$ of distributions u that have the property $\varphi u \in \mathcal{W}^{s,\gamma_1}(\mathbb{R} \times W)$ for every $\varphi \in C_0^\infty(\mathbb{R}_{r_2})$.

In fact, it suffices to set

$$\mathcal{W}^{s,\gamma_1}_{\text{loc}}(\mathbb{R} \times W) = \{\text{space of all locally finite sums } \sum_{\iota \in I} \varphi_\iota u_\iota\} \tag{13}$$

for arbitrary $\varphi_\iota \in C_0^\infty(\mathbb{R})$, $u_\iota \in \mathcal{W}^{s,\gamma}(\mathbb{R}\times W)$; locally finite means that $\varphi_\iota(p) \neq 0$ only holds for finitely many $\iota \in I$ when p varies in a compact set $\subset \mathbb{R}$.

In order to define the space $\mathcal{W}^{s,\gamma_1}_{\text{cone}}(W^\wedge)$ we set $B := \{y_1 \in \mathbb{R}^{q_1} : |y_1| < 1\}$ and consider a singular chart

$$V \to X^\Delta \times B$$

on W near a point $y \in Y$ and the induced diffeomorphism $U \to B$, $y \to y_1$, for $U := V \cap Y$. Moreover, we set $\Gamma := \{(r_2, \widetilde{y}_1) : r_2 \in \mathbb{R}_+,\ \widetilde{y}_1 = r_2 y_1, y_1 \in B\}$,

$$\beta_U : (r_1, x, r_2, y) \to (r_2 r_1, x, r_2, r_2 y_1) =: (\widetilde{r}_1, x, r_2, \widetilde{y}_1),$$

$$\beta_U : (\mathbb{R}_+ \times X) \times (\mathbb{R}_+ \times U) \to X^\wedge_{\widetilde{r}_1,x} \times \Gamma_{r_2,\widetilde{y}_1} \subset X^\wedge_{\widetilde{r}_1,x} \times \mathbb{R}^{1+q_1}_{r_2,\widetilde{y}_1}.$$

The space $\mathcal{W}^{s,\gamma_1}_{\text{cone}}(W^\wedge)$ is defined to be the set of all $u(r_2,\cdot) \in \mathcal{W}^{s,\gamma_1}_{\text{loc}}(\mathbb{R}\times W)|_{\mathbb{R}_+\times W}$ such that

(i) For every chart $U \to B$, $y \to y_1$ as mentioned before, we have

$$(1-\omega_2)\varphi\omega_1 u \circ \beta_U^{-1} \in \mathcal{W}^s\Big(\mathbb{R}^{1+q_1}_{r_2,\widetilde{y}_1}, \mathcal{K}^{s,\gamma_1}(X^\wedge_{\widetilde{r}_1,x})\Big)$$

for every $\varphi \in C_0^\infty(U)$ and cut-off functions $\omega_1(r_1), \omega_2(r_2)$;

(ii)

$$(1-\omega_1)u \in H^s_{\text{cone}}((2\mathbb{W})^\wedge).$$

Definition 3.3 *We set*

(i)

$$\mathcal{H}^{s,(\gamma_1,\gamma_2)}(W^\wedge) := \Big(S_{\gamma_2-\frac{1}{2}(\dim W)}\Big)^{-1}\mathcal{W}^{s,\gamma_1}(\mathbb{R}\times W);$$

(ii)

$$\mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge) := \omega_2\mathcal{H}^{s,(\gamma_1,\gamma_2)}(W^\wedge) + (1-\omega_2)\mathcal{W}^{s,\gamma_1}_{\text{cone}}(W^\wedge)$$

for any cut-off function ω_2 in the variable $r_2 \in \mathbb{R}_+$.

Remark 3.4 *The spaces of Definiton 3.3 are independent of the choice of ω_2, and they are Hilbert spaces with natural scalar products. Setting $(\kappa_\lambda u)(r_2,\cdot) := \lambda^{\frac{1+\dim W}{2}} u(\lambda r_2,\cdot)$, $\lambda \in \mathbb{R}_+$, we obtain a strongly continuous group of isomorphisms*

$$\kappa_\lambda : \mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge) \to \mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge)$$

for every $s, \gamma_1, \gamma_2 \in \mathbb{R}$.

Theorem 3.5 *Let*

$$A = r_2^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r_2, y_2)\Big(-r_2\frac{\partial}{\partial r_2}\Big)^j (r_2 D_{y_2})^\alpha$$

be an operator with coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega_2, \mathrm{Diff}^{\mu-(j+|\alpha|)}_{\deg}(W))$, $\Omega_2 \subseteq \mathbb{R}^{q_2}$ open. Then

$$\sigma_\wedge(A)(y_2,\eta_2) := r_2^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(0, y_2)\Big(-r_2\frac{\partial}{\partial r_2}\Big)^j (r_2\eta_2)^\alpha$$

is a family of continuous operators

$$\sigma_\wedge(A)(y_2,\eta_2):\mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge)\to\mathcal{K}^{s-\mu,(\gamma_1-\mu,\gamma_2-\mu)}(W^\wedge)$$

for every $s,\gamma_1,\gamma_2\in\mathbb{R}$, $(y_2,\eta_2)\in T^*\Omega_2\setminus 0$, *and we have*

$$\sigma_\wedge(A)(y_2,\lambda\eta_2)=\lambda^\mu\kappa_\lambda\sigma_\wedge(A)(y_2,\eta_2)\kappa_\lambda^{-1}$$

for all $\lambda\in\mathbb{R}_+$.

The proof of Theorem 3.5 is connected with a specific variant of operator-valued symbols. If H is a Hilbert space, endowed with a strongly continuous group of isomorphisms $\kappa_\lambda : H\to H$, $\lambda\in\mathbb{R}_+$, such that $\kappa_{\lambda\delta}=\kappa_\lambda\kappa_\delta$ for all $\lambda,\delta\in\mathbb{R}$, we say that H is endowed with a group action.

Definition 3.6 *Let H and $\widetilde{H}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\widetilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Then*

$$S^\mu(\Omega\times\mathbb{R}^q;H,\widetilde{H})$$

for $\mu=\mathbb{R}$, $\Omega\subseteq\mathbb{R}^p$ open, denotes the space of all $a(y,\eta)\in C^\infty(\Omega\times\mathbb{R}^q,\mathcal{L}(H,\widetilde{H}))$ such that

$$\sup_{\substack{y\in K\\ \eta\in\mathbb{R}^q}}\langle\eta\rangle^{\mu-|\beta|}\left\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa_{\langle\eta\rangle}\right\|_{\mathcal{L}(H,\widetilde{H})}<\infty$$

for all multi-indices $\alpha\in\mathbb{N}^p$, $\beta\in\mathbb{N}^q$ and all $K\subset\subset\Omega$.

The proof of Theorem 3.5 is based on the continuity of pseudo-differential operators with operator-valued symbols in abstract Sobolev spaces.

Another observation is the following relation. Assume that the coefficients $a_{j\alpha}(r_2,y_2)$ are independent of r_2 for $r_2>R$ for some $R>0$. Then

$$a(y_2,\eta_2):=r_2^{-\mu}\sum_{j+|\alpha|\le\mu}a_{j\alpha}(r_2,y_2)\Big(-r_2\frac{\partial}{\partial r_2}\Big)^j(r_2\eta_2)^\alpha$$

is an element of $S^\mu(\Omega_2\times\mathbb{R}^{q_2};H,\widetilde{H})$ for

$$H=\mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge),\widetilde{H}=\mathcal{K}^{s-\mu,(\gamma_1-\mu,\gamma_2-\mu)}(W^\wedge)$$

for every $s,\gamma_1,\gamma_2\in\mathbb{R}$.

Applying Definition 3.2 and Remark 3.4 we can define edge spaces of second generation

$$\mathcal{W}^{s,(\gamma_1,\gamma_2)}(W^\wedge\times\mathbb{R}^{q_2}):=\mathcal{W}^s\Big(\mathbb{R}^{q_2},\mathcal{K}^{s,(\gamma_1,\gamma_2)}(W^\wedge)\Big)\tag{14}$$

and their global versions

$$H^{s,(\gamma_1,\gamma_2)}(M)\tag{15}$$

on every compact $M\in\mathfrak{M}_2$. In (15) do not employ notation like $\mathcal{H},\mathcal{K}$ or $\mathcal{W}$, since these letters are reserved for specific features of the spaces, as in Definition 3.3 or (14). Another reason for the notation (15) is that we do not exclude edges of dimension 0. In this case the role of the group action disappears because corners of that kind are modelled on cones with singular base spaces.

4 Constructions for higher corners

We now show how the constructions are iterative, i.e., admit the step from the singularity order m to $m+1$. To this end we summarise what we need as an input for the iteration. We start from a manifold $M \in \mathfrak{M}_m$, the associated stretched manifold $\mathbb{M}$ and the double $2\mathbb{M} \in \mathfrak{M}_{m-1}$. We assume to have constructed the spaces

$$\mathcal{K}^{s,\gamma}(X_{m-1}^{\wedge}) \quad \text{for} \quad s \in \mathbb{R}, \gamma \in \mathbb{R}^m$$

with $X_{m-1} \in \mathfrak{M}_{m-1}$ being the base of the local model cones for M near $M^{(m)}$. We then need the spaces

$$\mathcal{W}^{s,\gamma}(\mathbb{R}_p \times M), \quad \mathcal{W}^{s,\gamma}_{\text{loc}}(\mathbb{R}_p \times M) \quad \text{and} \quad \mathcal{W}^{s,\gamma}_{\text{cone}}(M^{\wedge}). \tag{16}$$

The definition of $\mathcal{W}^{s,\gamma}(\mathbb{R} \times M)$ employs that we already possess $\mathcal{W}^{s,\gamma'}(\mathbb{R} \times 2\mathbb{M})$ for $\gamma' = (\gamma_1, \ldots, \gamma_{m-1})$ which is the case because $\mathbb{R} \times 2\mathbb{M} \in \mathfrak{M}_{m-1}$. Then $u \in \mathcal{W}^{s,\gamma}(\mathbb{R} \times M)$ is defined by the following conditions:

(i)

$$(1-\omega_m)u \in \mathcal{W}^{s,\gamma'}(\mathbb{R} \times 2\mathbb{M})|_{\mathbb{R}\times\mathbb{M}_{\text{reg}}}$$

for every cut-off function ω_m in the axial variable r_m from the local model cone

$$X_{m-1}^{\Delta} \quad \text{near} \quad Y = M^{(m)};$$

(ii) for every singular chart $\alpha : V \to X_{m-1}^{\Delta} \times \mathbb{R}^{q_m}$ on M near Y,

cf. the formula (3), and the induced map

$$1 \times \alpha_{\text{reg}} : \mathbb{R} \times (V \setminus Y) \to \mathbb{R} \times \mathbb{R}_+ \times X_{m-1} \times \mathbb{R}^{q_m},$$

$$1 \times \alpha_{\text{reg}} : (p, \cdot) \to (p, r_m, x, y_m),$$

we have

$$\varphi(1 \times \alpha_{\text{reg}}^{-1})^* \omega_1 u \in \mathcal{W}^s\Big(\mathbb{R}_p \times \mathbb{R}^{q_m}, \mathcal{K}^{s,\gamma}(X_{m-1}^{\wedge})\Big)$$

for every $\varphi \in C_0^\infty(\mathbb{R}^{q_m})$ and the cut-off function ω_1 from (i).

A slight modification of this construction gives us the space $\mathcal{W}^{s,\gamma}_{\text{loc}}(\mathbb{R} \times M)$ as the space of locally finite sums $\sum_{\iota \in I} \varphi_\iota u_\iota$, $\varphi_\iota \in C_0^\infty(\mathbb{R})$ $u_\iota \in \mathcal{W}^{s,\gamma}(\mathbb{R} \times M)$, similarly to (13).

For the definition of the space $\mathcal{W}^{s,\gamma}_{\text{cone}}(M^{\wedge})$ we set $B := \{y_m \in \mathbb{R}^{q_m} : |y_m| < 1\}$ and consider a singular chart

$$V \to X_{m-1}^{\Delta} \times B$$

on M near a point $y \in Y := M^{(m)}$ and the induced chart $U \to B$, $y \to y_m$, for $U := V \cap Y$. We set $\Gamma := \{(r_{m+1}, \widetilde{y}_m) \in \mathbb{R}^{1+q_m} : r_{m+1} \in \mathbb{R}_+, \widetilde{y}_m = r_{m+1} y_m, y_m \in B\}$ and form

$$\beta_U : (r_m, x, r_{m+1}, y) \to (r_{m+1} r_m, x, r_{m+1}, r_{m+1} y_m) =: (\widetilde{r}_m, x, r_{m+1}, \widetilde{y}_m),$$

$$\beta_U : (\mathbb{R}_+ \times X_{m-1}) \times (\mathbb{R}_+ \times U) \to X^{\wedge}_{\widetilde{r}_m, x} \times \Gamma_{r_{m+1}, \widetilde{y}_m} \subset X^{\wedge}_{\widetilde{r}_m, x} \times \mathbb{R}^{1+q_m}_{r_{m+1}, \widetilde{y}_m}.$$

The space $\mathcal{W}^{s,\gamma}_{\text{cone}}(M^{\wedge})$ is defined to be the set of all

$$u(r_{m+1}, \cdot) \in \mathcal{W}^{s,\gamma}_{\text{loc}}(\mathbb{R} \times M)|_{\mathbb{R}_+ \times M}$$

such that

(i) for every chart $U \to B$, $y \to y_m$ as mentioned before, we have

$$(1-\omega_{m+1})\varphi\omega_m u \circ \beta_U^{-1} \in \mathcal{W}^s\Big(\mathbb{R}^{1+q_m}_{r_{m+1},\tilde{y}_m}, \mathcal{K}^{s,\gamma}(X^\wedge_{m-1})_{\tilde{r}_m,x}\Big)$$

for every $\varphi \in C_0^\infty(U)$ and cut-off functions $\omega_m(r_m), \omega_{m+1}(r_{m+1})$;

(ii)

$$(1-\omega_m)u \in \mathcal{W}^{s,\gamma'}_{\text{cone}}((2\mathbb{M})^\wedge)$$

for a cut-off function $\omega_m(r_m)$.

Definition 4.1 *For every $s \in \mathbb{R}$, $(\gamma, \gamma_{m+1}) \in \mathbb{R}^{m+1}$ we set*

(i)

$$\mathcal{H}^{s,(\gamma,\gamma_{m+1})}(M^\wedge) := \Big(S_{\gamma_{m+1}-\frac{1}{2}(\dim M)}\Big)^{-1}\mathcal{W}^{s,\gamma}(\mathbb{R}_p \times M);$$

(ii)

$$\mathcal{K}^{s,(\gamma,\gamma_{m+1})}(M^\wedge) := \omega_{m+1}\mathcal{H}^{s,(\gamma,\gamma_{m+1})}(M^\wedge) + (1-\omega_{m+1})\mathcal{W}^{s,\gamma}_{\text{cone}}(M^\wedge).$$

Remark 4.2 *There is an anologue of Remark 3.4 with the group action*

$$\kappa_\lambda : \mathcal{K}^{s,(\gamma,\gamma_{m+1})}(M^\wedge) \to \mathcal{K}^{s,(\gamma,\gamma_{m+1})}(M^\wedge), \tag{17}$$

$$\kappa_\lambda : u(r_{m+1},\cdot) \to \lambda^{\frac{1+\dim M}{2}} u(\lambda r_{m+1},\cdot), \quad \lambda \in \mathbb{R}_+. \tag{18}$$

Clearly the factor $\lambda^{\frac{1+\dim M}{2}}$ can be replaced by λ^δ for any other $\delta \in \mathbb{R}$, but we employ $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ in the form (18) because of its role in the definition of higher edge spaces.

Theorem 4.3 *Let*

$$A = r_{m+1}^{-\mu}\sum_{j+|\alpha|\le\mu} a_{j,\alpha}(r_{m+1}, y_{m+1})\Big(-r_{m+1}\frac{\partial}{\partial r_{m+1}}\Big)^j (r_{m+1}D_{y_{m+1}})^\alpha$$

be an operator with coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}_{\deg}(M))$, $\Omega \subseteq \mathbb{R}^{q_{m+1}}$ open. Then

$$\sigma_\wedge(A)(y_{m+1},\eta_{m+1}) := r_{m+1}^{-\mu}\sum_{j+|\alpha|\le\mu} a_{j,\alpha}(0, y_{m+1})\Big(-r_{m+1}\frac{\partial}{\partial r_{m+1}}\Big)^j (r_{m+1}\eta_{m+1})^\alpha$$

represents a family of continuous operators

$$\sigma_\wedge(A)(y_{m+1},\eta_{m+1}) : \mathcal{K}^{s,(\gamma,\gamma_{m+1})}(M^\wedge) \to \mathcal{K}^{s-\mu,(\gamma-\mu,\gamma_{m+1}-\mu)}(M^\wedge)$$

for every $s \in \mathbb{R}$, $(\gamma,\gamma_{m+1}) \in \mathbb{R}^{m+1}$, $(y_{m+1},\eta_{m+1}) \in T^*\Omega \setminus 0$, and we have

$$\sigma_\wedge(A)(y_{m+1},\lambda\eta_{m+1}) = \lambda^\mu\kappa_\lambda\sigma_\wedge(A)(y_{m+1},\eta_{m+1})\kappa_\lambda^{-1}$$

for all $\lambda \in \mathbb{R}_+$.

More details may be found in [18].

Applying Definition 3.2 to $H = \mathcal{K}^{s,(\gamma,\gamma_{m+1})}(M^\wedge)$ and $q := q_{m+1}$ we obtain the higher edge space

$$\mathcal{W}^{s,(\gamma,\gamma_{m+1})}\Big(M^\wedge \times \mathbb{R}^{q_{m+1}}\Big) = \mathcal{W}^s\Big(\mathbb{R}^{q_{m+1}}, \mathcal{K}^{s,(\gamma,\gamma_{m+1})}(M^\wedge)\Big)$$

and we can start the iteration procedure all over again.

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