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Arithmeticity of modular forms over an algebraic number field of finite degree

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0 The case of symplectic group

In this lecture we will consider arithmeticities of modular forms over algebraic number fields which are finite extensions of $\mathbb{Q}$. First let us review the case of symplectic group, which was mainly researched by Shimura in 1970's. (See, [2], [3], [4] and [5].)

For a totally real algebraic number field F of finite degree, define

$$\mathrm{Sp}(l,F)=\left\{\gamma\in\mathrm{GL}(2l,F)\,\middle|\,{}^t\gamma\begin{pmatrix}0&1_l\\-1_l&0\end{pmatrix}\gamma=\begin{pmatrix}0&1_l\\-1_l&0\end{pmatrix}\right\}.$$

As is well known, we have $\det(\gamma)=1$ for any $\gamma\in\mathrm{Sp}(l,F)$. Set

$$\mathfrak{H}_l^{\mathbf{a}}=\left\{z=(z_v)_{v\in\mathbf{a}}\in(\mathbb{C}_l^l)^{\mathbf{a}}\,\middle|\,{}^tz_v=z_v,\ \mathrm{Im}(z_v)>0\text{ for each }v\in\mathbf{a}\right\},$$

where >0 means positive definite and $\mathbf{a}$ denotes the set of all archimedean primes of F. Then $\mathrm{Sp}(l,F)$ acts on $\mathfrak{H}_l^{\mathbf{a}}$ as $\alpha((z_v)_{v\in\mathbf{a}})=((a_vz_v+b_v)(c_vz_v+d_v)^{-1})_{v\in\mathbf{a}}$ with $\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{Sp}(l,F)$ and $a,b,c,d\in F_l^l$. The automorphic factor is defined by

$$\mu_v^{(l)}(\alpha,(z_v)_{v\in\mathbf{a}})=c_vz_v+d_v$$

for each $v\in\mathbf{a}$. For any $k=(k_v)_{v\in\mathbf{a}}\in\mathbb{Z}^{\mathbf{a}}$ and any congruence subgroup Γ of $\mathrm{Sp}(l,F)$, we denote by $\mathcal{M}_k^{(l)}(\Gamma)$ the space of holomorphic functions f on $\mathfrak{H}_l^{\mathbf{a}}$ which satisfy $(f|_k\gamma)=f$ for any $\gamma\in\Gamma$ (and are holomorphic at every cusp if $l=1$ and $F=\mathbb{Q}$). Here $f|_k\gamma$ denotes the holomorphic function on $\mathfrak{H}_l^{\mathbf{a}}$ defined by $(f|_k\gamma)(z)=f(\gamma(z))\prod_{v\in\mathbf{a}}\det(\mu_v^{(l)}(\gamma,z))^{-k_v}$. Let $\mathcal{M}_k^{(l)}$ denote the union of $\mathcal{M}_k^{(l)}(\Gamma)$ for all congruence subgroups Γ of $\mathrm{Sp}(l,F)$.

We also need to recall the Galois action on modular forms for symplectic groups. As is well known, any $f \in \mathcal{M}_k^{(l)}$ has a Fourier expansion as

$$f((z_v)_{v\in\mathbf{a}}) = \sum_{h\in L} c(f,h)\exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(h_v z_v)\right), \qquad (0.1)$$

where L is a certain lattice in the space of symmetric matrices of degree l with coefficients in F. Note that $c(f,h) \neq 0$ only if the real symmetric matrix h_v is semi-positive definite for each $v \in \mathbf{a}$. For $f \in \mathcal{M}_k^{(l)}$ as (0.1) and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists $f^\sigma \in \mathcal{M}_{k^\sigma}^{(l)}$ whose Fourier expansion is

$$f^\sigma((z_v)_{v\in\mathbf{a}}) = \sum_{h\in L} c(f,h)^\sigma \exp\left(2\pi\sqrt{-1}\sum_{v\in\mathbf{a}} \mathrm{tr}(h_v z_v)\right). \qquad (0.2)$$

This fact is stated in §25 of [5], or essentially in [2], [3] and [4].

Then this Galois action has a relation with Hecke operators as follows.

For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, take $\chi(\sigma) \in \prod_p \mathbb{Z}_p^\times \subset \mathbb{Q}_A^\times$ by $[\chi(\sigma)^{-1}, \mathbb{Q}] = \sigma|_{\mathbb{Q}_{ab}}$. Then the following proposition is proved in §26.10 of [5], or essentially in [2], [3] and [4].

Proposition 0.1. *For any $f \in \mathcal{M}_k^{(l)}$ and any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$f^\sigma|_{k^\sigma}\tilde{\alpha} = (f|_k\alpha)^\sigma$$

for any $\alpha, \tilde{\alpha} \in \mathrm{Sp}(l,F)$ such that

$$\alpha \in C\cdot\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)\end{pmatrix}\tilde{\alpha}\begin{pmatrix} 1_l & 0 \\ 0 & \chi(\sigma)\end{pmatrix}^{-1},$$

where C is an open subgroup of $\mathrm{Sp}(l,F_A)$ which fixes f.

We can naturally define the arithmeticity of modular forms using their Fourier coefficients. For any subfield Ω of $\mathbb{C}$, put

$$\begin{aligned}\mathcal{M}_k^{(l)}(\Omega) &= \left\{f \in \mathcal{M}_k^{(l)} \,|\, c(f,h) \in \Omega \text{ for any } h\right\} \\ &= \left\{f \in \mathcal{M}_k^{(l)} \,|\, f^\sigma = f \text{ for any } \sigma \in \mathrm{Aut}(\mathbb{C}/\Omega)\right\}.\end{aligned}$$

In [4], the following was stated.

Proposition 0.2. *For any $f \in \mathcal{M}_k^{(l)}$, the field $\mathbb{Q}(\{c(f,h)\}_h)$ is finitely generated over $\mathbb{Q}$.*

This implies that any $f \in \mathcal{M}_k^{(l)}(\overline{\mathbb{Q}})$ is contained in $\mathcal{M}_k^{(l)}(\Omega)$ for some algebraic number field Ω of finite degree.

Using Proposition 0.1, we can easily verify that this Galois action is compatible with Hecke operators. If (an adelized modular form) $\mathbf{f}$ is a Hecke common eigenform of eigenvalues $\{\lambda(\mathfrak{a})\}_{\mathfrak{a}}$, then $\mathbf{f}^\sigma$ is also a Hecke common eigenform of eigenvalues $\{\lambda(\mathfrak{a})^\sigma\}_{\mathfrak{a}}$. This implies that the eigenvalues of Hecke common eigen cusp forms are contained in some CM-fields. We naturally have the following conjecture.

Conjecture. *Let $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ be non-zero Hecke common eigen cusp forms of weight k with respect to a same congruence subgroup having same eigenvalues. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have*

$$\frac{<\mathbf{g}^{\rho\sigma\rho}, \mathbf{h}^\sigma>}{<\mathbf{f}^{\rho\sigma\rho}, \mathbf{f}^\sigma>} = \left(\frac{<\mathbf{g}, \mathbf{h}>}{<\mathbf{f}, \mathbf{f}>}\right)^\sigma,$$

where $<,>$ means the Peterson inner product and ρ denotes the complex conjugation.

This conjecture was proved by Shimura in [4], in case of Hilbert modular form of weight $k \in 2\mathbb{Z}^{\mathbf{a}}$ or $k - (1,1,\dots,1) \in 2\mathbb{Z}^{\mathbf{a}}$ except for the case $k = (1,1,\dots,1)$. The case of general $\mathrm{Sp}(l,F)$ $(l > 1)$ was proved by Garrett in [1] when $k = (\kappa,\kappa,\dots,\kappa)$ with $\kappa > 2l+1$.

1 The case of unitary group

Now we can consider the analogues for unitary case using the Galois action constructed in [6]. Let the notations be in [7]. Then we have the following relation between the Galois action and the action of unitary group.

Proposition 1.1. *Take any $f \in \mathcal{M}_k(T,\Psi)$ and $(\sigma;T,\Psi;\underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$. Then we have*

$$(f|_k\alpha)^{(\sigma;T,\Psi;\underline{a})} = f^{(\sigma;T,\Psi;\underline{a})}|_{k^\sigma}\tilde{\alpha},$$

for any $\alpha \in \mathrm{U}(T,\Psi)$, $\tilde{\alpha} \in \mathrm{U}(\tilde{T},\Psi\sigma)$ satisfying the following relation $(*)$.

$$(*) \qquad \alpha_{\mathbf{h}} \in C_{\mathbf{h}} \cdot B(\sigma;T,\Psi;\underline{a})\tilde{\alpha}_{\mathbf{h}} B(\sigma;T,\Psi;\underline{a})^{-1},$$

where $\alpha_{\mathbf{h}}$ and $\tilde{\alpha}_{\mathbf{h}}$ denote the non-archimedean components of α and $\tilde{\alpha}$, and $C_{\mathbf{h}}$ is some open compact subgroup of $\mathrm{U}(T,\Psi)_{\mathbf{h}}$ so that $f|_k\gamma = f$ for any $\gamma \in \mathrm{U}(T,\Psi) \cap (\mathrm{U}(T,\Psi)_{\mathbf{a}} \times C_{\mathbf{h}})$.

This can be easily verified from the definition of the Galois action by $(\sigma; T, \Psi; \underline{a})$.

We can naturally define the arithmeticity on the space of modular forms by

$$\mathcal{M}_k(T,\Psi)(\overline{\mathbb{Q}}) = \left\{ f \in \mathcal{M}_k(T,\Psi) \,\middle|\, \begin{array}{l} (f|_k\alpha)|\varepsilon_0(T,\Psi) \in \mathcal{M}_k^{(q)}(\overline{\mathbb{Q}}) \\ \text{for any } \alpha \in \mathrm{U}(T,\Psi) \end{array} \right\}.$$

Then we easily obtain the following lemma. (See, [6].)

Lemma 1.2. $\mathcal{M}_k(T,\Psi) = \mathcal{M}_k(T,\Psi)(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.

Using this Galois action, we can define the arithmeticity of modular forms over an algebraic number field of finite degree. Let $K^*_{(T,\Psi)}$ be the composite field of K^*_Ψ and all $K^*_{\Psi(T,j)}$ $(1 \le j \le m-2q)$, where $\Psi(T,j)$ is the CM-type of K as in [7] and $K^*_{\Psi(T,j)}$ is its reflex field. Clearly the field $K^*_{(T,\Psi)}$ is a CM-field since K^*_Ψ and $K^*_{\Psi(T,j)}$ are so. We denote by ${}^H K^*_{(T,\Psi)}$ the Hilbert class field of $K^*_{(T,\Psi)}$, which is finite degree over $\mathbb{Q}$, but not generally a CM-field. Set

$$\mathcal{G} = \left\{ (\sigma, c) \in \mathrm{Gal}(\overline{\mathbb{Q}}/{}^H K^*_{(T,\Psi)}) \times (K^*_{(T,\Psi)})^\times_A \,\middle|\, \begin{array}{l} c \in \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \\ \left[c^{-1}, K^*_{(T,\Psi)}\right] = \sigma|_{(K^*_{(T,\Psi)})_{ab}} \end{array} \right\}.$$

Then $\mathcal{G}$ is a compact group and can be embedded in $C_{(T,\Psi)}(\mathbb{C})$ by $(\sigma, c) \to (\sigma; T, \Psi; N_{(T,\Psi)}(c))$, where

$$N_{(T,\Psi)}(c) = \begin{pmatrix} & & N'_\Psi \circ N_{K^*_{(T,\Psi)}/K^*_\Psi}(c) \\ & N'_{\Psi(T,1)} \circ N_{K^*_{(T,\Psi)}/K^*_{\Psi(T,1)}}(c) & \\ & & \vdots \\ N'_{\Psi(T,m-2q)} \circ N_{K^*_{(T,\Psi)}/K^*_{\Psi(T,m-2q)}}(c) & & \end{pmatrix}.$$

Then the image of $f \in \mathcal{M}_k(T,\Psi)(\overline{\mathbb{Q}})$ by an element of $\mathcal{G}$ is also contained in $\mathcal{M}_k(T,\Psi)(\overline{\mathbb{Q}})$. We can obtain the following proposition.

Proposition 1.3. *For any $f \in \mathcal{M}_k(T,\Psi)(\overline{\mathbb{Q}})$, the stabilizer of f in $\mathcal{G}$ is an open subgroup.*

This means that the set $f^{\mathcal{G}}$ is finite. Hence we can define the arithmeticity of modular forms over an algebraic number field of finite degree.

Next let us consider Hecke operators. Using Proposition 1.1, we can prove that this action of $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$ is compatible with Hecke operators, that is,

$$(\mathbf{f}|\mathfrak{T}(\mathfrak{a}))^{(\sigma;T,\Psi;\underline{a})} = \mathbf{f}^{(\sigma;T,\Psi;\underline{a})}|\mathfrak{T}(\mathfrak{a}),$$

where $\mathbf{f}$ is an adelized modular form for $\mathrm{U}(T, \Psi)$ and $\mathfrak{T}(\mathfrak{a})$ is a Hecke operator. Note that $\mathfrak{T}(\mathfrak{a})$ in the left hand side is a Hecke operator for $\mathrm{U}(T, \Psi)$, while $\mathfrak{T}(\mathfrak{a})$ in the right hand side is the one for $\mathrm{U}(\tilde{T}, \Psi\sigma)$. Let $\mathbf{f}$ be a cusp form for $\mathrm{U}(T, \Psi)$ and a common eigenform of $\mathfrak{T}(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$ with eigenvalues $\{\lambda(\mathfrak{a})\}_{\mathfrak{a}}$. Then $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}$ is a common eigenform with eigenvalues $\{\lambda(\mathfrak{a})^{\sigma}\}_{\mathfrak{a}}$. Since $\lambda(\mathfrak{a}^{\rho}) = \overline{\lambda(\mathfrak{a})}$ holds, we have $(\overline{\lambda(\mathfrak{a})})^{\sigma} = \overline{\lambda(\mathfrak{a})^{\sigma}}$ for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. This implies $\lambda(\mathfrak{a})$ is contained in a CM-field.

We naturally have the following conjecture.

Conjecture *Let* $0 \neq \mathbf{f}, \mathbf{g}_1$ *and* $\mathbf{g}_2$ *be cusp forms for* $\mathrm{U}(T, \Psi)$ *and common eigenforms of* $\{\mathfrak{T}(\mathfrak{a})\}_{\mathfrak{a}}$ *with respect to a same congruence subgroup, having the same eigenvalue for each* $\mathfrak{a}$. *For any* $(\sigma; T, \Psi; \underline{a}) \in C_{(T,\Psi)}(\mathbb{C})$, *we have*

$$\frac{< \mathbf{g}_1^{(\rho\sigma\rho;T,\Psi;\underline{a}^{\rho})}|C(\sigma;T,\Psi;\underline{a}), \mathbf{g}_2^{(\sigma;T,\Psi;\underline{a})} >}{< \mathbf{f}^{(\rho\sigma\rho;T,\Psi;\underline{a}^{\rho})}|C(\sigma;T,\Psi;\underline{a}), \mathbf{f}^{(\sigma;T,\Psi;\underline{a})} >} = \left\{\frac{< \mathbf{g}_1, \mathbf{g}_2 >}{< \mathbf{f}, \mathbf{f} >}\right\}^{\sigma},$$

where $C(\sigma; T, \Psi; \underline{a}) = B(\rho\sigma\rho; T, \Psi; \underline{a}^{\rho})^{-1}B(\sigma; T, \Psi; \underline{a}) \in \mathrm{U}(\tilde{T}, \Psi\sigma)_{\mathbf{h}}$.

Note that $\mathbf{f}^{(\rho\sigma\rho;T,\Psi;\underline{a}^{\rho})}|C(\sigma; T, \Psi; \underline{a})$ and $\mathbf{f}^{(\sigma;T,\Psi;\underline{a})}$ are modular forms with respect to a same congruence subgroup.

This conjecture is the first step to more precise research of special values of L-functions. If this conjecture is proved, we will be able to show that special values belong to a specified algebraic number field of finite degree.

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