

# STRONG CONVERGENCE THEOREMS AND SUNNY NONEXPANSIVE RETRACTIONS IN BANACH SPACES

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## 1. INTRODUCTION

Let  $E$  be a real Banach space with the topological dual  $E^*$  and let  $C$  be a closed convex subset of  $E$ . Then, a mapping  $T$  of  $C$  into itself is called nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for each  $x, y \in C$ . In 1967, Browder [5] introduced the following iterative scheme for finding a fixed point of a nonexpansive mapping  $T$  in a Banach space:  $x \in C$  and

$$(1.1) \quad x_n = \alpha_n x + (1 - \alpha_n)Tx_n \quad \text{for } n = 1, 2, \dots,$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ . Then, he studied the strong convergence of the sequence. This result for nonexpansive mappings was extended to strong convergence theorems for accretive operators in a Banach space by Reich [16] and Takahashi and Ueda [31]. Reich also [17] studied the following iterative scheme for nonexpansive mappings:  $x = x_1 \in C$  and

$$(1.2) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \quad \text{for } n = 1, 2, \dots,$$

where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$ ; see originally Halpern [9]. Wittmann [32] showed that the sequence generated by (1.2) in a Hilbert space converges strongly to the point of  $F(T)$  which is nearest to  $x$  if  $\{\alpha_n\}$  satisfies  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ . Shioji and Takahashi [21] extended this result to that of a Banach space. In 1997, Shimizu and Takahashi [19, 20] introduced the first iterative schemes for finding common fixed points of families of nonexpansive mappings and obtained convergence theorems for the families. Since then, many authors also have studied iterative schemes for families of various mappings (cf. [1, 3, 24, 25, 29]). In particular, Shioji and Takahashi [23] established strong convergence theorems of the types (1.1) and (1.2) for families of mappings in uniformly convex

Banach spaces with a uniformly Gâteaux differentiable norm by using the theory of means of abstract semigroups; for the theory of means, see [8, 10, 12, 18, 26, 27, 28].

In this paper, motivated by Shioji and Takahashi [21], we study the iterative schemes for commutative nonexpansive semigroups defined on compact sets of general Banach spaces. Using these results, we prove some strong convergence theorems in cases of discrete and one-parameter semigroups.

## 2. PRELIMINARIES

Throughout this paper, we denote by  $\mathbb{N}$  and  $\mathbb{R}_+$  the set of all positive integers and the set of all nonnegative real numbers, respectively. We also denote by  $E$  a real Banach space with the topological dual  $E^*$  and by  $J$  the duality mapping of  $E$ , that is, a multivalued mapping  $J$  of  $E$  into  $E^*$  such that for each  $x \in E$ ,

$$J(x) = \{f \in E^* : f(x) = \|x\|^2 = \|f\|^2\}.$$

A Banach space  $E$  is said to be *smooth* if the duality mapping  $J$  of  $E$  is single-valued. We know that if  $E$  is smooth, then  $J$  is norm to weak-star continuous; for more details, see [30].

Let  $S$  be a semigroup. We denote by  $l^\infty(S)$  the Banach space of all bounded real-valued functions on  $S$  with supremum norm. For each  $s \in S$ , we define two operators  $l(s)$  and  $r(s)$  on  $l^\infty(S)$  by  $(l(s)f)(t) = f(st)$  and  $(r(s)f)(t) = f(ts)$  for each  $t \in S$  and  $f \in l^\infty(S)$ , respectively. Let  $X$  be a subspace of  $l^\infty(S)$  containing 1. An element  $\mu$  of the topological dual  $X^*$  of  $X$  is said to be a *mean* on  $X$  if  $\|\mu\| = \mu(1) = 1$ . For  $s \in S$ , we can define a *point evaluation*  $\delta_s$  by  $\delta_s(f) = f(s)$  for each  $f \in X$ . A convex combination of point evaluations is called a *finite mean* on  $X$ . As is well known,  $\mu$  is a mean on  $X$  if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each  $f \in X$ . Suppose that  $l(s)X \in X$  and  $r(s)X \in X$  for each  $s \in S$ . Then, a mean  $\mu$  on  $X$  is said to be *left invariant* (resp. *right invariant*) if  $\mu(l(s)f) = \mu(f)$  (resp.  $\mu(r(s)f) = \mu(f)$ ) for each  $s \in S$  and  $f \in X$ . A mean  $\mu$  on  $X$  is said to be *invariant* if  $\mu$  is both left and right invariant.  $X$  is said to be *amenable* if there exists an invariant mean on  $X$ . For fixed point theorems for the semigroups, see [13]. We know from [8] that if  $S$  is commutative, then  $X$  is amenable. Let  $\{\mu_\alpha\}$  be a net of means on  $X$ . Then  $\{\mu_\alpha\}$  is said to be *asymptotically invariant* (or *strongly regular*) if for each  $s \in S$ , both  $l^*(s)\mu_\alpha - \mu_\alpha$  and  $r^*(s)\mu_\alpha - \mu_\alpha$  converge to 0 in the weak-star topology (or the norm

topology), where  $l^*(s)$  and  $r^*(s)$  are the adjoint operators of  $l(s)$  and  $r(s)$ , respectively. Such nets were first studied by Day in [8]; see [8, 30] for more details.

Let  $C$  be a closed convex subset of  $E$  and let  $T$  be a mapping of  $C$  into itself. Then  $T$  is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for each  $x, y \in C$ . Let  $S$  be a commutative semigroup with identity  $0$  and let  $ne(C)$  be the set of all nonexpansive mappings of  $C$  into itself. Then  $\mathcal{S} = \{T(s) : s \in S\}$  is called a *nonexpansive semigroup* on  $C$  if  $T(s) \in ne(C)$  for each  $s \in S$ ,  $T(0) = I$  and  $T(s + t) = T(s)T(t)$  for each  $s, t \in S$ . We denote by  $F(\mathcal{S})$  the set of common fixed points of  $\{T(s) : s \in S\}$ .

We denote by  $l^\infty(S, E)$  the Banach space of bounded mappings of  $S$  into  $E$  with supremum norm, and by  $l_c^\infty(S, E)$  the subspace of elements  $f \in l^\infty(S, E)$  such that  $f(S) = \{f(s) : s \in S\}$  is a relatively weakly compact subset of  $E$ . Let  $X$  be a subspace of  $l^\infty(S)$  containing 1 such that  $l(s)X \subset X$  for each  $s \in S$  and let  $X^*$  be the topological dual of  $X$ . Then, for each  $\mu \in X^*$  and  $f \in l_c^\infty(S, E)$ , let us define a continuous linear functional  $\tau(\mu)f$  on  $E^*$  by

$$\tau(\mu)f : x^* \mapsto \mu\langle f(\cdot), x^* \rangle.$$

It follows from the bipolar theorem that  $\tau(\mu)f$  is contained in  $E$ . We know that if  $\mu$  is a mean on  $X$ , then  $\tau(\mu)f$  is contained in the closure of convex hull of  $\{f(s) : s \in S\}$ . We also know that for each  $\mu \in X^*$ ,  $\tau(\mu)$  is a bounded linear mapping of  $l_c^\infty(S, E)$  into  $E$  such that for each  $f \in l_c^\infty(S, E)$ ,  $\|\tau(\mu)f\| \leq \|\mu\|\|f\|$ ; see [11]. Let  $\mathcal{S} = \{T(s) : s \in S\}$  be a nonexpansive semigroup on  $C$  such that  $T(\cdot)x \in l_c^\infty(S, E)$  for some  $x \in C$ . If for each  $x^* \in E^*$  the function  $s \mapsto \langle T(s)x, x^* \rangle$  is contained in  $X$ , then there exists a unique point  $x_0$  of  $E$  such that  $\mu\langle T(\cdot)x, x^* \rangle = \langle x_0, x^* \rangle$  for each  $x^* \in E^*$ ; see [26] and [10]. We denote such a point  $x_0$  by  $T(\mu)x$ . Note that  $T(\mu)x$  is contained in the closure of convex hull of  $\{T(s)x : s \in S\}$  for each  $x \in C$  and  $T(\mu)$  is a nonexpansive mapping of  $C$  into itself; see [26] for more details.

Let  $D$  be a subset of  $C$  and let  $P$  be a retraction of  $C$  onto  $D$ , that is,  $Px = x$  for each  $x \in D$ . Then  $P$  is said to be *sunny* [15] if for each  $x \in C$  and  $t \geq 0$  with  $Px + t(x - Px) \in C$ ,

$$P(Px + t(x - Px)) = Px.$$

A subset  $D$  of  $C$  is said to be a *sunny nonexpansive retract* of  $C$  if there exists a sunny nonexpansive retraction  $P$  of  $C$  onto  $D$ . We know that if  $E$  is smooth and  $P$  is a retraction of  $C$  onto  $D$ , then  $P$  is sunny and nonexpansive if and only if for each  $x \in C$  and  $z \in D$ ,

$$\langle x - Px, J(z - Px) \rangle \leq 0.$$

For more details, see [30].

In order to prove our main theorems, we need the following propositions:

**Proposition 1** ([14]). *Let  $E$  be a Banach space, let  $C$  be a compact convex subset of  $E$ , let  $S$  be a commutative semigroup with identity 0, let  $\mathcal{S} = \{T(s) : s \in S\}$  be a nonexpansive semigroup on  $C$ , let  $X$  be a subspace of  $l^\infty(S)$  containing 1 such that  $l(s)X \subset X$  for each  $s \in S$  and the functions  $s \mapsto \langle T(s)x, x^* \rangle$  and  $s \mapsto \|T(s)x - y\|$  are contained in  $X$  for each  $x, y \in C$  and  $x^* \in E^*$  and let  $\{\mu_n\}$  be an asymptotically invariant sequence of means on  $X$ . If  $z \in C$  and  $\liminf_{n \rightarrow \infty} \|T(\mu_n)z - z\| = 0$ , then  $z$  is a common fixed point of  $\mathcal{S}$ .*

In particular, we can deduce the following result from Proposition 1.

**Proposition 2.** *Let  $E$  be a Banach space, let  $C$  be a compact convex subset of  $E$ , let  $S$  be a commutative semigroup with identity 0, let  $\mathcal{S} = \{T(s) : s \in S\}$  be a nonexpansive semigroup on  $C$ , let  $X$  be a subspace of  $l^\infty(S)$  containing 1 such that  $l(s)X \subset X$  for each  $s \in S$  and the functions  $s \mapsto \langle T(s)x, x^* \rangle$  and  $s \mapsto \|T(s)x - y\|$  are contained in  $X$  for each  $x, y \in C$  and  $x^* \in E^*$  and let  $\mu$  be an invariant mean on  $X$ . Then  $F(\mathcal{S})$  is nonempty and  $F(T(\mu)) = F(\mathcal{S})$ .*

*Proof.* Let  $\mu$  be an invariant mean on  $X$ . It is clear that  $F(\mathcal{S})$  is contained in  $F(T(\mu))$ . So, it suffices to show that  $F(T(\mu)) \subset F(\mathcal{S})$ . Let  $z \in F(T(\mu))$ . Putting  $\mu_n = \mu$  for each  $n \in \mathbb{N}$ ,  $\{\mu_n\}$  is an asymptotically invariant sequence of means on  $X$ . Since we have

$$\liminf_{n \rightarrow \infty} \|T(\mu_n)z - z\| = \|T(\mu)z - z\| = 0,$$

it follows from Proposition 1 that  $z$  is a common fixed point of  $\mathcal{S}$ . This completes the proof.  $\square$

### 3. MAIN RESULTS

Before proving a strong convergence theorem (Theorem 2) of Browder's type for nonexpansive semigroups defined on compact sets in Banach spaces, we establish the following result for sunny nonexpansive retractions.

**Theorem 1.** *Let  $C$  be a compact convex subset of a smooth Banach space  $E$ , let  $S$  be a commutative semigroup with identity 0 and let  $\mathcal{S} = \{T(s) : s \in S\}$  be a nonexpansive semigroup on  $C$ . Suppose that  $X$  is a subspace of  $l^\infty(S)$  containing 1 such that  $l(s)X \subset X$  for each  $s \in S$  and the functions  $s \mapsto \langle T(s)x, x^* \rangle$  and  $s \mapsto \|T(s)x - y\|$  are contained in  $X$  for each  $x, y \in C$  and  $x^* \in E^*$ . Then  $F(\mathcal{S})$  is a sunny*

nonexpansive retract of  $C$  and a sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$  is unique.

*Proof.* Let  $x \in C$  be fixed and let  $\mu$  be an invariant mean on  $X$ . Then, by the Banach contraction principle, we get a sequence  $\{x_n\}$  in  $C$  such that

$$(3.1) \quad x_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T(\mu)x_n$$

for each  $n \in \mathbb{N}$ . We shall show that the sequence  $\{x_n\}$  converges strongly to an element of  $F(\mathcal{S})$ . We have, for each  $z \in F(\mathcal{S})$  and  $n \in \mathbb{N}$ ,

$$\langle x_n - x, J(x_n - z) \rangle \leq 0,$$

where  $J$  is the duality mapping of  $E$ . Indeed, we have, for each  $z \in F(\mathcal{S})$  and  $x^* \in E^*$ ,

$$\langle T(\mu)z, x^* \rangle = \mu \langle T(\cdot)z, x^* \rangle = \mu \langle z, x^* \rangle = \langle z, x^* \rangle$$

and hence  $z = T(\mu)z$  for each  $z \in F(\mathcal{S})$ . Therefore, from (3.1), we have

$$\begin{aligned} \langle x_n - x, J(x_n - z) \rangle &= (n-1) \langle T(\mu)x_n - x_n, J(x_n - z) \rangle \\ &= (n-1) (\langle T(\mu)x_n - T(\mu)z, J(x_n - z) \rangle \\ &\quad + \langle z - x_n, J(x_n - z) \rangle) \\ &\leq (n-1) (\|T(\mu)x_n - T(\mu)z\| \|x_n - z\| - \|x_n - z\|^2) \\ &\leq (n-1) (\|x_n - z\|^2 - \|x_n - z\|^2) \\ &= 0. \end{aligned}$$

Further, from (3.1), we have, for each  $n \in \mathbb{N}$ ,

$$\|x_n - T(\mu)x_n\| = \frac{1}{n} \|x - T(\mu)x_n\|$$

and hence  $\lim_{n \rightarrow \infty} \|x_n - T(\mu)x_n\| = 0$ . Let  $\{x_{n_i}\}$  and  $\{x_{n_j}\}$  be subsequences of  $\{x_n\}$  such that  $\{x_{n_i}\}$  and  $\{x_{n_j}\}$  converges strongly to  $y$  and  $z$ , respectively. We have, for each  $i \in \mathbb{N}$ ,

$$\begin{aligned} \|y - T(\mu)y\| &\leq \|y - x_{n_i}\| + \|x_{n_i} - T(\mu)x_{n_i}\| + \|T(\mu)x_{n_i} - T(\mu)y\| \\ &\leq 2\|y - x_{n_i}\| + \|x_{n_i} - T(\mu)x_{n_i}\|, \end{aligned}$$

and hence  $y = T(\mu)y$ . By Proposition 2, we have  $y \in F(\mathcal{S})$ . Similarly, we have  $z \in F(\mathcal{S})$ . Further, we have

$$\langle y - x, J(y - z) \rangle = \lim_{i \rightarrow \infty} \langle x_{n_i} - x, J(x_{n_i} - z) \rangle \leq 0.$$

Similarly, we have  $\langle z - x, J(z - y) \rangle \leq 0$  and hence  $y = z$ . Thus,  $\{x_n\}$  converges strongly to an element of  $F(\mathcal{S})$ . Let us define a mapping  $P$  of  $C$  into itself by  $Px = \lim_{n \rightarrow \infty} x_n$ . Then, we have, for each  $z \in F(\mathcal{S})$ ,

$$(3.2) \quad \langle x - Px, J(z - Px) \rangle = \lim_{n \rightarrow \infty} \langle x_n - x, J(x_n - z) \rangle \leq 0.$$

It follows from [30, Lemma 5.1.6] that  $P$  is a sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ .

Let  $Q$  be another sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ . Then, from [30, p.199], we have, for each  $x \in C$  and  $z \in F(\mathcal{S})$ ,

$$(3.3) \quad \langle x - Qx, J(z - Qx) \rangle \leq 0.$$

Putting  $z = Qx$  in (3.2) and  $z = Px$  in (3.3), we have

$$\langle x - Px, J(Qx - Px) \rangle \leq 0 \quad \text{and} \quad \langle x - Qx, J(Px - Qx) \rangle \leq 0$$

and hence  $\langle Qx - Px, J(Qx - Px) \rangle \leq 0$ . This implies  $Qx = Px$ . This completes the proof.  $\square$

**Theorem 2.** *Let  $C$  be a compact convex subset of a smooth Banach space  $E$ , let  $S$  be a commutative semigroup with identity 0, let  $\mathcal{S} = \{T(s) : s \in S\}$  be a nonexpansive semigroup on  $C$ , let  $X$  be a subspace of  $l^\infty(S)$  containing 1 such that  $l(s)X \subset X$  for each  $s \in S$  and the functions  $s \mapsto \langle T(s)x, x^* \rangle$  and  $s \mapsto \|T(s)x - y\|$  are contained in  $X$  for each  $x, y \in C$  and  $x^* \in E^*$  and let  $\{\mu_n\}$  be an asymptotically invariant sequence of means on  $X$ . Let  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  such that  $0 < \alpha_n < 1$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Let  $x \in C$  and let  $\{x_n\}$  be the sequence defined by*

$$(3.4) \quad x_n = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad n = 1, 2, 3, \dots$$

*Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ .*

*Proof.* Let  $P$  be a sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ . Let  $\{x_{n_k}\}$  be a subsequence of  $\{x_n\}$  such that  $\{x_{n_k}\}$  converges strongly to an element  $y$  of  $C$ . Then, we have  $y \in F(\mathcal{S})$ . Indeed, from (3.4), we have

$$\|x_n - T(\mu_n)x_n\| = \frac{\alpha_n}{1 - \alpha_n} \|x_n - x\|$$

and hence  $\lim_{n \rightarrow \infty} \|x_n - T(\mu_n)x_n\| = 0$ . Thus, we have

$$\begin{aligned} \|y - T(\mu_{n_k})y\| &\leq \|y - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| + \|T(\mu_{n_k})x_{n_k} - T(\mu_{n_k})y\| \\ &\leq 2\|y - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\|, \end{aligned}$$

and hence  $0 \leq \liminf_{n \rightarrow \infty} \|y - T(\mu_n)y\| \leq \lim_{k \rightarrow \infty} \|y - T(\mu_{n_k})y\| = 0$ . This implies  $\liminf_{n \rightarrow \infty} \|y - T(\mu_n)y\| = 0$ . By Proposition 1, we have  $y \in F(\mathcal{S})$ .

On the other hand, as in the proof of Theorem 1, we have, for each  $z \in F(\mathcal{S})$ ,

$$\langle x_n - x, J(x_n - z) \rangle = \frac{1 - \alpha_n}{\alpha_n} \langle T(\mu_n)x_n - x_n, J(x_n - z) \rangle \leq 0$$

and hence  $\langle y - x, J(y - z) \rangle \leq 0$ . Then, since  $P$  is a sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ , we have

$$\begin{aligned} \|y - Px\|^2 &= \langle y - Px, J(y - Px) \rangle \\ &= \langle y - x, J(y - Px) \rangle + \langle x - Px, J(y - Px) \rangle \\ &\leq \langle x - Px, J(y - Px) \rangle \\ &\leq 0 \end{aligned}$$

and hence  $y = Px$ . Therefore,  $\{x_n\}$  converges strongly to  $Px$ . It completes the proof.  $\square$

Next, we prove a strong convergence theorem of Halpern's type for nonexpansive semigroups defined on compact sets in Banach spaces.

**Theorem 3.** *Let  $C$  be a compact convex subset of a strictly convex and smooth Banach space  $E$ , let  $S$  be a commutative semigroup with identity 0 and let  $\mathcal{S} = \{T(s) : s \in S\}$  be a nonexpansive semigroup on  $C$ , let  $X$  be a subspace of  $l^\infty(S)$  containing 1 such that  $l(s)X \subset X$  for each  $s \in S$  and the functions  $s \mapsto \langle T(s)x, x^* \rangle$  and  $s \mapsto \|T(s)x - y\|$  are contained in  $X$  for each  $x, y \in C$  and  $x^* \in E^*$  and let  $\{\mu_n\}$  be a strongly regular sequence of means on  $X$ . Let  $\{\alpha_n\}$  be a sequence in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be the sequence defined by*

$$(3.5) \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \quad n = 1, 2, \dots$$

*Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  denotes a unique sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ .*

*Proof.* We know from [6, 7, 2] that for each  $n \in \mathbb{N}$ , there exists a strictly increasing, continuous and convex function  $\gamma$  of  $[0, \infty)$  into itself with  $\gamma(0) = 0$  such that for each  $T \in ne(C)$ ,  $x_i \in C$  ( $n = 1, \dots, n$ ), and  $c_i \geq 0$  ( $n = 1, \dots, n$ ) with  $\sum_{i=1}^n c_i = 1$ ,

$$\gamma \left( \left\| \sum_{i=1}^n c_i T x_i - T \sum_{i=1}^n c_i x_i \right\| \right) \leq \max_{1 \leq i, j \leq n} (\|x_i - x_j\| - \|T x_i - T x_j\|)$$

where  $ne(C)$  denotes the set of nonexpansive mappings of  $C$  into itself. Let  $\epsilon > 0$ . As in the proof of Shioji and Takahashi [22, Lemma 1], there exists  $\delta > 0$  such that for each  $T \in ne(C)$ ,

$$\overline{\text{co}}(F_\delta(T)) + 2B_\delta \subset F_\epsilon(T),$$

where for  $r > 0$ ,  $F_r(T) = \{x \in E : \|x - Tx\| \leq r\}$ ,  $B_r = \{x \in E : \|x\| \leq r\}$  and  $\overline{\text{co}}A$  denotes the closure of convex hull of a subset  $A$  of  $E$ . We also know from [2, Corollary 2.8] that

$$\lim_{n \rightarrow \infty} \sup_{T \in ne(C)} \sup_{x \in C} \left\| \frac{1}{n+1} \sum_{i=0}^n T^i x - T \left( \frac{1}{n+1} \sum_{i=0}^n T^i x \right) \right\| = 0.$$

Then, there exists  $N \in \mathbb{N}$  such that for each  $n \geq N$ ,  $T \in ne(C)$  and  $x \in C$ ,

$$\left\| \frac{1}{n+1} \sum_{i=0}^n T^i x - T \left( \frac{1}{n+1} \sum_{i=0}^n T^i x \right) \right\| \leq \delta.$$

Hence, we have, for each  $n \geq N$ ,  $s, t \in S$  and  $x \in C$ ,

$$\left\| \frac{1}{n+1} \sum_{i=0}^n (T(s))^i T(t)x - T(s) \left( \frac{1}{n+1} \sum_{i=0}^n (T(s))^i T(t)x \right) \right\| \leq \delta.$$

Let  $s \in S$  be fixed and let us define a finite mean  $\lambda$  on  $X$  by

$$\lambda = \frac{1}{N+1} \sum_{i=0}^N \delta(is),$$

where  $0s = 0$ . Then, we have, for each  $t \in S$ ,

$$T(l^*(t)\lambda)x = \frac{1}{N+1} \sum_{i=0}^N (T(s))^i T(t)x \in F_\delta(T(s)).$$

If  $\mu$  is a mean on  $X$ , then we have, for each  $x \in C$ ,

$$\begin{aligned} \tau(\lambda)(T(l^*(\cdot)\mu)x) &= \frac{1}{N+1} \sum_{i=0}^N T(l^*(is)\mu)x \\ &= \frac{1}{N+1} \sum_{i=0}^N \tau(l^*(is)\mu)(T(\cdot)x) \\ &= \frac{1}{N+1} \sum_{i=0}^N \tau(\mu)(T(is + \cdot)x) \\ &= \tau(\mu) \left( \frac{1}{N+1} \sum_{i=0}^N T(is + \cdot)x \right) \\ &= \tau(\mu)(T(l^*(\cdot)\lambda)x) \\ &\in \overline{\text{co}}F_\delta(T(s)). \end{aligned}$$



Let  $R = \sup_{x \in C} \|x\|$ . Since  $\{\mu_n\}$  is strongly regular, there exists  $N_1 \geq N$  such that for each  $n \geq N_1$  and  $i = 0, \dots, N$ ,

$$\|\mu_n - l^*(is)\mu_n\| \leq \delta/R.$$

Then, we have, for each  $n \geq N_1$ ,

$$\begin{aligned} & \|T(\mu_n)x_n - \tau(\lambda)(T(l^*(\cdot)\mu_n)x_n)\| \\ &= \sup_{\|x^*\|=1} \left( \mu_n \langle T(\cdot)x_n, x^* \rangle - \frac{1}{N+1} \sum_{i=0}^N \mu_n \langle T(is + \cdot)x_n, x^* \rangle \right) \\ &\leq \frac{1}{N+1} \sum_{i=0}^N \sup_{\|x^*\|=1} (\mu_n \langle T(\cdot)x_n, x^* \rangle - \mu_n \langle T(is + \cdot)x_n, x^* \rangle) \\ &\leq \frac{1}{N+1} \sum_{i=0}^N \|\mu_n - l^*(is)\mu_n\| R \\ &\leq \delta. \end{aligned}$$

From  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , we can take  $N_2 \geq N_1$  such that for each  $n \geq N_2$ ,  $\alpha_n \leq \delta/2R$ . Then, we have, for each  $n \geq N_2$ ,

$$\|x_{n+1} - T(\mu_n)x_n\| \leq \alpha_n \|x - T(\mu_n)x_n\| \leq \delta$$

and hence

$$\begin{aligned} x_{n+1} &= (x_{n+1} - T(\mu_n)x_n) + (T(\mu_n)x_n - \tau(\lambda)(T(l^*(\cdot)\mu_n)x_n)) \\ &\quad + \tau(\lambda)(T(l^*(\cdot)\mu_n)x_n) \\ &\in \overline{\text{co}}(F_\delta(T(s))) + 2B_\delta \\ &\subset F_\epsilon(T(s)). \end{aligned}$$

This implies that for each  $s \in S$ ,

$$\limsup_{n \rightarrow \infty} \|T(s)x_n - x_n\| \leq \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, we have, for each  $s \in S$ ,

$$\lim_{n \rightarrow \infty} \|T(s)x_n - x_n\| = 0.$$

On the other hand, we know from Theorem 1 that there exists a unique sunny nonexpansive retraction  $P$  of  $C$  onto  $F(S)$ . Next, we shall show that

$$\limsup_{n \rightarrow \infty} \langle x - Px, J(x_n - Px) \rangle \leq 0.$$

Let

$$r = \limsup_{n \rightarrow \infty} \langle x - Px, J(x_n - Px) \rangle.$$

Since  $C$  is compact, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\lim_{k \rightarrow \infty} \langle x - Px, J(x_{n_k} - Px) \rangle = r$$

and  $\{x_{n_k}\}$  converges strongly to some  $z \in C$ . Then, we have, for each  $k \in \mathbb{N}$  and  $s \in S$ ,

$$\begin{aligned} \|T(s)z - z\| &\leq \|T(s)z - T(s)x_{n_k}\| + \|T(s)x_{n_k} - x_{n_k}\| + \|x_{n_k} - z\| \\ &\leq 2\|x_{n_k} - z\| + \|T(s)x_{n_k} - x_{n_k}\| \end{aligned}$$

and hence

$$\|T(s)z - z\| \leq 2 \lim_{k \rightarrow \infty} \|x_{n_k} - z\| + \lim_{k \rightarrow \infty} \|T(s)x_{n_k} - x_{n_k}\| \leq 0.$$

This implies  $z \in F(S)$ . Since  $E$  is smooth, we have

$$r = \lim_{k \rightarrow \infty} \langle x - Px, J(x_{n_k} - Px) \rangle = \langle x - Px, J(z - Px) \rangle \leq 0.$$

From (3.5) and [30, p.99], we have, for each  $n \in \mathbb{N}$ ,

$$(1 - \alpha_n)^2 \|T_{\mu_n} x_n - Px\|^2 - \|x_{n+1} - Px\|^2 \geq -2\alpha_n \langle x - Px, J(x_{n+1} - Px) \rangle.$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - Px\|^2 &\leq (1 - \alpha_n) \|T_{\mu_n} x_n - Px\|^2 + 2\alpha_n \langle x - Px, J(x_{n+1} - Px) \rangle. \end{aligned}$$

Let  $\epsilon > 0$ . Then, there exists  $m \in \mathbb{N}$  such that

$$\langle x - Px, J(x_n - Px) \rangle \leq \frac{\epsilon}{2}$$

for each  $n \geq m$ . We have, for each  $n \geq m$ ,

$$\begin{aligned} \|x_{n+1} - Px\|^2 &\leq (1 - \alpha_n) \|x_n - Px\|^2 + \epsilon(1 - (1 - \alpha_n)) \\ &\leq (1 - \alpha_n) ((1 - \alpha_{n-1}) \|x_{n-1} - Px\|^2 + \epsilon(1 - (1 - \alpha_{n-1}))) \\ &\quad + \epsilon(1 - (1 - \alpha_n)) \\ &\leq (1 - \alpha_n)(1 - \alpha_{n-1}) \|x_{n-1} - Px\|^2 \\ &\quad + \epsilon(1 - (1 - \alpha_n)(1 - \alpha_{n-1})) \\ &\leq \prod_{k=m}^n (1 - \alpha_k) \|x_m - Px\|^2 + \epsilon(1 - \prod_{k=m}^n (1 - \alpha_k)). \end{aligned}$$

Thus, we have

$$\limsup_{n \rightarrow \infty} \|x_n - Px\|^2 \leq \prod_{k=m}^{\infty} (1 - \alpha_k) \|x_m - Px\|^2 + \epsilon(1 - \prod_{k=m}^{\infty} (1 - \alpha_k))$$

and hence  $\limsup_{n \rightarrow \infty} \|x_n - Px\|^2 \leq \epsilon$ . Since  $\epsilon > 0$  is arbitrary, we have  $\lim_{n \rightarrow \infty} \|x_n - Px\|^2 = 0$ . This completes the proof.  $\square$

## 4. SOME STRONG CONVERGENCE THEOREMS

In this section, applying generalized strong convergence theorems for nonexpansive semigroups in Section 3, we obtain some strong convergence theorems for nonexpansive mappings and one-parameter nonexpansive semigroups in general Banach spaces.

**Theorem 4.** *Let  $C$  be a compact convex subset of a smooth Banach space  $E$  and let  $S$  and  $T$  be nonexpansive mappings of  $C$  into itself with  $ST = TS$ . Let  $x \in C$  and let  $\{x_n\}$  be a sequence defined by*

$$x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n$$

for each  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $0 < \alpha_n < 1$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(S) \cap F(T)$ .

*Proof.* Let  $T(i, j) = S^i T^j$  for each  $i, j \in \mathbb{N} \cup \{0\}$ . Since  $S^i$  and  $T^j$  are nonexpansive for each  $i, j \in \mathbb{N} \cup \{0\}$  and  $ST = TS$ ,  $\{T(i, j) : i, j \in \mathbb{N} \cup \{0\}\}$  is a nonexpansive semigroup on  $C$ . For each  $n \in \mathbb{N}$ , let us define

$$\mu_n(f) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(i, j)$$

for each  $f \in l^\infty((\mathbb{N} \cup \{0\})^2)$ . Then,  $\{\mu_n\}$  is an asymptotically invariant sequence of means; for more details, see [30]. Next, for each  $x \in C$ ,  $x^* \in E^*$  and  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \mu_n \langle T(\cdot)x, x^* \rangle &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle S^i T^j x, x^* \rangle \\ &= \left\langle \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x, x^* \right\rangle. \end{aligned}$$

Then, we have

$$T(\mu_n)x = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x$$

for each  $n \in \mathbb{N}$ . Therefore, it follows from Theorem 2 that  $\{x_n\}$  converges strongly to  $Px$ . This completes the proof.  $\square$

**Theorem 5.** *Let  $C$  be a compact convex subset of a smooth Banach space  $E$  and let  $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$  be a strongly continuous nonexpansive semigroup on  $C$ . Let  $x \in C$  and let  $\{x_n\}$  be a sequence define*

by

$$x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds$$

for each  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $0 < \alpha_n < 1$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\{t_n\}$  is an increasing sequence in  $(0, \infty]$  such that  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $\lim_{n \rightarrow \infty} t_n/t_{n+1} = 1$ . Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ .

*Proof.* For  $n \in \mathbb{N}$ , let us define

$$\mu_n(f) = \frac{1}{t_n} \int_0^{t_n} f(t) dt$$

for each  $f \in C(\mathbb{R}_+)$ , where  $C(\mathbb{R}_+)$  denote the space of all real-valued bounded continuous functions on  $\mathbb{R}_+$  with supremum norm. Then,  $\{\mu_n\}$  is an asymptotically invariant sequence of means; for more details, see [30]. Further, for each  $x \in C$  and  $x^* \in E^*$ , we have

$$\begin{aligned} \mu_n \langle T(\cdot)x, x^* \rangle &= \frac{1}{t_n} \int_0^{t_n} \langle T(s)x, x^* \rangle ds \\ &= \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)x ds, x^* \right\rangle. \end{aligned}$$

Then, we have

$$T(\mu_n)x = \frac{1}{t_n} \int_0^{t_n} T(s)x ds.$$

Therefore, it follows from Theorem 2 that  $\{x_n\}$  converges strongly to  $Px$ . This completes the proof.  $\square$

**Theorem 6.** Let  $C$  be a compact convex subset of a smooth Banach space  $E$  and let  $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$  be a strongly continuous nonexpansive semigroup on  $C$ . Let  $x \in C$  and let  $\{x_n\}$  be a sequence define by

$$x_n = \alpha_n x + (1 - \alpha_n) r_n \int_0^\infty \exp(-r_n s) T(s)x_n ds$$

for each  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $0 < \alpha_n < 1$  and  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\{r_n\}$  is a decreasing sequence in  $(0, \infty]$  such that  $\lim_{n \rightarrow \infty} r_n = 0$ . Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ .

*Proof.* For  $n \in \mathbb{N}$ , let us define

$$\mu_n(f) = r_n \int_0^\infty \exp(-r_n t) f(t) dt$$

for each  $f \in C(\mathbb{R}_+)$ . Then,  $\{\mu_n\}$  is an asymptotically invariant sequence of means; for more details, see [30]. Further, for each  $x \in C$  and  $x^* \in E^*$ , we have

$$\begin{aligned}\mu_n \langle T(\cdot)x, x^* \rangle &= r_n \int_0^\infty \exp(-r_n t) \langle T(t)x, x^* \rangle dt \\ &= \left\langle r_n \int_0^\infty \exp(-r_n t) T(t)x dt, x^* \right\rangle.\end{aligned}$$

Then, we have

$$T(\mu_n)x = r_n \int_0^\infty \exp(-r_n t) T(t)x dt.$$

Therefore, it follows from Theorem 2 that  $\{x_n\}$  converges strongly to  $Px$ . This completes the proof.  $\square$

**Theorem 7.** *Let  $C$  be a compact convex subset of a strictly convex and smooth Banach space  $E$  and let  $S$  and  $T$  be nonexpansive mappings of  $C$  into itself with  $ST = TS$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n$$

for each  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^\infty \alpha_n = \infty$ . Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(S) \cap F(T)$ .

*Proof.* Let  $T(i, j) = S^i T^j$  for each  $i, j \in \mathbb{N} \cup \{0\}$ . Since  $S^i$  and  $T^j$  are nonexpansive for each  $i, j \in \mathbb{N} \cup \{0\}$  and  $ST = TS$ ,  $\{T(i, j) : i, j \in \mathbb{N} \cup \{0\}\}$  is a nonexpansive semigroup on  $C$ . For each  $n \in \mathbb{N}$ , let us define

$$\mu_n(f) = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(i, j)$$

for each  $f \in l^\infty((\mathbb{N} \cup \{0\})^2)$ . Then,  $\{\mu_n\}$  is a strongly regular sequence of means; for more details, see [30]. Next, for each  $x \in C$ ,  $x^* \in E^*$  and  $n \in \mathbb{N}$ , we have

$$\begin{aligned}\mu_n \langle T(\cdot)x, x^* \rangle &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle S^i T^j x, x^* \rangle \\ &= \left\langle \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x, x^* \right\rangle.\end{aligned}$$

Then, we have

$$T(\mu_n)x = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x$$

for each  $n \in \mathbb{N}$ . Therefore, it follows from Theorem 3 that  $\{x_n\}$  converges strongly to  $Px$ . This completes the proof.  $\square$

**Theorem 8.** *Let  $C$  be a compact convex subset of a strictly convex and smooth Banach space  $E$  and let  $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$  be a strongly continuous nonexpansive semigroup on  $C$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds$$

for each  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and  $\{t_n\}$  is an increasing sequence in  $(0, \infty]$  such that  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $\lim_{n \rightarrow \infty} t_n/t_{n+1} = 1$ . Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(\mathcal{S})$ .

*Proof.* For  $n \in \mathbb{N}$ , let us define

$$\mu_n(f) = \frac{1}{t_n} \int_0^{t_n} f(t) dt$$

for each  $f \in C(\mathbb{R}_+)$ , where  $C(\mathbb{R}_+)$  denote the space of all real-valued bounded continuous functions on  $\mathbb{R}_+$  with supremum norm. Then,  $\{\mu_n\}$  is a strongly regular sequence of means; for more details, see [30]. Further, for each  $x \in C$  and  $x^* \in E^*$ , we have

$$\begin{aligned} \mu_n \langle T(\cdot)x, x^* \rangle &= \frac{1}{t_n} \int_0^{t_n} \langle T(s)x, x^* \rangle ds \\ &= \left\langle \frac{1}{t_n} \int_0^{t_n} T(s)x ds, x^* \right\rangle. \end{aligned}$$

Then, we have

$$T(\mu_n)x = \frac{1}{t_n} \int_0^{t_n} T(s)x ds.$$

Therefore, it follows from Theorem 3 that  $\{x_n\}$  converges strongly to  $Px$ . This completes the proof.  $\square$

**Theorem 9.** *Let  $C$  be a compact convex subset of a strictly convex and smooth Banach space  $E$  and let  $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$  be a strongly*

continuous nonexpansive semigroup on  $C$ . Let  $x_1 = x \in C$  and let  $\{x_n\}$  be a sequence defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) r_n \int_0^\infty \exp(-r_n s) T(s) x_n ds$$

for each  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^\infty \alpha_n = \infty$  and  $\{r_n\}$  is a decreasing sequence in  $(0, \infty]$  such that  $\lim_{n \rightarrow \infty} r_n = 0$ . Then  $\{x_n\}$  converges strongly to  $Px$ , where  $P$  is a unique sunny nonexpansive retraction of  $C$  onto  $F(S)$ .

*Proof.* For  $n \in \mathbb{N}$ , let us define

$$\mu_n(f) = r_n \int_0^\infty \exp(-r_n t) f(t) dt$$

for each  $f \in C(\mathbb{R}_+)$ . Then,  $\{\mu_n\}$  is a strongly regular sequence of means; for more details, see [30]. Further, for each  $x \in C$  and  $x^* \in E^*$ , we have

$$\begin{aligned} \mu_n \langle T(\cdot)x, x^* \rangle &= r_n \int_0^\infty \exp(-r_n t) \langle T(t)x, x^* \rangle dt \\ &= \left\langle r_n \int_0^\infty \exp(-r_n t) T(t)x dt, x^* \right\rangle. \end{aligned}$$

Then, we have

$$T(\mu_n)x = r_n \int_0^\infty \exp(-r_n t) T(t)x dt.$$

Therefore, it follows from Theorem 3 that  $\{x_n\}$  converges strongly to  $Px$ . This completes the proof.  $\square$

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