数理解析研究所講究録 1400 巻 2004 年 39-87

COMPACTIFICATION OF THE ISOSPECTRAL VARIETIES OF NILPOTENT TODA LATTICES

LUIS CASIAN AND YUJI KODAMA

ABSTRACT. The paper concerns a compactification of the isospectral varieties of nilpotent Toda lattices for real split simple Lie algebras. The compactification is obtained by taking the closure of unipotent group orbits in the flag manifolds. The unipotent group orbits are called the Peterson varieties and can be used in the complex case to describe the quantum cohomology of Grassmannian manifolds. We construct a chain complex based on a cell decomposition consisting of the subsystems of Toda lattices. Explicit formulae for the incidence numbers of the chain complex are found, and encoded in a graph containing an edge whenever an incidence number is non-zero. We then compute rational cohomology, and show that there are just three different patterns in the calculation of Betti numbers.

Although these compactified varieties are singular, they resemble certain smooth Schubert varieties e.g. they both have a cell decomposition consiting of unipotent group orbits of the same dimensions. In particular, for the case of a Lie algebra of type A the rational homology/cohomology obtained from the compactified isospectral variety of the nilpotent Toda lattice equals that of the corresponding Schubert variety.

1. INTRODUCTION

Let $\mathfrak{g}$ denote a real split semisimple Lie algebra of rank l. We fix a split Cartan subalgebra $\mathfrak{h}$ with root system $\Delta = \Delta(\mathfrak{g},\mathfrak{h}) = \Delta^+ \cup \Delta^-$, real root vectors e_{α_i} associated with simple roots $\Pi = \{\alpha_i \mid i = 1, \cdots, l\}$. We also denote $\{h_{\alpha_i}, e_{\pm\alpha_i}\}$ the Cartan-Chevalley basis of the algebra $\mathfrak{g}$ which satisfies the relations,

$$[h_{\alpha_i}, h_{\alpha_j}] = 0, \quad [h_{\alpha_i}, e_{\pm\alpha_j}] = \pm C_{j,i} e_{\pm\alpha_j}\ , \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_j},$$

where $(C_{i,j})$ is the $l \times l$ Cartan matrix of the Lie algebra $\mathfrak{g}$ and $C_{i,j} = \alpha_i(h_{\alpha_j})$. The Lie algebra $\mathfrak{g}$ admits the decomposition,

$$\mathfrak{g} = \mathcal{N}^- \oplus \mathfrak{h} \oplus \mathcal{N}^+ = \mathcal{N}^- \oplus \mathcal{B}^+ = \mathcal{B}^- \oplus \mathcal{N}^+\,,$$

where $\mathcal{N}^\pm$ are nilpotent subalgebras defined as $\mathcal{N}^\pm = \sum_{\alpha\in\Delta^\pm} \mathbb{R} e_\alpha$ with root vectors e_α, and $\mathcal{B}^\pm = \mathcal{N}^\pm \oplus \mathfrak{h}$ are Borel subalgebras of $\mathfrak{g}$

1.1. **The generalized Toda lattices.** The Toda lattice equation related to the Lie algebra $\mathfrak{g}$ is defined by the Lax equation, [3, 13],

$$\frac{dL}{dt} = [L, A] \tag{1.1}$$

Key words and phrases. integrable systems, algebraic geometry, representation theory.

where L is a Jacobi element of $\mathfrak{g}$, and A is the $\mathcal{N}^-$-projection of L, denoted by $\Pi_{\mathcal{N}^-}L$,

$$(1.2)\qquad \begin{cases} L(t) = \sum_{i=1}^{l} b_i(t)h_{\alpha_i} + \sum_{i=1}^{l}(a_i(t)e_{-\alpha_i} + e_{\alpha_i}) \\ A(t) = \Pi_{\mathcal{N}^-}L = \sum_{i=1}^{l} a_i(t)e_{-\alpha_i} \end{cases}$$

The Lax equation (1.1) then gives the equations of the functions $\{(a_i(t), b_i(t)) \mid i = 1, \cdots, l\}$,

$$(1.3)\qquad \begin{cases} \dfrac{db_i}{dt} = a_i \\ \dfrac{da_i}{dt} = -\left(\sum_{j=1}^{l} C_{i,j}b_j\right) a_i \end{cases}$$

The integrability of the system can be shown by the existence of the Chevalley invariants, $\{I_k(L) : k = 1, \cdots, l\}$, which are given by the homogeneous polynomial of $\{(a_i, b_i) : i = 1, \cdots, l\}$. Those invariant polynomials also define the commutative equations of the Toda equation (1.1),

$$(1.4)\qquad \frac{\partial L}{\partial t_k} = [L, \Pi_{\mathcal{N}^-}\nabla I_k(L)] \quad \text{for} \quad k = 1, \cdots, l,$$

where ∇ is the gradient with respect to the Killing form, i.e. for any $x \in \mathfrak{g}$, $dI_k(L)(x) = K(\nabla I_k(L), x)$. For example, in the case of $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$, the invariants $I_k(L)$ and the gradients $\nabla I_k(L)$ are given by

$$I_k(L) = \frac{1}{k+1}\mathrm{tr}(L^{k+1}) \quad \text{and} \quad \nabla I_k(L) = L^k.$$

The set of commutative equations is called the Toda lattice hierarchy.

In this paper we are concerned with the *real* isospectral manifold defined by

$$Z(\gamma)_{\mathbb{R}} = \left\{(a_1, \cdots, a_l, b_1, \cdots, b_l) \in \mathbb{R}^{2l} \ : \ I_k(L) = \gamma_k \in \mathbb{R}, \ k = 1, \cdots, l\right\}.$$

The manifold $Z(\gamma)_{\mathbb{R}}$ can be compactified by adding the set of points corresponding to the *blow-ups* of the solution $\{(a_i, b_i)\}$. The set of blow-ups has been shown to be characterized by the intersections with the Bruhat cells of the flag manifold G/B^+, which are referred to as the *Painlevé divisors*, and the compactification is described in the flag manifold [9]. In order to explain some details of this fact, we first define the set $\mathcal{F}_\gamma$,

$$\mathcal{F}_\gamma := \{L \in e_+ + \mathcal{B}^- \mid I_k(L) = \gamma_k, \ k = 1, \cdots, l\},$$

where $e_+ = \sum_{i=1}^{l} e_{\alpha_i} \in \mathcal{N}^+$. Then there exists a unique element $n_0 \in N^-$, the unipotent subgroup with $\mathrm{Lie}(N^-) = \mathcal{N}^-$, such that $L \in \mathcal{F}_\gamma$ can be conjugated to the normal form C_γ, $L = n_0 C_\gamma n_0^{-1}$ [12]. In the case of $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$, C_γ has a representation as the companion matrix given by

$$C_\gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ (-1)^l\gamma_l & \cdots & \cdots & -\gamma_1 & 0 \end{pmatrix},$$

where the Chevalley invariants are given by the elementary symmetric polynomials of the eigenvalues of L. In this paper, we are particularly interested in the case where *all* $\gamma_k = 0$, which implies L is a (regular) nilpotent element, and we denote C_0 as a representation of the element e_+. In order to discuss a compactification of the isospectral manifold, $\tilde{Z}(\gamma)_{\mathbb{R}}$, let us recall:

Definition 1.1. [9]: The companion embedding of $\mathcal{F}_\gamma$ is defined as the map,

$$\begin{array}{rccc} c_\gamma : & \mathcal{F}_\gamma & \longrightarrow & G/B^+ \\ & L & \longmapsto & n_0^{-1} \bmod B^+ \end{array}$$

where $L = n_0 C_\gamma n_0^{-1}$ with $n_0 \in N^-$.

The isospectral manifold $Z(\gamma)_{\mathbb{R}}$ can be considered as a subset of $\mathcal{F}_\gamma$ with the element L in the form of (1.2). Then a compactification of $Z(\gamma)_{\mathbb{R}}$ can be obtained by the closure of the image of the companion embedding c_γ in the flag manifold G/B^+,

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \overline{c_\gamma(Z(\gamma)_{\mathbb{R}})}\,.$$

One can also define the Toda flow on $\mathcal{F}_\gamma$ as follows: First we make a factorization of $e^{tL^0} \in G$,

$$\exp(tL^0) = n(t)b(t), \quad \text{with} \quad n(t) \in N^-\,, \; b(t) \in B^+\,. \tag{1.5}$$

where L^0 is the initial element of $L(t)$, i.e. $L(0) = L^0$ and B^+ is the Borel subgroup with Lie$(B^+) = \mathcal{B}^+$. Then the solution $L(t)$ can be expressed as

$$L(t) = n(t)^{-1} L^0 n(t) = b(t) L^0 b(t)^{-1}\,. \tag{1.6}$$

Here one should note that the factorization is not always possible, and the general form is given by the Bruhat decomposition, that is, for some $t = t_*$,

$$\exp(t_* L^0) \in N^- w B^+ \quad \text{for some } w \in W\,,$$

where W is the Weyl group of reflections on $\Delta(\mathfrak{g}, \mathfrak{h})$. We will discuss this in more detail in the following section (see also [9, 1]). Then one can show:

Proposition 1.1. [9] *With the embedding c_γ, the Toda flow maps to the flag manifold as*

$$\begin{array}{ccc} L^0 & \xrightarrow{\;c_\gamma\;} & n_0^{-1} \bmod B^+ \\ {\scriptstyle Ad(n(t)^{-1})}\big\downarrow & & \big\downarrow \\ L(t) & \xrightarrow{\;c_\gamma\;} & \left\{ \begin{array}{l} n_0^{-1} n(t) \bmod B^+ \\ = n_0^{-1} e^{tL^0} \bmod B^+ \\ = e^{tC_\gamma} n_0^{-1} \bmod B^+ \end{array} \right. \end{array}$$

where $L^0 = n_0 C_\gamma n_0^{-1}$, and $n(t) \in N^-$ is given by the factorization (1.5).

The commuting flows (1.4) can be also embedded in the same way, and taking the closure of the Toda orbit generated by all the flows, we can obtain the compactified manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ in terms of the Toda orbit. Then the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ for a generic $\gamma \in \mathbb{R}^l$ is described by a union of 2^l convex polytopes Γ_ϵ with $\epsilon =$

$(\epsilon_1, \cdots, \epsilon_l)$, $\epsilon_i = \mathrm{sign}(a_i)$, and each Γ_ϵ is expressed as the closure of the orbit of a Cartan subgroup with the connected component of the identity G^{C_γ}:

Proposition 1.2. *(Theorem 8.9 in [7])*

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \bigcup_{\epsilon \in \{\pm\}^l} \Gamma_\epsilon$$

with

$$\Gamma_\epsilon = \overline{\{\, g n_\epsilon^{-1} \bmod B^+ \mid g \in G^{C_\gamma} \,\}}, \quad G^{C_\gamma} := \left\{ \exp\left(\sum_{k=1}^{l} t_k \nabla I_k(C_\gamma) \right) \;\middle|\; t_k \in \mathbb{R} \right\},$$

where $n_\epsilon \in N^-$ is a generic element given by $L_\epsilon = n_\epsilon C_\gamma n_\epsilon^{-1}$ for each set of the signs $\epsilon = (\epsilon_1, \cdots, \epsilon_l)$ with $\epsilon_i = sgn(a_i)$.

Here note that G^{C_γ} is the connected component including the identity element. Thus in an ad-diagonalizable case with distinct eigenvalues, the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is a toric variety, i.e. G^{C_γ}-orbit defines $(\mathbb{R}^*)^l$-action, and the convexity of Γ_ϵ is a consequence of the Atiyah's convexity theorem in [2]. The smooth compactification is done uniquely by gluing the boundaries of the polytopes according to the action of the Weyl group on the signs $(\epsilon_1, \ldots, \epsilon_l)$ (Theorem 8.14 in [7]). The W-action is defined as follows:

Definition 1.2. (Proposition 3.16 in [7]) : For any set of signs $(\epsilon_1, \cdots, \epsilon_l) \in \{\pm\}^l$, a simple reflection $s_i := s_{\alpha_i} \in W$ acts on the sign ϵ_j by

$$s_i \;:\; \epsilon_j \longmapsto \epsilon_j \epsilon_i^{-C_{j,i}}.$$

The sign change is defined on the group character χ_{α_i} with $\epsilon_i = \mathrm{sign}(\chi_{\alpha_i})$ (recall $s_i \cdot \alpha_j = \alpha_j - C_{j,i}\alpha_i$). We also identify the sign ϵ_i as that of a_i, since the condition $\chi_{\alpha_i} = 0$ corresponds to the subsystem defined by $a_i = 0$.

Note that each polytope Γ_ϵ is identifiable with a connected component of a Cartan subgroup, and the construction of the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ given in [7] is an extension of the work of Kostant [13] where the signs of the off diagonal elements a_i's in L are assumed to be positive, i.e. only considered the polytope $\Gamma_{+\cdots+}$.

The compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ can be also considered as the real part of the complex variety $\tilde{Z}(\gamma)_{\mathbb{C}}$ (Theorem 3.3 in [9]),

$$\tilde{Z}(\gamma)_{\mathbb{C}} := \overline{G_{\mathbb{C}}^{C_\gamma} w_* B_{\mathbb{C}}^+ / B_{\mathbb{C}}^+},$$

where w_* is the longest element of the Weyl group. Since $w_* B_{\mathbb{C}}^+ / B_{\mathbb{C}}^+ = w_* B^+ / B^+$, the real point $w_* B^+ / B^+$ is considered as the center of the manifold which corresponds to the blow-up point (see Section 3 for more detail). In particular, the polytope Γ_ϵ with $\epsilon = (- \ldots -)$ can be identified as the G^{C_γ}-orbit of the point $w_* B^+ / B^+$,

$$\Gamma_{-\cdots-} = \overline{G^{C_\gamma} w_* B^+ / B^+}.$$

In the generic case of $\gamma \in \mathbb{R}^l$, the G^{C_γ}-orbit defines a toric variety, and then following the paper [7], we have:

Proposition 1.3. *The polytope Γ_ϵ has a cell decomposition using the Weyl group action on the polytope,*

$$\Gamma_\epsilon = \bigsqcup_{J\subseteq\Pi} \bigsqcup_{[w]\in W/W^J} \langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle\,. \tag{1.7}$$

Here W^J is the Weyl subgroup defined by $W^J := W_{\Pi\setminus J} = \langle s_{\alpha_i} | \alpha_i \in \Pi\setminus J\rangle$, and $\sigma_J(\epsilon) = \sigma_J(\epsilon_1,\dots,\epsilon_l) = (\sigma_1,\dots,\sigma_l)$ is defined as

$$\sigma_k = \begin{cases} 0 & \text{if} \quad \alpha_k \in J\,, \\ \epsilon_k & \text{if} \quad \alpha_k \notin J\,. \end{cases}$$

The unique l-cell $\langle\emptyset;[e];\epsilon\rangle = G^{C_\gamma}w_*B^+/B^+$ labels the top cell of Γ_ϵ. Each cell $\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle$ has the dimension $l-|J|$, and the number of those cells are given by $|W|/|W^J|$. Each cell $\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle$ can be also associated to the subsystem of the Toda lattice having the signs and zeros,

$$\mathrm{sign}(a_j(t)) = \left(\sigma_J(w^{-1}\cdot\epsilon)\right)_j \quad \text{for} \quad t\ll 0\,.$$

One can also define the orientation of each cell by the length of the Weyl group element, that is, we denote

$$o(\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle) = (-1)^{\ell(w)}\,, \tag{1.8}$$

where $\ell(w)$ is the length of w.

Example 1.3. $\mathfrak{sl}(2,\mathbb{R})$: The compact maifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is a union of two line segments,

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \Gamma_- \cup \Gamma_+\,,$$

with the decompositions,

$$\begin{aligned} \Gamma_- &= \langle\emptyset;[e];(-)\rangle \sqcup \langle\{\alpha_1\};e;(0)\rangle \sqcup \langle\{\alpha_1\};s_1;(0)\rangle\,, \\ \Gamma_+ &= \langle\emptyset;[e];(+)\rangle \sqcup \langle\{\alpha_1\};e;(0)\rangle \sqcup \langle\{\alpha_1\};s_1;(0)\rangle\,, \end{aligned}$$

Thus the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is diffeomorphic to the circle.

Example 1.4. $\mathfrak{sl}(3,\mathbb{R})$: The polytope Γ_ϵ is given by a hexagon having the decomposition with the following cells: For example in the case of $\epsilon = (--)$, we have

- 2-cell: this is the top cell $\langle\emptyset;[e];(--)\rangle$
- 1-cell: there are six 1-cells having either $J=\{\alpha_1\}$ or $J=\{\alpha_2\}$;
  $\langle\{\alpha_1\};[e];(0-)\rangle$, $\langle\{\alpha_1\};[s_1];(0+)\rangle$, $\langle\{\alpha_1\};[s_2s_1];(0-)\rangle$
  $\langle\{\alpha_2\};[e];(-0)\rangle$, $\langle\{\alpha_2\};[s_2];(+0)\rangle$, $\langle\{\alpha_2\};[s_1s_2];(-0)\rangle$
- 0-cell: there are six 0-cells, $\langle\Pi;w;(00)\rangle$ for each $w\in W$.

(See also Figure 1, from which one can easily label the boundaries of the hexagons.)

In the case of the nilpotent Toda lattice ($\gamma=0$), the compactified isospectral variety is given by

$$\tilde{Z}(0)_{\mathbb{R}} = \overline{G^{C_0}w_*B^+/B^+}\,,$$

that is, the variety is the compactification of unipotent group orbit of a regular nilpotent element $C_0 \in \mathcal{N}^+$ in the flag G/B^+. One should note that the G^{C_0}-orbit defines an $\mathbb{R}^l$-action and it can be obtained by a nilpotent limit of the polytope $\Gamma_{-\dots-}$ with several identification of the boundaries. The compactified variety $\tilde{Z}(0)_{\mathbb{R}}$ is singular, which will be also discussed in the paper. The study of the topological structure of this variety $\tilde{Z}(0)_{\mathbb{R}}$ is the main purpose of the present paper.

Remark 1.5. The G^{C_0}-orbit has been studied in the context of the quantum cohomology of the Grassmann manifold (see e.g. [16]), and it is called the Peterson variety [14]. Then Peterson's theorem identifies the quantum cohomology ring of the Grassmaniann $Gr(k, l+1)$ in $\mathbb{C}^{l+1}$ denoted as $QH^*(Gr(k, l+1)) \otimes \mathbb{C}$ with the coordinate ring of a particular variety $\mathcal{V}_{k,l+1}$ (Definition 3.1 in [16]) which is the Painlevé divisor $\mathcal{D}^{\{\alpha_k\}}$ defined in Section 3 of the present paper. The varieties $\mathcal{D}^{\{\alpha_k\}}$ play a crucial rule in the compactification of the G^{C_0}-orbit in this paper. We also discuss singular structure of the Painlevé divisors.

It is also known that the solution $\{a_j(t), b_j(t)\}$ of the Toda lattice equation (1.3) can be expressed in terms of the τ-functions [13],

$$a_j(t) = a_j^0 \prod_{k=1}^{l} (\tau_k(t))^{-C_{j,k}}\,, \quad b_j(t) = \frac{d}{dt} \ln \tau_j(t)\,, \tag{1.9}$$

where the τ-functions, $\tau_j(t)$, are defined by (Definition 2.1 in [9])

$$\tau_j(t) = \left\langle e^{tL^0} v^{\omega_j}, v^{\omega_j} \right\rangle . \tag{1.10}$$

Here v^{ω_j} is the highest weight vector in a fundamental representation of G, and $\langle \cdot, \cdot \rangle$ is a pairing on the representation space. Note from (1.9) that the τ-functions satisfy the bilinear equation,

$$\tau_j \tau_j'' - (\tau_j')^2 = a_j^0 \prod_{k \neq j} (\tau_k(t))^{-C_{j,k}} . \tag{1.11}$$

In the next section, we consider the case of $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$ in the matrix representation, and give explicit formulae of the τ-functions.

1.2. **Toda lattice of type A_l.** Here we consider a matrix (adjoint) representation of $\mathfrak{sl}(l+1, \mathbb{R})$ on $\mathbb{R}^{l+1}$. With the factorization (1.5), one can construct an explicit solution $\{a_j, b_j\}$ in the matrix form of $L(t)$ which is given by a tridiagonal matrix,

$$L_A = \begin{pmatrix} b_1 & 1 & 0 & \cdots & 0 \\ a_1 & b_2 - b_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_l - b_{l-1} & 1 \\ 0 & \cdots & \cdots & a_l & -b_l \end{pmatrix} \tag{1.12}$$

In order to construct the explicit solution, we start with the following obvious Lemma which can be also applied to other Lie algebras.

Lemma 1.1. *The diagonal element $b_{j,j}$ of the upper trianguler matrix $b(t) \in B^+$ in the factorization $\exp(tL^0) = n(t)b(t)$ is expressed by*

$$b_{j,j}(t) = \frac{D_j[\exp(tL^0)]}{D_{j-1}[\exp(tL^0)]}$$

where $D_j[\exp(tL^0)]$ is the determinant of the j-th principal minor of $\exp(tL^0)$, that is, with a pairing $\langle\cdot,\cdot\rangle$ on the exterior product space $\bigwedge^j \mathbb{R}^{l+1}$,

$$D_j[\exp(tL^0)] = \left\langle e^{tL^0} e_0 \wedge \cdots \wedge e_{j-1},\ e_0 \wedge \cdots \wedge e_{j-1} \right\rangle. \tag{1.13}$$

Here $\{e_i\}$ is the standard basis of $\mathbb{R}^{l+1}$.

Here the pairing $\langle\cdot,\cdot\rangle$ on $\bigwedge^j \mathbb{R}^{l+1}$ is defined by

$$\langle v_1 \wedge \cdots \wedge v_j,\ w_1 \wedge \cdots \wedge w_j \rangle = \det\left[(\langle v_m, w_n \rangle)_{1 \le m,n \le j}\right],$$

where $\langle v_m, w_n \rangle$ is the standard inner product of v_m, $w_n \in \mathbb{R}^{l+1}$.

The group $G = SL(l+1, \mathbb{R})$ has l fundamental representations; these are defined on the j-fold exterior product of $\mathbb{R}^{l+1}$ for $j = 1, \cdots, l$. Then the heighest weight vector on the representation space $\bigwedge^j \mathbb{R}^{l+1}$ is given by

$$v^{\omega_j} = e_0 \wedge e_1 \wedge \cdots \wedge e_{j-1}.$$

We then obtain the following Proposition which gives the solution formula (1.9) in the case of $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$:

Proposition 1.4. *The solution $\{a_i(t), b_i(t)\}$ in the matrix $L(t)$ in (1.12) can be given by*

$$a_i(t) = a_i^0 \frac{D_{i+1} D_{i-1}}{D_i^2}, \quad b_i(t) = \frac{d}{dt} \ln D_i,$$

that is, $\tau_j(t) = D_j[\exp(tL^0)]$ of (1.13).

Proof. From $L = bL^0 b^{-1}$ in (1.6), we have

$$a_j = a_j^0 \frac{b_{j+1,j+1}}{b_{j,j}},$$

Then using Lemma 1.1 for the diagonal element $b_{j,j}$ of $b \in B^+$, and (1.3) for the equation of b_j, we obtain the above formulae. □

Note that the solution for the Toda lattice hierarchy containing all the commuting flows (1.4) can be expressed by the same formula with the τ-functions,

$$\tau_j(t_1, \cdots, t_l) = \langle g(t_1, \cdots, t_l) \cdot e_0 \wedge \cdots \wedge e_{j-1}, e_0 \wedge \cdots \wedge e_{j-1} \rangle,$$

where $g(t_1, \cdots, t_l) \in SL(l+1, \mathbb{R})$ is given by

$$g = \exp\left(\sum_{k=1}^{l} t_k (L^0)^k\right).$$

(Recall that $\nabla I_j = L^j$ for $\mathfrak{sl}(l+1, \mathbb{R})$.) The Toda orbit $g \cdot e_0 \wedge \cdots \wedge e_{j-1}$ on the representation space $\bigwedge^j \mathbb{R}^{l+1}$ plays an essential role for the study of the topology of compactified isospectral manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ (see Proposition 1.2). The Toda orbit of the generic element is given by

$$\pm G^{C_\gamma} \cdot e_l \wedge \cdots \wedge e_{l-j+1}, \quad \text{with} \quad G^{C_\gamma} = \left\{ \exp\left(\sum_{k=1}^{l} t_k C_\gamma^k\right) \;\middle|\; t_k \in \mathbb{R} \right\}.$$

Here the highest weight vector $v_j = e_0 \wedge \cdots \wedge e_{j-1}$ is mapped by the longest element w_* to the lowest weight vector $w_* v_j = (-1)^{j(j-1)/2} e_l \wedge \cdots \wedge e_{l-j+1}$.

In the case of (regular) nilpotent L, G^{C_0} has a representation,

$$G^{C_0} = \left\{ \exp\left(\sum_{k=1}^{l} t_k C_0^k\right) = \begin{pmatrix} 1 & p_1 & p_2 & \cdots & p_l \\ 0 & 1 & p_1 & \cdots & p_{l-1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & p_1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \subset N^+. \tag{1.14}$$

Namely this is an N^+-orbit given by the stabilizer of the regular nilpotent element $C_0 \in N^+$. Here $\{p_k(t) \mid k = 1, \cdots, l\}$ are the Schur polynomials of $(t_1, \cdots, t_l)$ defined as

$$\exp\left(\sum_{k=1}^{l} t_k \lambda^k\right) = \sum_{k=0}^{\infty} p_k(t) \lambda^k,$$

where $p_0 = 1$. Those Schur polynomials $p_k(t)$ are complete homogeneous symmetric functions in terms of $\{x_k \mid k = 1, \cdots, l\}$ defined by $t_k = (\sum_{i=1}^{l} x_i^k)/k$, and they are expressed by

$$\begin{aligned} p_k(t_1, \cdots, t_k) &= \sum_{k_1+2k_2+\cdots+nk_n=k} \frac{t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}}{k_1! k_2! \cdots k_n!} \\ &= \frac{t_1^k}{k!} + \frac{t_1^{k-2} t_2}{(k-2)!} + \cdots + t_{k-1} t_1 + t_k. \end{aligned} \tag{1.15}$$

The τ-functions corresponding to the generic orbit are then given by

$$\tau_j(t_1, \cdots, t_l) = \langle g w_* \cdot e_0 \wedge \cdots \wedge e_{j-1}, \, e_0 \wedge \cdots \wedge e_{j-1} \rangle, \quad g \in G^{C_0}. \tag{1.16}$$

In terms of the Schur polynomials, those are given by the Hankel determinants,

$$\tau_1 = p_l, \quad \tau_2 = \begin{vmatrix} p_l & p_{l-1} \\ p_{l-1} & p_{l-2} \end{vmatrix}, \quad \tau_3 = \begin{vmatrix} p_l & p_{l-1} & p_{l-2} \\ p_{l-1} & p_{l-2} & p_{l-3} \\ p_{l-2} & p_{l-3} & p_{l-4} \end{vmatrix} \cdots.$$

(Note $\partial^k p_l / \partial t_1^k = p_{l-k}$, and see the next section for the representation of those Wronskian determinants using the Young diagrams.) Then the corresponding nilpotent matrix $L(t)$ evaluated at $t = (1, 0, \ldots, 0)$ is given by

$$L(1, 0, \ldots, 0) = \begin{pmatrix} l & 1 & 0 & \cdots & 0 \\ -l & l-2 & 1 & \cdots & 0 \\ 0 & -2(l-1) & l-4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & -l \end{pmatrix}$$

The τ-functions are also computed as

$$\tau_k(t_1, 0, \dots, 0) = (-1)^{\frac{k(k-1)}{2}} \prod_{j=1}^{k} \frac{(k-j)!}{(l-k+1)!} t_1^{k(l-k+1)} .$$

Here note the multiplicity of the zero at $t_1 = 0$ (this will be discussed more details in Section 3). Also note that $\tau_k \neq 0$ if $t_1 \neq 0$, and the corresponding functions $a_j = \tau_{j+1}\tau_{j-1}/\tau_j^2$ are all negative.

Example 1.6. $\mathfrak{sl}(2, \mathbb{R})$: The Lax matrix L and the companion matrix C_γ are given by

$$L = \begin{pmatrix} b & 1 \\ a & -b \end{pmatrix}, \quad C_\gamma = \begin{pmatrix} 0 & 1 \\ -\gamma & 0 \end{pmatrix} \quad \text{with} \quad \gamma = -a - b^2 .$$

For the semisimple case, i.e. $\gamma \neq 0$, the C_γ with $\gamma = -\lambda^2$ can be diagonalized as,

$$C_\gamma = V \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} V^{-1}, \quad \text{with} \quad V = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} .$$

Then the τ-function is given by

$$\tau_1(t) = \langle e^{tC_\gamma} w_* e_0, e_0 \rangle = \frac{1}{\lambda} \sinh(\lambda t) .$$

The correspondng solution $(a(t), b(t))$ is given by

$$a(t) = -\lambda^2 \mathrm{csh}^2(\lambda t), \quad b(t) = \lambda \coth(\lambda t) ,$$

which blows up at $t = 0$, and as $t \to \pm\infty$ the solution approaches to the fixed points $(a = 0, b = \pm\lambda)$. This describes the Γ_- polytope in Example 1.3. The nilpotent case $(\gamma = 0)$ can be also obtained by the limit $\lambda \to 0$, that is, we have

$$\tau_1(t) = t .$$

The Γ_+ polytope is obtained by the G^{C_γ}-orbit of the point eB^+/B^+;

$$\tau_1(t) = \langle e^{tC_\gamma} e_0, e_0 \rangle = \cosh(\lambda t) .$$

The solution $(a(t), b(t))$ is given by

$$a(t) = \lambda^2 \mathrm{sech}^2(\lambda t), \quad b(t) = \lambda \tanh(\lambda t).$$

Notice that in the nilpotent limit $\lambda \to 0$, the τ-function takes $\tau_1 = 1$, and the corresponding orbit is just the unique fixed point $(a = 0, b = 0)$. Thus the polytope Γ_+ is squeezed into the 0-cell. This is true for the general case, that is, the polytope Γ_ϵ having at least one positive sign in ϵ is squeezed into a lower dimensional cell in the nilpotent limit. Then the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ can be obtained by glueing the boundaries of the $\Gamma_{-\dots-}$ polytope in the nilpotent limit. This is a key idea for the compactification of the unipotent orbit G^{C_0}, and will be explained more deails through the present paper.

2. Flag manifold G/B^+ and the Bruhat decomposition

In this section, we summarize the basics of the flag manifold G/B^+ and the Bruhat decomposition for $G = SL(l+1,\mathbb{R})$. The purpose of this section is to fix the notation and to make the present paper accessible to the reader who is not familiar with Lie theory and algebraic geometry. Those subjects can be found in the standard books, for example [10].

2.1. Grassmannian and cell decomposition.

Let $Gr(k+1,l+1)$ be a real Grassmannian of the set of $(k+1)$-dimensional subspaces of $\mathbb{R}^{l+1}$. A point ξ of the Grassmannian is expressed by the $(k+1)$-frame of vectors,

$$\xi = [\xi_0, \xi_1, \cdots, \xi_k], \quad \text{with} \quad \xi_j = \sum_{i=0}^{l} \xi_{ij} e_i \in \mathbb{R}^{l+1},$$

where $\{e_i \mid i = 0, 1, \cdots, l\}$ is the standard basis of $\mathbb{R}^{l+1}$, and (ξ_{ij}) defines a $(l+1) \times (k+1)$ matrix. Then the Grassmannian $Gr(k+1,l+1)$ can be embedded to the projectivization of the exterior space $\bigwedge^{k+1}\mathbb{R}^{l+1}$, which is called the Plücker embedding,

$$\begin{array}{ccc} Gr(k+1,l+1) & \hookrightarrow & \mathbb{P}(\bigwedge^{k+1}\mathbb{R}^{l+1}) \\ \xi = [\xi_0, \cdots, \xi_k] & \mapsto & \xi_0 \wedge \cdots \wedge \xi_k \end{array}$$

Here the element on $\mathbb{P}(\bigwedge^{k+1}\mathbb{R}^{l+1})$ is expressed as

$$\xi_0 \wedge \cdots \wedge \xi_k = \sum_{0 \le i_0 < \cdots < i_k \le l} \xi_{(i_0,\cdots,i_k)}\, e_{i_0} \wedge \cdots \wedge e_{i_k},$$

where the coefficients $\xi_{(i_0,\cdots,i_k)}$ give the Plücker coordinates defined by the determinant,

$$\xi_{(i_0,\cdots,i_k)} = ||\xi_{i_0,0}, \cdots, \xi_{i_k,0}|| := \begin{vmatrix} \xi_{i_0,0} & \cdots & \xi_{i_k,0} \\ \vdots & \ddots & \vdots \\ \xi_{i_0,k} & \cdots & \xi_{i_k,k} \end{vmatrix}.$$

It is also well known that the Grassmannian can have the cellular decomposition [10],

$$Gr(k+1,l+1) = \bigsqcup_{0 \le i_0 < \cdots < i_k \le l} W^{k+1}_{(i_0,\cdots,i_k)} \tag{2.1}$$

where the cells are defined by

$$W^{k+1}_{(i_0,\cdots,i_k)} = \left\{ \xi = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ * & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ * & * & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in \overbrace{\mathbb{R}^{l+1} \times \cdots \times \mathbb{R}^{l+1}}^{k+1} \right\}$$

$$= \{ \text{ the set of } (l+1) \times (k+1) \text{ matrices in the echelon form whose pivot ones are at } (i_0, \cdots, i_k) \text{ positions } \}$$

Namely an element $\xi = [\xi_0, \cdots, \xi_k] \in W^{k+1}_{(i_0,\cdots,i_k)}$ is described by

$$\xi \in W^{k+1}_{(i_0,\cdots,i_k)} \Leftrightarrow \begin{cases} \text{(i)} & \xi_{(i_0,\cdots,i_k)} \neq 0. \\ \text{(ii)} & \xi_{(j_0,\cdots,j_k)} = 0 \text{ if } j_n < i_n \text{ for some } n \in \{0, \cdots, k\}. \end{cases}$$

Each cell $W^{k+1}_{(i_0,\cdots,i_k)}$ is labeled by the Young diagram $Y = (i_0, \cdots, i_k)$ where the number of boxes are given by $\ell_j = i_j - j$ for $j = 0, \cdots, k$ (counted from the bottom), which expresses a partition $(\ell_k, \ell_{k-1}, \cdots, \ell_0)$ of the number $|Y| := \sum_{i=0}^{k} \ell_i$, the *size* of Y. We then denote it as $W^{k+1}_{(i_0,\cdots,i_k)} = W_{Y_{k+1}}$. The codimension of $W^{k+1}_{(i_0,\cdots,i_k)}$ is then given by the size of Y,

$$\operatorname{codim} W_{Y_{k+1}} = |Y_{k+1}|,$$

and the dimension is given by the number of free variables in the echelon form. Note that the top cell of $Gr(k+1, l+1)$ is labeled by $Y = (0, 1, \cdots, k)$, i.e. $|Y| = 0$, and

$$\dim W_{(0,1,\cdots,k)} = \dim Gr(k+1, l+1) = (k+1)(l-k).$$

2.2. **The Bruhat decomposition of G/B^+.** We now consider a diagonal embedding of the flag manifold G/B^+ into the product of the Grassmannians $Gr(k, l+1)$,

$$\begin{array}{ccccccccc} G/B^+ & \hookrightarrow & Gr(1, l+1) & \times & Gr(2, l+1) & \times & \cdots & \times & Gr(l, l+1) \\ x & \mapsto & (W^1, & & W^2, & & \cdots, & & W^l) \end{array} \tag{2.2}$$

where the subspaces $\{W^j \mid j = 1, \cdots, l\}$ define a complete flag,

$$\{0\} \subset W^1 \subset W^2 \subset \cdots \subset W^l \subset \mathbb{R}^{l+1},$$

This defines the Bruhat decomposition of the flag manifold G/B^+,

$$G/B^+ = \bigsqcup_{Y_1 \prec \cdots \prec Y_l} W[Y_1, \cdots, Y_l], \quad \text{with} \quad W[Y_1, \cdots, Y_l] := (W^1_{Y_1}, \cdots, W^l_{Y_l}),$$

where the order $\prec$ is defined by

$$Y_k \prec Y_{k+1} \overset{\text{def}}{\Longleftrightarrow} W_{Y_k} \subset W_{Y_{k+1}}.$$

In terms of $Y_k = (i_0, i_1, \cdots, i_{k-1})$ and $Y_{k+1} = (j_0, j_1, \cdots, j_k)$, the order $Y_k \prec Y_{k+1}$ implies the inclusion between the non-ordered sets,

$$\{i_0, i_1, \cdots, i_{k-1}\} \subset \{j_0, j_1, \cdots, j_k\}.$$

The Bruhat cell $W[Y_1,\cdots,Y_l]$ is also expressed as

$$W[Y_1,\cdots,Y_l]=N^-wB^+/B^+\,,$$

where the corresponding Weyl element w can be found by the W-action on the Young diagrams which is defined as follows: Let $s_k:=s_{\alpha_k}\in W$ be a simple reflection. Then the W-action is defined by

$$\begin{aligned}s_k:\quad &[\{j_0\},\cdots,\{j_0,\cdots,j_{k-1}\},\{j_0,\cdots,j_{k-1},j_k\},\cdots,\{j_0,\cdots,j_{l-1}\}]\\ &\mapsto[\{j_0\},\cdots,\{j_0,\cdots,j_{k-2},j_k\},\{j_0,\cdots,j_{k-1},j_k\},\cdots,\{j_0,\cdots,j_{l-1}\}]\end{aligned}$$

where we have expressed the Young diagram $Y_{k+1}=(i_0,\cdots,i_k)$ as the non-ordered set $\{j_0,\cdots,j_k\}=\{i_0,\cdots,i_k\}$. Thus the s_k-action gives the exchange, $j_{k-1}\leftrightarrow j_k$. Thus the Weyl element w associated with the Bruhat cell $W[Y_1,\cdots,Y_l]$ is expressed by the permutation, $\begin{pmatrix}0 & 1 & \cdots & l\\ j_0 & j_1 & \cdots & j_l\end{pmatrix}$, that is, $j_k=w(k)$,

$$w=\sum_{k=0}^{l}E_{k,j_k}\,,$$

where E_{ij} is the $(l+1)\times(l+1)$ matrix with ±1 at (i,j) entry ($\pm$ needed for $\det(w)=1$). Also the codimension of the Bruhat cell $W[Y_1,\cdots,Y_l]=N^-wB^+/B^+$ is given by

$$\text{codim }W[Y_1,\cdots,Y_l]=\ell(w)=|Y_1\cup\cdots\cup Y_l|\,.$$

Example 2.1. The top cell is given by

$$N^-B^+/B^+=W[(0),(0,1),(0,1,2),\cdots,(0,1,2,\cdots,l-1)]\,,$$

where all the Young diagrams have no boxes, i.e. $Y_k=\emptyset$ for $k=1,\cdots,l$. Then for example it is obvious to get the following cells,

$$\left\{\begin{array}{lcl}N^-s_1B^+/B^+ &=& W[(1),(0,1),(0,1,2),\cdots,(0,1,\cdots,l-1)]\,,\\ N^-s_2s_1B^+/B^+ &=& W[(1),(1,2),(0,1,2),\cdots,(0,1,\cdots,l-1)]\,,\\ N^-s_1s_2s_1B^+/B^+ &=& W[(2),(1,2),(0,1,2),\cdots,(0,1,\cdots,l-1)]\,,\end{array}\right.$$

The unique 0-cell is corresponding to the longest element $w_*\in W$ with $\ell(w_*)=\frac{1}{2}l(l+1)$,i.e.

$$N^-w_*B^+/B^+=w_*B^+/B^+=W[(l),(l-1,l),\cdots,(1,2,\cdots,l)]\,,$$

where each Young diagram $Y_k=(l-k+1,\cdots,l-1,l)$ has a rectangular shape with k stack of $(l-k+1)$ number of boxes in the horizontal direction.

3. Toda orbit and the τ-functions

Here we consider the Toda orbit given by a G^{C_0}-orbit on the flag manifold G/B^+, and give explicit representations of the τ-functions for $G=SL(l+1,\mathbb{R})$. The discussions in this section can be also applied to the generic case of γ with some trivial modifications. The main purpose in this section is to give an elementary proof of Theorem 3.3 in [9] (Theorem 3.1 below), which provides an explicit description of the Painlevé divisors (the sets of zeros of τ-functions) as the sets in the flag manifold G/B^+.

3.1. **Generic orbit and the τ-functions.** Through the diagonal embedding (2.2), we consider the orbit of the highest weight vector on the representation space $\bigwedge^{k+1}\mathbb{R}^{l+1}$, whose projectivization defines an orbit on the Grassmannian $Gr(k+1, l+1)$, i.e.

$$gw_* \cdot e_0 \wedge e_1 \wedge \cdots \wedge e_k =: \xi_0 \wedge \xi_1 \wedge \cdots \xi_k, \quad \text{for} \quad g \in G^{C_0},$$

where $\xi_k := gw_* \cdot e_k$. In terms of the τ-function, $\tau_1 := \langle \xi_0, e_0 \rangle = p_l$ (see (1.16)), the orbit ξ_k on $\mathbb{RP}^l$ is given by

$$\xi_k = \sum_{j=0}^{l-k} \tau_1^{(j+k)} e_j, \quad \text{with} \quad \tau_1^{(n)} = \frac{\partial^n \tau_1(t)}{\partial t_1^n} = p_{l-n}(t),$$

and gw_* has the form,

$$gw_* = \begin{pmatrix} \tau_1^{(0)} & \tau_1^{(1)} & \cdots & \tau_1^{(l)} \\ \tau_1^{(1)} & \tau_1^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1^{(l)} & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} p_l & p_{l-1} & \cdots & 1 \\ p_{l-1} & p_{l-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.1}$$

which is the Wronskian of $\{\, \tau_1^{(k)} \mid k = 0, 1, \cdots, l\,\}$. Then we have

$$\xi_0 \wedge \cdots \wedge \xi_k = \sum_{0 \le i_0 < \cdots < i_k \le l} \xi_{(i_0, \cdots, i_k)}\, e_{i_0} \wedge \cdots \wedge e_{i_k}.$$

Here the Plučker coordinate $\xi_{(i_0, \cdots, i_k)}$ is given by

$$\xi_{(i_0, \cdots, i_k)} := ||\tau_1^{(i_0)}, \cdots, \tau_1^{(i_k)}||. \tag{3.2}$$

Here $||\tau_1^{(i_0)}, \cdots, \tau_1^{(i_k)}||$ becomes the Wronskian of $\{\tau_1^{(i_j)} \mid j = 0, 1, \cdots, k\}$. In particular, note that (1.16) becomes

$$\tau_{k+1} = ||\tau_1^{(0)}, \cdots, \tau_1^{(k)}|| = ||p_l, \cdots p_{l-k}||,$$

where p_k are the Schur polynomials in (1.15). Thus the τ_{k+1}-function is given by the Schur polynomial associated with the rectangular Young diagram having $k+1$ stack of $l-k$ horizontal boxes, i.e.

$$\tau_{k+1}(t_1, \cdots, t_l) = (-1)^{\frac{k(k+1)}{2}} S_{(l-k, l-k+1, \cdots, l)}(t_1, \cdots, t_l), \tag{3.3}$$

The Schur polynomial $S_Y(t_1, \cdots, t_l)$ associated with the Young diagram $Y = (i_0, i_1, \cdots, i_k)$ is defined by

$$S_{(i_0, i_1, \cdots, i_k)} := ||p_{i_0}, p_{i_1}, \cdots, p_{i_k}||.$$

Note here that the Young diagram of the Schur polynomial $p_{i_k} = S_{(i_k)}$ is the i_k horizontal boxes. With the duality between the Grassmannians $Gr(k+1, l+1)$ and $Gr(l-k, l+1)$, i.e. $\bigwedge^{k+1}\mathbb{R}^{l+1} \cong \bigwedge^{l-k}\mathbb{R}^{l+1}$, one can express τ_{k+1} in terms of $S_{(1,2,\cdots,l)} = \pm\tau_l$ (instead of τ_1): Let us denote the Schur polynomial with $Y = (1, \cdots, k)$ as $p_{\overline{k}}$, which is related to the elementary symmetric function whose Young diagram has k vertical boxes, i.e.

$$p_{\overline{k}} = S_{(1,2,\cdots,k)} = ||p_1, p_2, \cdots, p_k||.$$

Define the dual τ-functions, denoted as $\overline{\tau}_{k+1}$, by

$$\overline{\tau}_{k+1} := ||p_{\overline{l}}, p_{\overline{l-1}}, \cdots, p_{\overline{k+1}}||. \tag{3.4}$$

Then we have

$$\tau_{k+1} = \pm\overline{\tau}_{k+1}\,.$$

This can be shown by using the duality given in [15] where the Schur polynomial has a dual expression associated with the conjugate Young diagrams, $Y' = (j_0, \cdots, j_m)$, where $(j_0, j_1 - 1 \cdots, j_m - m)$ represent the numbers of boxes in the Young diagram in the vertical direction, that is,

$$\begin{aligned} S_{(i_0,i_1,\cdots,i_k)} &= \|p_{i_0}, p_{i_1}, \cdots, p_{i_k}\| \\ &= S_{(\overline{j_0},\overline{j_1}\cdots,\overline{j_m})} := \|p_{\overline{j_0}}, p_{\overline{j_1}}, \cdots, p_{\overline{j_m}}\|\,. \end{aligned}$$

For examples, $p_{\overline{l}} = S_{(1,2,\cdots,l)}$ and $p_l = S_{(\overline{1},\overline{2},\cdots,\overline{l})}$. One should note that the dual τ-functions are defined by the fundamental (lowest weight) representation,

$$\overline{\tau}_{k+1} = \pm\langle \overline{g}w_* \cdot e_l \wedge \cdots \wedge e_{l-k},\ e_l \wedge \cdots \wedge e_{l-k} \rangle, \tag{3.5}$$

where $\overline{g} = (g^{-1})^T \in N^-$ and $\overline{g}w_*$ is given by

$$\overline{g}w_* = \begin{pmatrix} 0 & \cdots & 0 & \pm 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \pm p_{\overline{l-2}} & \mp p_{\overline{l-1}} \\ 1 & \cdots & \mp p_{\overline{l-1}} & \pm p_{\overline{l}} \end{pmatrix}$$

3.2. **The Painlevé divisors.** Now we consider how the G^0-orbit intersects with the Bruhat cells. We first collect the information on the zeros of τ-functions and their multiplicities.

For each $J = \{\alpha_{i+1}, \cdots, \alpha_{i+s}\} \subset \Pi$, we define $\mathcal{T}_J$ as the set of zeros of τ-functions given by

$$\mathcal{T}_J := \left\{ t = (t_1, \cdots, t_l) \in \mathbb{R}^l \;\middle|\; \tau_j(t) = 0 \text{ if } \alpha_j \in J \right\}.$$

Then we have:

Lemma 3.1. *For each simple root $\alpha_j \in J$, $\tau_j(t)$ has the following form near its zero $t = t_J \in \mathcal{T}_J$ with $t_J = (t_{J1}, \ldots, t_{Jl})$,*

$$\tau_{i+k}(t_1, \cdots) \simeq (t_1 - t_{J1})^{m_k} + \cdots, \quad \text{with} \quad m_k = k(s+1-k),\ 1 \le k \le s. \tag{3.6}$$

Proof. Substituting (3.6) into (1.11), and using $\tau_i(t_J) \neq 0$, we have $m_k = k(m_1 + 1 - k)$. Then $\tau_{i+s+1}(t_J) \neq 0$ implies $m_1 = s$. □

We then have the following Proposition on the cell, with which the Painlevé divisor intersects:

Proposition 3.1. *For all $t \in \mathcal{T}_J$ with $J \in \Pi$, the orbit $g(t)w_*B^+/B^+$ stays on the cell $W[Y_1, \cdots, Y_l]$ where the Young diagrams Y_j are given by*

$$\begin{cases} Y_k &= \emptyset, & \text{for} \quad k = 1, \cdots, i \\ Y_{i+k} &= (s-k+1, \cdots, s) & \text{for} \quad k = 1, \cdots, s \\ Y_{i+s+k} &= \emptyset, & \text{for} \quad k = 1, \cdots, l-(i+s) \end{cases}$$

Proof. Let us first consider the case with $i = 0$, i.e. $J = \{\alpha_1, \cdots, \alpha_s\}$. Since $\tau_1(t) = 0$ has the multiplicity s (Lemma 3.1), $\tau_1^{(s)} \neq 0$. This implies

$$\xi_0 = gw_* \cdot e_0 = \tau_1^{(s)} e_s + \tau_1^{(s+1)} e_{s+1} + \cdots + \tau_1^{(l)} e_l \in W^1_{(s)},$$

where $g \in G^{C_0}$ and $W^1_{(s)}$ is a cell of $Gr(1, l+1)$ in (2.1). From the Plücker coordinates (3.2) of the G^{C_0}-orbit, one can see that the first nonzero coordinate including the $Y_1 = (s)$ is given by

$$\xi_{(s-1,s)} = ||\tau_1^{(s-1)}, \tau_1^{(s)}|| = -(\tau_1^{(s)})^2 \neq 0 .$$

This implies

$$\xi_0 \wedge \xi_1 = \sum_{s-1 \leq i < j \leq l} \xi_{(i,j)}\, e_i \wedge e_j \in W^2_{(s-1,s)} .$$

Note here that the multiplicity of $\tau_2(t) = 0$ is $2(s-1)$, and the term $\xi_{(s-1,s)}$ appears in the derivative $\tau_2^{(2(s-1))} \neq 0$. Now following the above argument, we can see

$$\xi_{(s-k+1,\cdots,s-1,s)} = ||\tau_1^{(s-k+1)}, \cdots, \tau_1^{(s-1)}, \tau_1^{(s)}|| = (-1)^{\frac{k(k-1)}{2}} \left(\tau_1^{(s)}\right)^k \neq 0 ,$$

and

$$\xi_0 \wedge \cdots \wedge \xi_k = \sum_{s-k+1 \leq j_0 < \cdots < j_k \leq l} \xi_{(j_0,\cdots,j_k)}\, e_{j_0} \wedge \cdots \wedge e_{j_k} .$$

This implies

$$\xi_0 \wedge \cdots \wedge \xi_k \in W^k_{(s-k+1,\cdots,s-1,s)} .$$

In the general case with $i \neq 0$, from $\tau_k \neq 0$ for $k = 1, \cdots, i$, we first have

$$\xi_0 \wedge \cdots \wedge \xi_k \in W^{k+1}_{(0,1,\cdots,k)} \quad \text{for} \quad k = 0, 1, \cdots, i-1 .$$

Note here that all of the Young diagrams $Y_{k+1} = (0, 1, \cdots, k)$ represent $Y_{k+1} = \emptyset$. Since $\tau_{i+1}(t) = 0$ has the multiplicity s, we have $\tau_{i+1}^{(s)} \neq 0$. This leads to $||\tau_1^{(0)}, \cdots, \tau_1^{(i-1)}, \tau_1^{(i+s)}|| \neq 0$, which implies

$$\xi_0 \wedge \cdots \wedge \xi_i \in W^{i+1}_{(0,1,\cdots,i-1,i+s)} .$$

Then using the multiplicity of τ_{i+2}, which is $2(s-1)$, we have

$$\xi_0 \wedge \cdots \wedge \xi_{i+1} \in W^{i+2}_{(0,\cdots,i-1,i+s-1,i+s)} .$$

Now it is straightforward to conclude the assertion of this Proposition. □

Note here that we have represented $Y_{i+k} = (0, 1, \cdots, i-1, i+s-k+1, \cdots, i+s)$ as $(s-k+1, \cdots, s)$ which both give the same rectangular diagram having k stack of $(s-k+1)$ boxes (see Example 2.1), and the multiplicity of the zero for τ_{i+k} is given by the total number of boxes in Y_{i+k}, i.e. $|Y_{i+k}| = k(s-k+1)$. Proposition 3.1 leads to the following Corollary:

Corollary 3.1. *The cell given in Proposition 3.1 is identified as*

$$W[Y_1, \cdots, Y_l] = N^- w_J B^+ / B^+ , \quad \text{with} \quad w_\emptyset = id ,$$

where w_J is the longest element of the Weyl subgroup $W_J := \langle s_j \mid \alpha_j \in J \rangle$.

Proof. We consider the case with $i = 0$, i.e. $J = \{\alpha_1, \cdots, \alpha_s\}$. The other cases are obvious by making the shift $\alpha_k \to \alpha_{k+s}$. The Young diagrams $[Y_1, \cdots, Y_l]$ corresponding to this J are given by

$$[(s),\ (s-1,s),\ \cdots,\ (1,\cdots,s),\ (0,1,\cdots,s),\ \cdots,\ (0,1,\cdots,l-1)]\,.$$

Then it is easy to see that the Young diagrams $[Y_1^0, \cdots, Y_l^0]$ with $Y_{k+1}^0 = (0, \cdots, k)$ is transformed to the above $[Y_1, \cdots, Y_l]$ by the longest element w_J given by

$$w_J = s_1 s_2 \cdots s_s s_1 s_2 \cdots s_{s-1} s_1 s_2 \cdots s_{s-2} \cdots s_1 s_2 s_1\,.$$

□

Corollary 3.1 then proves the following theorem found in [1, 9]:

Theorem 3.1. *(Theorem 3.3 in [9]) The compactified isospectral manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ has a decomposition in terms of the Bruhat cells,*

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \bigsqcup_{J \subseteq \Pi} \mathcal{D}_J, \quad \text{with} \quad \mathcal{D}_J = \tilde{Z}(\gamma)_{\mathbb{R}} \bigcap (N^- w_J B^+/B^+)\,.$$

Here $\mathcal{D}_J$ is called the Painlevé divisor associated with J which can be redefined as

$$\lim_{t \to t_J} c_\gamma(L(t)) \in \mathcal{D}_J \overset{\text{def}}{\Longleftrightarrow} \tau_k(t_J) = 0, \text{ iff } k \in J. \tag{3.7}$$

We also define the set Θ_J as a disjoint union of $\mathcal{D}_{J'}$,

$$\Theta_J := \bigsqcup_{J' \supseteq J} \mathcal{D}_{J'} \quad \text{with} \dim \Theta_J = l - |J|\,.$$

Then we have a stratification of $\tilde{Z}(\gamma)_{\mathbb{R}}$,

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \Theta^{(l)} \supset \Theta^{(l-1)} \supset \cdots \supset \Theta^{(0)} \quad \text{with} \quad \Theta^{(k)} = \bigcup_{|J| = l-k} \Theta_J\,.$$

Note here that the 0-cell $\Theta^{(0)} = \mathcal{D}_\Pi = w_* B^+/B^+$ describes a center of the manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$, and it is included in the $\Gamma_{-\cdots-}$ polytope where all the Painlevé divisors meet at this point.

Example 3.2. $\mathfrak{sl}(3,\mathbb{R})$: This case is illustrated in Figure 1, in which there are four hexagons Γ_ϵ which are glued into the compact manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$. The compactification can be done uniquely by identifying the boundaries given by the subsystems $\langle J; [w]; \sigma_J(w^{-1}\cdot\epsilon)\rangle$ (see Example 1.4). One example of $\langle\{\alpha_1\}; [s_1]; (0+)\rangle$ is shown in the Figure, and those two subsystems should be identified. One can also compute the boundary of the manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ by taking account of the orientations of the subsystems (see (1.8)), i.e

$$\begin{aligned} \partial \tilde{Z}(\gamma)_{\mathbb{R}} \;&=\; 2\langle\{\alpha_1\}; [s_1]; (0-)\rangle - 2\langle\{\alpha_2\}; [s_2]; (-0)\rangle \\ &\quad -2\langle\{\alpha_1\}; [s_1]; (0+)\rangle + 2\langle\{\alpha_2\}; [s_2]; (+0)\rangle\,. \end{aligned}$$

The manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is non-orientable, and it was shown in Theorem 8.14 of [7] (also see [11]) that the manifold is smooth and topologically equivalent to a connected sum of two Klein bottles.

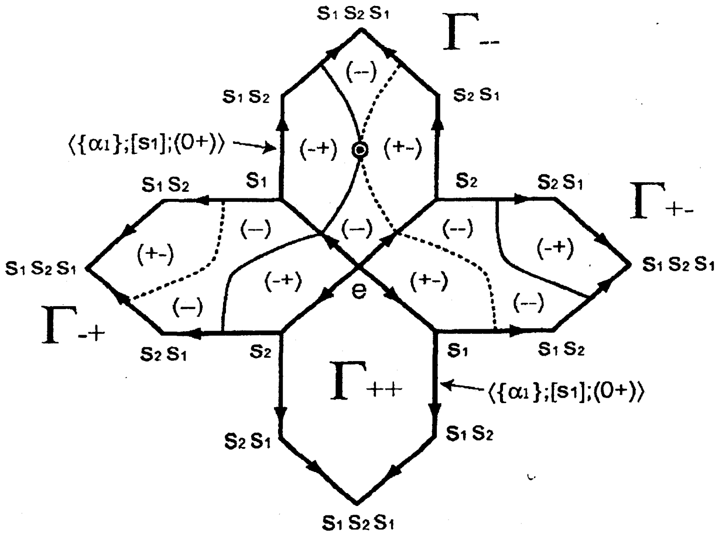

FIGURE 1. The hexagons Γ_ϵ and the Painlevé divisors for $\mathfrak{sl}(3,\mathbb{R})$ Toda lattice. The Painlevé divisors are indicated with a solid curve for $\mathcal{D}_{\{1\}}$ and with a dashed curve for $\mathcal{D}_{\{2\}}$. The double circle at the center of the Γ_{--} polytope is $\mathcal{D}_\Pi$. The arrows in the boundaries of Γ_ϵ's show the flow direction of the Toda orbit.

Notice that each signed hexagon except Γ_{++} further breaks into regions whose boundaries are given by the Painlevé divisors. These regions have also signs given by the pair of $\epsilon_i = \text{sign}(a_i)$, $i = 1, 2$. The second set of signs attached to a region with signs $(\epsilon_1\epsilon_2)$ is simply the W-orbit, $W \cdot (\epsilon_1\epsilon_2)$. The W-actions label the vertices in terms of the elements. The Γ_{--} hexagon is important for the nilpotent cases which will be discussed in some detail below.

In the case of nilpotent L, i.e. $\gamma = 0$, since the G^{C_o}-orbit is an N^+-orbit, the Painlevé divisor $\mathcal{D}_J$ is determined by the intersection between the "opposite" Bruhat cells, that is, N^-- and N^+-orbits. This observation will be a key point in the next section where we discuss the cell decomposition based on the subsystems which consist of smaller Toda equations associated with the subalgebras of the original $\mathfrak{g}$. Then each 1-dimensional Painlevé divisor Θ_J with $|J| = l - 1$ intersects with the corresponding subsystem marked by the compliment of J, i.e. $J^c = \Pi \setminus J$. The intersection occurs at one point which corresponds to the longest element of the Weyl subgroup W_{J^c}, that is, the center of the subsystem.

4. Cell decomposition with the subsystems

In this section, we define the subsystems of Toda lattice and a chain complex based on the subsystems.

4.1. Subsystems.

The subsystems of the Toda lattice is defined as

Definition 4.1. Let $J \subset \Pi$. The subsystem associated with J is defined by

$$\mathcal{S}_J := \{ \ L \in \mathcal{F}_\gamma \subset \mathfrak{g} \ \mid \ a_j = 0 \text{ iff } \alpha_j \in J \ \} .$$

Since the condition $a_j = 0$ is invariant under the Toda flow (see (1.9), i.e. $a_j^0 = 0$ implies $a_j(t) = 0$, $\forall t \in \mathbb{R}$), $\mathcal{S}_J$ defines invariant subvarieties of $Z(\gamma)_{\mathbb{R}}$ which correspond to the Toda lattice defined on the Lie algebra associated with the Dynkin (sub)diagram $(* \cdots * 0 \cdots * 0 * \cdots *)$ where "0" is located at the jth place for $\alpha_j \in J$, and indicates the elimination of j-th dot in the original diagram. Let denote the (sub)algebra associated to the Dynkin diagram of $\mathcal{S}_J$ by

$$\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \subset \mathfrak{g}$$

where m is the number of connected diagrams in $J^c := \Pi \setminus J = \Pi_1 \cup \cdots \cup \Pi_m$, and $\mathfrak{g}_k$ is the simple algebra whose Dynkin diagram is the connected diagram associated to Π_k. Then the subsystem $\mathcal{S}_J$ can be expressed as a product of smaller Toda lattices,

$$\mathcal{S}_J = \overset{\circ}{Z}_{\Pi_1} \times \cdots \times \overset{\circ}{Z}_{\Pi_m} \,,$$

where $\overset{\circ}{Z}_{\Pi_k}$ is the Toda lattice associated to $\mathfrak{g}_k$ with $a_j \neq 0$, $\forall \alpha_j \in \Pi_k$. We then add the Painlevé divisors (blow-ups) to $\mathcal{S}_J$ by the companion embedding $c_\gamma : \mathcal{F}_\gamma \to G/B^+$ (Definition 1.1. A connected set in the image $c_\gamma(\mathcal{S}_J)$ then corresponds to a cell $\langle J; [w]; \sigma_J(w^{-1} \cdot \epsilon)\rangle$ in the decomposition (1.7), which we also refer as a subsystem.

We now express each subsystem as a group orbit: Let P_J be a parabolic subgroup associated with the simple root system J^c containing B^+. Then each subsystem $\langle J; [w]; \sigma_J(w^{-1} \cdot \epsilon)\rangle$ can be expressed by a group orbit of the parabolic subgroup of the normal form $C_\gamma^J(w) \in \mathrm{Lie}(P_J)$,

$$\langle J; [w]; \sigma_J(w^{-1} \cdot \epsilon)\rangle = G^{C_\gamma^J(w)} n_J^{-1} B^+/B^+ \,,$$

where $n_J \in N^- \cap P_J$ is a generic element defined by $L^0 = n_J C_\gamma^J(w) n_J^{-1}$, and the connected subgroup $G^{C_\gamma^J(w)}$ is given by the stabilizer of the element $C_\gamma^J(w)$,

$$G^{C_\gamma^J(w)} := \{\ g \in P_J \mid \mathrm{Ad}_g(C_\gamma^J(w)) = C_\gamma^J(w)\ \}_0 \,,$$

where the suffix "0" indicates the connected component. For example, in the case of $\mathfrak{sl}(l+1, \mathbb{R})$, the element $C_\gamma^J(w)$ with $J = \{\alpha_{n_1+1}\}$ is given by the matrix,

$$C_\gamma^J(w) = \left(\begin{array}{cccc|cccc} 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots & \vdots & \ddots & 0 \\ \xi_{n_1} & \xi_{n_1-1} & \cdots & \xi_0 & 1 & 0 & \cdots & 0 \\ \hline & & & & 0 & 1 & \cdots & 0 \\ & & & & 0 & 0 & \ddots & \vdots \\ & & 0 & & \vdots & \vdots & \ddots & 1 \\ & & & & \eta_{n_2} & \eta_{n_2-1} & \cdots & \eta_0 \end{array}\right),$$

where $\{\xi_k | k = 0, 1, \ldots, n_1\}$ and $\{\eta_j | j = 0, 1, \ldots, n_2\}$ are the symmetric polynomials of the eigenvalues $\{\lambda_{w(k)} | k = 0, 1, \ldots, n_1\}$ and $\{\lambda_{w(l-j)} | j = 0, 1, \ldots, n_2\}$, respectively.

We now consider a nilpotent limit of those subsystems: First recall that the top cell of the $\Gamma_{-\cdots-}$ polytope, $\langle \emptyset; [e]; (-\ldots-)\rangle$, is diffeomorphic to the top cell of the

$\tilde{Z}(0)_{\mathbb{R}}$, which we denote $\langle\emptyset\rangle$, i.e. we have in the limit $\gamma \to 0$,

$$\langle\emptyset;[e];(-\ldots-)\rangle \xrightarrow{\simeq} \langle\emptyset\rangle\,.$$

For the subsystems $\langle J;[w];\sigma_J(w^{-1}\cdot(-\ldots-))\rangle$ of $\Gamma_{-\ldots-}$, one can show:

Proposition 4.1. *For each $J \subset \Pi$ and $\epsilon = (-\ldots-)$, the following nilpotent limit is a diffeomorphism,*

$$\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle \xrightarrow{\simeq} G^{C_0}w^J B^+/B^+\,, \quad \text{if} \quad (\sigma_J(w^{-1}\cdot\epsilon))_j = -,\ \forall\alpha_j \notin J\,,$$

where $[w] \in W/W^J$ and w^J is the longest element in W^J.

Proof. In the nilpotent limit ($\gamma \to 0$), the normal form $C^J_\gamma(w)$ for any J and $[w] \in W/W^J$ converges to the unique element C_0. Also note that only the cells $\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle$ having $(\sigma_J(w^{-1}\cdot\epsilon))_j = -$, $\forall\alpha_j \notin J$ have the intersection with the Painlevé divisor $\mathcal{D}^J$ (the proof is similar to the case of the top cell). Since $\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle$ is the product of the top cells for smaller Toda lattices, it is obvious that each top cell in the subsystem is diffeomorphic to the corresponding nilpotent cell in $G^{C_0}w^J B^+/B^+$. □

One should remark here that the number of subsystems $\langle J;[w];\sigma_J(w^{-1}\cdot\epsilon)\rangle$ having the same limit can be obtained by counting the number of the Weyl elements satisfying the condition in Proposition 4.1. In particular, we have an explicit result for $J = \{\alpha_k\}$ $k = 1,2$ (or $k = l-1, l$) in the case of $\mathfrak{sl}(l+1,\mathbb{R})$ (Lemma 4.2 below). Other cases of simple Lie algebras will be discussed in the next section. This number is important for studying a chain complex of the variety $\tilde{Z}(0)_{\mathbb{R}}$ and its singular structure as will be explained below.

We also remark that the number of such subsystems of codimension one is related to the number of the *real* irreducible components in one dimensional divisor $\mathcal{D}^{\{\alpha_k\}}$. This can be seen by noting that each subsystem $\langle\{\alpha_k\};[w];(-\cdots-\overset{k}{0}-\cdots-)\rangle$ has a unique intersection with the real part $\mathcal{D}^{\{\alpha_k\}}_{\mathbb{R}}$ of the divisor $\mathcal{D}^{\{\alpha_k\}}$. Also each irreducible component in $\mathcal{D}^{\{\alpha_k\}}$ has the intersection with the subsystems at the boundaries of $\Gamma_{-\ldots-}$, i.e. two subsystems intersect with each component of $\mathcal{D}^{\{\alpha_k\}}$. Since there is no intersection between the subsystems with different $[w]$, the total number of subsystems is twice of the number of irreducible components in $\mathcal{D}^{\{\alpha_k\}}_{\mathbb{R}}$.

Now we can state the number of such subsystems. First let us define the following subset of the quotient W/W^J,

$$W^-_{[J]} := \left\{\ [w] \in W/W^J \ \middle|\ \left(\sigma_J(w^{-1}(-\cdots-))\right)_j = -,\ \forall\alpha_j \notin J\ \right\}. \tag{4.1}$$

In particular, as we mentioned above, the number of the elements in $W^-_{[\alpha_k]}$ is related to the number of real irreducible components in $\mathcal{D}^{\{\alpha_k\}}$ as $|W^-_{[\alpha_k]}| = 2|\mathcal{D}^{\{\alpha_k\}}_{\mathbb{R}}|$. Also the following Lemma is useful for finding the elements in $W^-_{[J]}$:

Lemma 4.1. *There exists a duality between two elements in $W^-_{[J]}$,*

$$x \in W^-_{[J]} \qquad \text{iff} \qquad w_* x w^J \in W^-_{[J]}\,.$$

Proof. The duality "$x \in W/W^J$ iff $w_* x w^J \in W/W^J$" is obvious (note that $\ell(xs_k) > \ell(x)$ and $\ell(xw^J s_k) < \ell(x)$ iff $\alpha_k \notin J$). This is a Poincaré duality of the Weyl element (consider a convex polytope whose vertices are the orbit of the

Weyl action, which is also a Morse complex (e.g. see [6])). Then it is easy to show that $w_* \cdot (-\dots-) = (-\dots-)$ and $\sigma_J(w^J \cdot (-\dots-)) = \sigma_J(-\dots-)$. This can be understood as the invariance of the Toda lattice in time $t \to -t$. This symmentry corresponds to the duality between the top and the bottom cells. □

Then one can show te following in the case of $\mathfrak{sl}(l+1, \mathbb{R})$:

Lemma 4.2. *Let $W = S_{l+1}$, the symmetry group of order $l+1$. Let $J = \{\alpha_k\}$ for $k = 1, 2$ (or $k = l-1, l$). Then we have*

- *For $J = \{\alpha_1\}$ (or $\{\alpha_l\}$),*
$$\left|W^-_{[J]}\right| = 2\,,$$
- *For $J = \{\alpha_2\}$ (or $\{\alpha_{l-1}\}$),*
$$\left|W^-_{[J]}\right| = 2\left\lfloor \frac{l+1}{2} \right\rfloor\,,$$
where $\lfloor x \rfloor$ is the maximum integer of x.

Proof. For $J = \{\alpha_1\}$, the following two Weyl elements are obviously in $W^-_{[\alpha_1]}$,
$$w = e, \quad s_l s_{l-1} \cdots s_2 s_1\,.$$
Note the duality $s_l s_{l-1} \cdots s_2 s_1 = w_* e w^{\{\alpha_1\}}$ (see Lemma 4.1). Since the subsystems $\langle\{\alpha_1\}; [w]; (0 - \dots -)\rangle$ intersect with the divisor $\mathcal{D}^{\{\alpha_1\}}$, one can show by counting the number of irreducible components in the divisor that those are only the elements in $W^-_{[\alpha_1]}$: First recall that the divisor $\mathcal{D}^{\{\alpha_1\}}$ is given by the condition,
$$\tau_k(t_1, \dots, t_l) = 0, \quad \text{for} \quad k = 2, 3, \dots, l\,.$$
For sufficiently small γ, this is equivalent to the conditions on the Schur polynomials,
$$p_{\bar{k}}(t_1, \dots, t_k) = 0, \quad \text{for} \quad k = 2, \dots, l\,.$$
This implies that the divisor has just one connected component parametrized by
$$\mathcal{D}^{\{\alpha_1\}} \cong \left\{ (t_1, \dots, t_l) \in \mathbb{R}^l \;\middle|\; t_k = \frac{1}{k} t_1^k \text{ for } k = 2, \dots, l \right\}.$$
Then those two subsystems intersect with the divisor $\mathcal{D}^{\{\alpha_1\}}$ in the limits $t_1 \to \pm\infty$, which shows that there is no other element in $W^-_{[\alpha_1]}$. The case for $J = \{\alpha_l\}$ is obvious.

For $J = \{\alpha_2\}$, one can easily find that the following elements are in $W^-_{[\alpha_2]}$:

- For $l =$ even, we find l elements,
$$w = e,\ s_1 s_2,\ s_2 s_3 s_1 s_2,\ \dots,\ \overbrace{s_{l-1} s_l \cdots s_1 s_2}^{2l-2}\,.$$
Here the first half elements are dual to the second half, e.g. $s_{l-1} s_l \cdots s_1 s_2 = w_* e w^{\{\alpha_2\}}$. Also note $\ell(w_* w^{\{\alpha_2\}}) = 2l - 2$.
- For $l =$ odd, we find $l+1$ elements with the same l elements as above plus one other element,
$$w = s_l s_{l-1} \cdots s_2\,.$$
This element is self-dual, i.e. $w = w_* w w^{\{\alpha_2\}}$ (note $\ell(w) = l - 1$).

Then from Lemma 4.3 below, the number of real components (loops) in $\mathcal{D}^{\{\alpha_2\}}$ is given by $\lfloor (l+1)/2 \rfloor$. This implies that all the elements in $W^-_{[\alpha_k]}$ are given by those we already found. □

The following Lemma gives the number of real irreducible components in the Painlevé divisor $\mathcal{D}^{\{\alpha_2\}}$ for the case of $\mathfrak{sl}(l+1,\mathbb{R})$.

Lemma 4.3. *All the irreducible components of $\mathcal{D}^{\{\alpha_2\}}$ are real, and the total number of the components is given by*

$$\left|\mathcal{D}^{\{\alpha_2\}}\right| = \left\lfloor \frac{l+1}{2} \right\rfloor .$$

Proof. First note that the divisor $\mathcal{D}^{\{\alpha_2\}}$ is given by the condition,

$$\tau_k(t_1, \dots, t_l) = 0, \quad \forall k \quad \text{except } k = 2 .$$

Then using (3.4) for the formulae of $\overline{\tau}_k$, one can see that this condition is equivalent to $p_{\overline{k}} = 0$ for $k = 3, 4, \dots, l$ and $\overline{\tau}_1 = 0$ which is the $l \times l$ determinant,

$$\begin{vmatrix} 0 & \cdots & 0 & p_{\overline{2}} & p_{\overline{1}} \\ 0 & \cdots & p_{\overline{2}} & p_{\overline{1}} & 1 \\ \vdots & \ddots & \vdots & 1 & 0 \\ p_{\overline{2}} & \cdots & \vdots & \vdots & \vdots \\ p_{\overline{1}} & 1 & \cdots & 0 & 0 \end{vmatrix} = 0 . \tag{4.2}$$

Now we show that this equation has $\lfloor (l+1)/2 \rfloor$ real roots:

For l =even, say $l = 2n$, first note that $p_{\overline{1}} (= p_1) = 0$ is not a solution of (4.2). Then setting $p_{\overline{2}} = x p_{\overline{1}}^2$, the determinant becomes a polynomial of x of degree $n = \lfloor (l+1)/2 \rfloor$. Thus n is the maximum number of real roots, that is, the number of irreducible components in $\mathcal{D}^{\{\alpha_2\}}$. On the other hand, in the proof of Lemma 4.2 we found that the number of the subsystems having the intersection with $\mathcal{D}^{\{\alpha_2\}}$ is at least $l = 2n$. This shows that n must be the number of real roots, that is, all the roots are real.

For l =odd, say $l = 2n - 1$, first note that $p_1 = 0$ is a simple solution of (4.2). For other solutions, we set $p_{\overline{2}} = x p_{\overline{1}}^2$. Then (4.2) gives a polynomial of x of degree $n - 1 = \lfloor l/2 \rfloor$. Thus the maximum number of real roots for (4.2) is $n = \lfloor (l+1)/2 \rfloor$. Again from the proof of Lemma 4.2, the number of the subsystems is at least $l + 1 = 2n$. This then implies that n must be the number of real roots. □

Remark 4.2. A. Nemethi informed us that the number of irreducible components in $\mathcal{D}^{\{\alpha_k\}}$ is given by the number of equivalent k-gons formed from the k vertices of a regular $(l+1)$-gon in which the equivalence is given by the rotation. The number of real components is then given by the number of k-gons having the reflective symmetry with respect to a line. The details will be reported elsewhere.

Example 4.3. For $\mathfrak{sl}(3,\mathbb{R})$, we have

$$W^-_{[\alpha_1]} = \{e, \ s_2 s_1\}, \quad W^-_{[\alpha_2]} = \{e, \ s_1 s_2\} .$$

This indicates that each divisor $\mathcal{D}^{\{\alpha_k\}}$ has one component intersecting with the subsystems marked by the elements in $W^-_{[\alpha_k]}$. Those subsystems have the same orientation (i.e. the lengths $\ell(w)$ are all even).

For $\mathfrak{sl}(4,\mathbb{R})$, we have

$$W^-_{[\alpha_1]} = \{e,\ s_3s_2s_1\}, \quad W^-_{[\alpha_2]} = \{e,\ s_1s_2,\ s_3s_2,\ s_2s_3s_1s_2\},$$
$$W^-_{[\alpha_3]} = \{e,\ s_1s_2s_3\}.$$

Notice that there are two components in $\mathcal{D}^{\{\alpha_2\}}$ intersecting with the subsystems having the same orientation.

We now denote the subsystem in the nilpotent limit as $\langle J\rangle$ for each $J \subset \Pi$, and then we have a cell decomposition of the compactified variety $\tilde{Z}(0)_{\mathbb{R}}$,

$$\tilde{Z}(0)_{\mathbb{R}} = \bigsqcup_{J\subseteq\Pi} \langle J\rangle\,, \quad \text{with} \quad \langle J\rangle := G^{C_0}w^J B^+/B^+\,.$$

The compactification of $\langle J\rangle$ is obtained in the similar way as in the case of the Painlevé divisor Θ_J, i.e.

$$\overline{\langle J\rangle} = \bigsqcup_{J'\supseteq J} G^{C_\gamma}w^{J'}B^+/B^+\,.$$

Then we have a stratification of the variety $\tilde{Z}(0)_{\mathbb{R}}$,

$$\tilde{Z}(0)_{\mathbb{R}} = \Sigma^{(l)}(\gamma) \supset \Sigma^{(l-1)} \supset \cdots \supset \Sigma^{(0)}, \quad \text{with } \Sigma^{(k)} := \bigcup_{|J|=l-k} \overline{\langle J\rangle}\,.$$

The number of components in each $\Sigma^{(k)}$ is given by

$$\left|\Sigma^{(k)}\right| = \binom{l}{k}\,.$$

For a convenience, let us denote each subsystem $\langle J\rangle$ as

$$\langle J\rangle = (*\cdots*0\cdots*0*\cdots*)\,,$$

where 0's are assigned at the vertices $\alpha_j \in J$. For example, $\langle\{\alpha_{n+1}\}\rangle = (\overbrace{*\cdots*}^{n}0*\cdots*)$. Thus each component can be uniquely labeled by $J \subset \Pi$ which gives the arrangement of the "0"s in the diagram (compare with the case of generic γ in the Introduction (see also [7])).

Example 4.4. $\mathfrak{sl}(3,\mathbb{R})$: In Figure 2, the left hexagon is the polytope Γ_{--} in Figure 1, which collapses to a square in the right as a limit of nilpotent case. In the limit, the subsystems $\langle\{\alpha_1\};[s_1];(0+)\rangle$ and $\langle\{\alpha_2\};[s_2];(+0)\rangle$ are squeezed to the point $\langle\Pi\rangle = (00)$, the 0-cell. The subsystems $\langle\{\alpha_1\};[e];(0-)\rangle$ and $\langle\{\alpha_1\};[s_2s_1];(0-)\rangle$ have the same limit to $\langle\{\alpha_1\}\rangle = (0*)$. This implies that the two sides of the square corresponding to the limit of those subsystems should be identified. The other two subsystems corresponding to $J = \{\alpha_2\}$ with the sign (-0) have also the same limit to $\langle\{\alpha_2\}\rangle = (*0)$, which are also identified. This process of identification provides the compactification of the G^{C_0}-orbit.

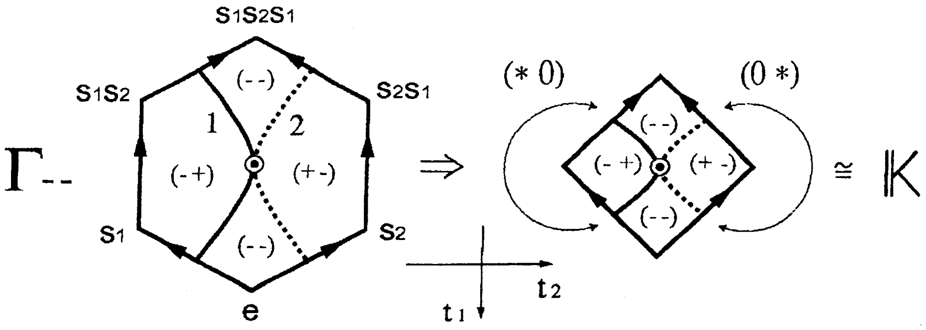

FIGURE 2. The $\mathfrak{sl}(3,\mathbb{R})$ Toda lattice in the nilpotent limit. The Painlevé divisors $\mathcal{D}_{\{1\}}$ and $\mathcal{D}_{\{2\}}$ are shown as the solid and the dashed curves, respectively. In the limit, the left two sides of the square are identified, so are the right two sides. The compact variety is then homologous to the Klein bottle $\mathbb{K}$.

Since those two subsytems for each $J = \{\alpha_k\}$ have the same orientation (i.e. both $\ell(w) = \text{even}$), the subsystem $\langle\{\alpha_k\}\rangle$ contributes as the boundary of $\tilde{Z}(0)_{\mathbb{R}}$, that is, the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ is nonorientable. The variety $\tilde{Z}(0)_{\mathbb{R}}$ is homologous to the Klein bottle $\mathbb{K}$.

Remark 4.5. As mentioned in [9], the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ for $\mathfrak{sl}(3,\mathbb{R})$ has the A_2-type singularity at the 0-cell $\langle\Pi\rangle$. This can be seen as follows: Let L be the Lax matrix,

$$L = \begin{pmatrix} b_1 & 1 & 0 \\ a_1 & b_2 - b_1 & 1 \\ 0 & a_2 & -b_2 \end{pmatrix} .$$

Then the Chevalley invariants $I_k(L)$ are

$$I_1 = b_1 b_2 - b_1^2 - b_2^2 - a_1 - a_2, \quad I_2 = b_1 b_2 (b_1 - b_2) + a_1 b_2 - a_2 b_1 .$$

In the nilpotent case ($I_1 = 0$ and $I_2 = 0$), we have

$$x^3 + yz = 0, \quad \text{with} \quad x = b_1,\ y = a_1,\ z = b_1 + b_2.$$

In the general case of $\mathfrak{sl}(l+1,\mathbb{R})$, one can show that the two dimensional Painlevé divisor $\mathcal{D}^J$ with $J = \{\alpha_1, \alpha_2\}$ (or $J = \{\alpha_{l-1}, \alpha_l\}$) gives an Arnold slice with the A_l-type surface singularity at the 0-cell. (see Proposition 4.2 in [8]). The details will be discussed in a future communication.

4.2. **The subsystems of codimension one.** Here we consider the case of $\mathfrak{sl}(l+1,\mathbb{R})$ Toda lattice, and give a detailed description of the subsystems of codimension one, $\langle\{\alpha_{n_1+1}\}\rangle$ for $n_1 = 0, 1, \ldots, l-1$ as boundaries of the top cell $\langle\emptyset\rangle = (*\cdots*)$. First recall that each cell $\langle\{\alpha_{n_1+1}\}\rangle$ is the orbit, $\langle\{\alpha_{n_1+1}\}\rangle = G^{C_0} w^{\{\alpha_{n_1+1}\}} B^+/B^+$. Here the longest element w^J takes the form,

$$w^J = \left(\begin{array}{c|c} R_1 & 0 \\ \hline 0 & R_2 \end{array} \right) ,$$

with the $(n_j+1)\times(n_j+1)$ matrices,

$$R_j := \begin{pmatrix} 0 & \cdots & \pm 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \quad \text{for} \quad j=1,2.$$

Thus with the element $g \in G^{C_0}$ in (1.14), we have for $J=\{\alpha_{n_1+1}\}$

$$(4.3) \quad gw^J = \left(\begin{array}{cccc|cccc} p_{n_1} & p_{n_1-1} & \cdots & \pm p_0 & & & & \\ p_{n_1-1} & p_{n_1-2} & \cdots & 0 & & & * & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ p_0 & 0 & \cdots & 0 & & & & \\ \hline & & & & p_{n_2} & p_{n_2-1} & \cdots & \pm p_0 \\ & & & & p_{n_2-1} & p_{n_2-2} & \cdots & 0 \\ & 0 & & & \vdots & \vdots & \ddots & \vdots \\ & & & & p_0 & 0 & \cdots & 0 \end{array}\right),$$

It is then obvious that the τ-functions (1.16) generated by (4.3) provide the solutions of the subsystem consisting of two smaller Toda systems associated with $\Pi_1 = \{\alpha_i \,|\, i=1,\cdots,n_1\}$ and $\Pi_2=\{\alpha_{n_1+1+j}\,|\,j=1,\cdots,n_2\}$: Note here that $\tau_{n_1+1}=1$ which implies $a_{n_1+1}=0$ as requested (recall $a_j = db_j/dt = d^2\ln\tau_j/dt^2$). One should also note that gw^J can be decomposed into actions on the Grassmannians $Gr(k,n_1+1)$ and $Gr(j,n_2+1)$ as

$$(4.4) \qquad g_1\times g_2 \curvearrowright \left(\bigwedge^k \mathbb{R}^{n_1+1}\right)\otimes\left(\bigwedge^j \mathbb{R}^{n_2+1}\right),$$

where g_j is given by

$$g_j := \begin{pmatrix} p_{n_j} & p_{n_j-1} & \cdots & \pm p_0 \\ p_{n_j-1} & p_{n_j-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_0 & 0 & \cdots & 0 \end{pmatrix}.$$

Example 4.6. The cell decomposition for $\mathfrak{sl}(3,\mathbb{R})$ Toda lattice: The $g\in G^{C_0}$ is given by

$$g=\begin{pmatrix} 1 & p_1 & p_2 \\ 0 & 1 & p_1 \\ 0 & 0 & 1\end{pmatrix}, \quad p_1=t_1,\ p_2=t_2+\frac{1}{2}t_1^2.$$

Then the cells in the decomposition of $\tilde{Z}(0)_{\mathbb{R}}$ are given by

- 2-cell: this is the top cell $\langle\emptyset\rangle = (**)$,

$$\langle\emptyset\rangle = \left\{\begin{pmatrix} p_2 \\ p_1 \\ 1\end{pmatrix},\ \begin{pmatrix} p_1 & p_2 \\ 1 & 0 \\ 0 & 1\end{pmatrix}\right\} \subset V[(2),(12)] := N^+ s_1 s_2 s_1 B^+/B^+,$$

- 1-cell: there are two 1-cells, i.e. $\langle\{\alpha_1\}\rangle = (0*)$ and $\langle\{\alpha_2\}\rangle = (*0)$,

$$\langle\{\alpha_1\}\rangle = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & p_1 \\ 0 & 1 \end{pmatrix} \right\} \subset V[(0),(02)] := N^+ s_2 B^+/B^+ ,$$

$$\langle\{\alpha_2\}\rangle = \left\{ \begin{pmatrix} p_1 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \subset V[(1),(01)] := N^+ s_1 B^+/B^+ ,$$

- 0-cell: the cell is the unique fixed point $\langle\emptyset\rangle = (00)$,

$$\langle\Pi\rangle = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \subset V[(0),(01)] := eB^+/B^+ .$$

Note here that each Bruhat cell $V[(i_0),(i_0i_1)] = N^+w^JB^+/B^+$ is complementary to the opposit cell $W[(i_0),(i_0i_1)] = N^-w^JB^+/B^+$ introduced in Section 3.

Now we express the subsystem $(\overbrace{*\cdots*}^{n_1}\,0\,\overbrace{*\cdots*}^{n_2})$ as a limit of the G^{C_0}-orbit in the top cell $G^{C_0}w_*B^+/B^+$: We first recall that the center of the subsystem $\langle\{\alpha_{n_1+1}\}\rangle$ is given by $w^{\{\alpha_{n_1+1}\}}B^+/B^+$ which is the intersection point with the 1-dimensional Panlevé divisor $\mathcal{D}^{\{\alpha_{n_1+1}\}}$. This implies that the 1-dimensional divisor $\mathcal{D}^{\{\alpha_{n_1+1}\}}$ connects the center of the variety, w_*B^+/B^+ with the center of the subsystem. Since the divisor intersects transversally with the subsystem, one can introduce a local coordinate system near the center of the subsystem. Let us recall

$$\mathcal{D}^{\{\alpha_{n_1+1}\}} \cong \{\ t = (t_1,\dots,t_l) \in \mathbb{R}^l \mid \tau_k(t) = 0\ \forall k \text{ except } k = n_1+1\ \} .$$

With (3.3), i.e. $\tau_k = \pm S_{(l-k+1,\dots,l)}$, the zero conditions for the τ-functions are also written as

$$\text{(4.5)} \qquad \begin{cases} p_{n_2+1+k} = 0 & \text{for} \quad 1 \le k \le n_1 , \\ p_{\overline{n_1+1+j}} = 0 & \text{for} \quad 1 \le j \le n_2 . \end{cases}$$

Notice that on the divisor $\mathcal{D}^{\{\alpha_{n_1+1}\}}$, we have $p_{n_2+1} \neq 0$ and $p_{\overline{n_1+1}} \neq 0$. Then a local coordinates, denoted as $(q_1,\dots,q_{n_1},r_1,\dots,r_{n_2})$, for a neiborhood of the center of the subsystem $\langle\{\alpha_{n_1+1}\}\rangle$ can be given by the following homogeneous functions,

$$\text{(4.6)} \qquad \begin{cases} q_k = \dfrac{p_{n_2+1+k}}{p_{n_2+1}} & \text{for} \quad 1 \le k \le n_1 , \\[2ex] r_{\bar{j}} = \dfrac{p_{\overline{n_1+1+j}}}{p_{\overline{n_1+1}}} & \text{for} \quad 1 \le j \le n_2 , \end{cases}$$

The variables q_k and $r_{\bar{k}}$ both have the weight k, and $\{\, r_j \,|\, j = 0,1,\cdots,n_2 \}$ are defined in the same way as in the case of p_k defined from $p_{\bar{j}}$, i.e.

$$r_k = \|r_{\bar{1}},\cdots,r_{\bar{k}}\| .$$

Then we have

Proposition 4.2. *Consider the limits $p_1 \to \infty$ (or $p_2 \to \infty$) so that the new variables (q_k, r_j) remain finite, and they give a coordinate system for the subsystem. Then the following limit is a diffeomorphism for the group element $g \in G^{C_0}$ in (3.1),*

$$gw_* \xrightarrow{\simeq} g_1^q \times g_2^r ,$$

where g_1^q and g_2^r are given by

$$g_1^q = \begin{pmatrix} q_{n_1} & q_{n_1-1} & \cdots & \pm q_0 \\ q_{n_1-1} & q_{n_1-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_0 & 0 & \cdots & 0 \end{pmatrix}, \qquad g_2^r = \begin{pmatrix} r_{n_2} & r_{n_2-1} & \cdots & \pm r_0 \\ r_{n_2-1} & r_{n_2-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & 0 & \cdots & 0 \end{pmatrix},$$

where $q_0 = r_0 = 1$.

Proof. Taking the limits with the change of variables (4.6), the τ-functions become

$$\begin{cases} \tau_{j+1} = \|p_l, \cdots, p_{l-j}\| = (p_{n_2+1})^{j+1}\|q_{n_1}, \cdots, q_{n_1-j}\| & + \quad O\left((p_{n_2+1})^j\right) , \\ \bar{\tau}_{l-k} = \|p_{\bar{l}}, \cdots, p_{\overline{l-k}}\| = (p_{\overline{n_1+1}})^{k+1}\|r_{\bar{n}_2}, \cdots, r_{\overline{n_2-k}}\| & + \quad O\left((p_{\overline{n_1+1}})^k\right) , \end{cases}$$

where $0 \le j \le n_1 - 1$ and $0 \le k \le n_2 - 1$. Note here that $\tau_{n_1+1} = \pm\bar{\tau}_{n_1+1}$ has the weight $(n_1+1)(n_2+1)$ and becomes a constant in the limit $p_1 \to \pm\infty$. Define the following homogeneous functions,

$$\begin{cases} \tau_{j+1}^q := \lim_{|p_1|\to\infty} \dfrac{\tau_{j+1}}{(p_{n_2+1})^{j+1}} = \|q_{n_1}, \cdots, q_{n_1-j}\| , & \text{for} \quad 0 \le j \le n_1 - 1 \\ \bar{\tau}_{n_2-k}^r := \lim_{|p_1|\to\infty} \dfrac{\bar{\tau}_{l-k}}{(p_{\overline{n_1+1}})^{k+1}} = \|r_{\bar{n}_2}, \cdots, r_{\overline{n_2-k}}\| , & \text{for} \quad 0 \le k \le n_2 - 1 . \end{cases}$$

In particular, the $\bar{\tau}_{n_2-k}^r$ functions can be equivalently written by

$$\tau_{k+1}^r = \|r_{n_2}, \cdots, r_{n_2-k}\| , \quad \text{for} \quad 0 \le k \le n_2 - 1 .$$

Those τ_j^q and τ_k^r define the τ-functions for the two smaller Toda systems associated with $\mathfrak{sl}(n_1+1, \mathbb{R})$ and $\mathfrak{sl}(n_2+1, \mathbb{R})$, which are separated by the condition $a_{n_1+1} = 0$ in the limit. This implies that gw_* takes the desired form in the limit, which provides those τ-functions. □

Example 4.7. We consider the case of $\mathfrak{sl}(3, \mathbb{R})$ for a detailed discussion of the limits, $(**) \to (*0)$ and $(**) \to (0*)$. In this case, the τ-functions are given by

$$\begin{cases} \tau_1(t_1, t_2) & = & p_2(t_1, t_2) & = & t_2 + \dfrac{1}{2}t_1^2 , \\ \tau_2(t_1, t_2) & = & p_{\bar{2}}(t_1, t_2) & = & t_2 - \dfrac{1}{2}t_1^2 \end{cases}$$

For the limit, $(**) \to (*0)$, from (4.6) we use

$$p_2 = q_1 p_1 , \quad \text{or} \quad t_2 = -\frac{1}{2}t_1^2 + q_1 t_1 .$$

In terms of the τ-functions, the limit gives

$$\tau_1^q = \lim_{|p_1|\to\infty} \frac{\tau_1}{p_1} = q_1, \quad \text{and} \quad \tau_2^q = \lim_{|p_1|\to\infty} \frac{\tau_2}{p_1^2} = -1,$$

which implies

$$a_1 = \frac{\tau_2}{\tau_1^2} \to -\frac{1}{q_1^2}, \quad \text{and} \quad a_2 = \frac{-\tau_1}{\tau_2^2} \to 0.$$

Thus the limit is the solution for $(*0)$ of $\mathfrak{sl}(2,\mathbb{R})$. Also for the limit, $(**) \to (0*)$, we use

$$p_{\bar{2}} = r_1 p_1 , \quad \text{or} \quad t_2 = \frac{1}{2}t_1^2 + r_1 t_1 .$$

Then in terms of the τ-functions, we have

$$\bar{\tau}_1^r = \lim_{|p_1|\to\infty} \frac{\overline{\tau_2}}{p_1} = r_1, \quad \text{and} \quad \bar{\tau}_0^r = \lim_{|p_1|\to\infty} \frac{\overline{\tau_1}}{p_1^2} = 1,$$

which implies

$$a_1 = \frac{\tau_2}{\tau_1^2} \to 0, \quad a_2 = \frac{-\tau_1}{\tau_2^2} \to -\frac{1}{r_1^2}.$$

The limit is the solution for $(0*)$ of $\mathfrak{sl}(2,\mathbb{R})$. Here the top cell $(**)$ is described as $\{(t_1,t_2) \in \mathbb{R}^2\}$, and the Painlevé divisors are given by $\Theta_{\{1\}} = \{t_2 + t_1^2/2 = 0\}$ and $\Theta_{\{2\}} = \{t_2 - t_1^2/2 = 0\}$ (see Figure 2, where the Painlevé divisors are shown as the graphs in the (t_1,t_2)-coordinates). The new variables q_1 and r_1 are the parameters for the subsystems $(*0)$ and $(0*)$.

5. The chain complex based on the subsystems

The $\mathbb{Z}$-modules of the set $\{\langle J\rangle \mid J \subset \Pi\}$ defines a chain complex $(\mathcal{C}_*, \partial_*)$,

(5.1) $$0 \longrightarrow \mathcal{C}_l \xrightarrow{\partial_l} \mathcal{C}_{l-1} \xrightarrow{\partial_{l-1}} \cdots \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \longrightarrow 0,$$

with

$$\mathcal{C}_k := \bigoplus_{|J|=l-k} \mathbb{Z}\langle J\rangle .$$

The boundary map ∂_k acts on $\langle J\rangle \in \mathcal{C}_k$, $(|J| = l-k)$ as follows:

$$\partial_k\langle J\rangle = \sum_{\alpha\notin J} [\, J \,;\, J\cup\{\alpha\}\,]\,\langle J\cup\{\alpha\}\rangle\, ,$$

where $[\,J\,;\,J'\,]$ with $J' = J\cup\{\alpha\}$ is the incidence number. In terms of the notation $\langle J\rangle = (*\cdots 0\cdots 0 * \cdots *)$, the operator ∂_k adds one more "0" at the place with $\alpha \notin J$. To compute the incidence number, it is sufficient to consider only the boundary map on the top cell $\langle\emptyset\rangle$ for each case of simple Lie algebra, and the general case can be computed inductively. We thus consider

$$\partial_l\langle\emptyset\rangle = \sum_{k=1}^{l} [\,\emptyset\,;\,\{\alpha_k\}\,]\,\langle\{\alpha_k\}\rangle\, .$$

The key ingredient for computing the incidence number $[\emptyset;\{\alpha_k\}]$ is to count the number of subsystems $\langle\{\alpha_k\};[w];(-\cdots-\overset{k}{0}-\cdots-)\rangle$ with different $[w] \in W/W^{\{\alpha_k\}}$ which have the same limit to $\langle\{\alpha_k\}\rangle$. Then taking account of the orientation of each subsystem which is given by the length $\ell(w)$ (see (1.8)), we have

Definition 5.1. The incidence numbers are defined by

$$[\emptyset\,;\{\alpha_k\}\,]=(-1)^{k-1}\Big|\sum_{w\in W_-^{[\alpha_k]}}(-1)^{\ell(w)}\Big|,$$

where $W_-^{[\alpha_k]}\subset W/W^{\{\alpha_k\}}$ is defined in (4.1). In the general case, for $\alpha_k\notin J$ the incidence number $[J;J']$ with $J'=J\cup\{\alpha_k\}$ is given by

$$[\,J\,;\,J'\,]=(-1)^{\nu(J;J')}\Big|[\,J\,;\,J'\,]\Big|,$$

where $\nu(J;J')$ is given by

$$\nu(J;J\cup\{\alpha_k\}):=\Big|\{\,\alpha_j\notin J\,|\,1\le j<k\,\}\Big|.$$

Here the number $|[J;J']|$ is given by $|[\emptyset;\{\alpha_k\}]|$ for the smaller system corresponding to the connected Dynkin subdiagram including α_k.

Note here that the sign $(-1)^{\nu(J;J')}$ is necessary to satisfy the chain complex condition, $\partial^2=0$: Applying ∂^2 to a cell $\langle J_1\rangle$, we have

$$\partial^2\langle J_1\rangle=([J_1\,;J_2][J_2;J_4]+[J_1\,;J_3][J_3;J_4])\,\langle J_4\rangle+\cdots,$$

where $J_2=J_1\cup\{\alpha_i\}$, $J_3=J_1\cup\{\alpha_j\}$ with $\alpha_i,\alpha_j\notin J_1$ $(i\ne j)$, and $J_4=J_2\cup J_3$. Then $\partial^2=0$ implies

$$[J_1\,;J_2][J_2;J_4]+[J_1\,;J_3][J_3;J_4]=0, \tag{5.2}$$

that is, the functions $\nu(J_n;J_m)$ have to satisfy the condition,

$$\nu(J_1;J_2)+\nu(J_2;J_4)+\nu(J_1;J_3)+\nu(J_3;J_4)=\text{odd}\,.$$

Assuming $i<j$, one can easily show that

$$\nu(J_1;J_1\cup\{\alpha_i\})=\nu(J_1\cup\{\alpha_j\};J_4),\quad \nu(J_1;J_1\cup\{\alpha_j\})=\nu(J_1\cup\{\alpha_i\};J_4)+1\,,$$

which give the above condition.

In Proposition 5.1 below, we give the explicit form of the incidence numbers for the case $\mathfrak{sl}(l+1,\mathbb{R})$. Other cases will be discussed in the next sections. The key of computing the incidence numbers is to find all the elements in $W^-_{[\alpha_k]}$ as shown in the case of $\mathfrak{sl}(l+1,\mathbb{R})$.

Now we state the following Proposition on the incidence numbers for the case of $\mathfrak{sl}(l+1,\mathbb{R})$:

Proposition 5.1. *The incidence numbers $[\emptyset;\{\alpha_k\}]$ are given by*

$$[\,\emptyset\,;\{\alpha_k\}\,]=\begin{cases}\left((-1)^{k-1}-1\right)B\left(\dfrac{l+1}{2},\left\lfloor\dfrac{k}{2}\right\rfloor\right) & \text{for}\quad l=\text{odd}\,,\\[2ex] 2(-1)^{k-1}B\left(\dfrac{l}{2},\left\lfloor\dfrac{k}{2}\right\rfloor\right) & \text{for}\quad l=\text{even}\end{cases}$$

where $B(n,m)$ is the binomial coefficient $\binom{n}{m}$.

Proof. We use the mathematical induction. The case of $l=1$ is trivial. The cases of $l=2,3$ can be shown directly from Example 4.3. Then we assume that the formulae are correct up to the rank $l-1$.

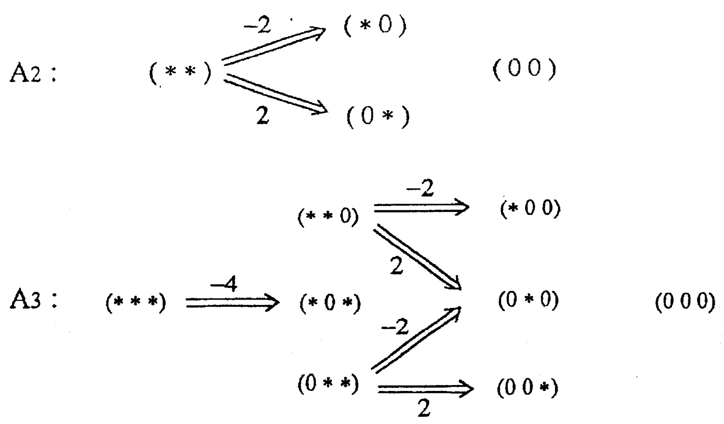

FIGURE 3. The weighted graphs $\mathcal{G}_{A_l}$ for $l = 2, 3$.

For the case l = odd, we first recall from Lemma 4.2 and its proof that $W^-_{[\alpha_1]}$ contains two elements with length 0 and l, and $W^-_{[\alpha_2]}$ contains $l+1$ elements whose length are all even. This implies $[\emptyset; \{\alpha_1\}] = 0$ and $[\emptyset; \{\alpha_2\}] = -(l+1)$ which agree with the above formula. Now from the chain complex condition (5.2) for $J_1 = \emptyset$, $J_2 = \{\alpha_1\}$, $J_3 = \{\alpha_k\}$ and $J_4 = \{\alpha_1, \alpha_k\}$, we see $[\emptyset; \{\alpha_k\}][\{\alpha_k\}; \{\alpha_1, \alpha_k\}] = 0$. Then for k = odd, the subsystem $\langle\{\alpha_k\}\rangle$ is the product of two smaller systems with rank $k-1$ and $l-k$. Since $[\{\alpha_k\}; \{\alpha_1, \alpha_k\}] \neq 0$ ($k-1$ is even), $[\emptyset; \{\alpha_k\}] = 0$ for all k. This agrees with the formula.

Now we consider the condition (5.2) for $J_1 = \emptyset$, $J_2 = \{\alpha_2\}$, $J_3 = \{\alpha_k\}$ and $J_4 = \{\alpha_2, \alpha_k\}$. Then if k = even, we have $[\{\alpha_k\}; \{\alpha_2, \alpha_k\}] = -k$ (using the result for the rank $k-1$) and $[\{\alpha_2\}; \{\alpha_2, \alpha_k\}] = 2B(\frac{l-1}{2}, \frac{k-2}{2})$ (using the result for the rank $l-2$ and the sign $(-1)^{k-2} = 1$). Writing $l = 2n-1$ and $k = 2m$, the condition (5.2) gives

$$2(2n)B(n-1, m-1) + 2m[\emptyset; \{\alpha_k\}] = 0\,,$$

which implies the formula for even k.

For the case of l = even, first note from Lemma 4.2 that $[\emptyset; \{\alpha_1\}] = 2$ and $[\emptyset; \{\alpha_2\}] = -l$. Then following the above argument, we can confirm the formula. □

Proposition 5.1 provides a sufficient information to compute the integral homology for the chain complex $\mathcal{C}_*$. To summarize the results in this section, we give Examples for the Lie algebras of type A_l ($\mathfrak{sl}(l+1, \mathbb{R})$) for $l = 2, 3$ which we present in a *weighted graph*.

Definition 5.2. The weighted graph $\mathcal{G}$ of the chain complex $(\mathcal{C}_*, \partial_*)$ consists of the vertices given by the cells $\langle J \rangle$ for $J \subset \Pi$ and *weighted* edges "$\overset{m}{\Longrightarrow}$" with $m \in \mathbb{Z}^*$ between the cells $\langle J \rangle$ and $\langle J' \rangle$ with $|J'| = |J| + 1$. The oriented edge is defined as

$$\langle J \rangle \overset{m}{\Longrightarrow} \langle J' \rangle \quad \text{implies} \quad m = [\,J;\, J'\,] \neq 0\,,$$

and if $[J; J'] = 0$, then there is no edge between $\langle J \rangle$ and $\langle J' \rangle$.

Example 5.3. From the graphs, we can find the integral homology $H_*(\tilde{Z}(0)_{\mathbb{R}}, \mathbb{Z})$:

For the case of A_2, we have

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z} \oplus \mathbb{Z}_2, \quad H_2 = 0.$$

For the case of A_3, we have

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z} \oplus 2\mathbb{Z}_2, \quad H_2 = \mathbb{Z}_4, \quad H_3 = 0.$$

The following is direct from Proposition 5.1,

Corollary 5.1. *The compactified isospectral variety $\tilde{Z}(0)_{\mathbb{R}}$ for $\mathfrak{sl}(l+1,\mathbb{R})$ Toda lattice is nonorientable in the sense, $H_l(\tilde{Z}(0)_{\mathbb{R}},\mathbb{Z}) = 0$, and in particular we have:*

- *for the case of type A_l with $l =$ even,*

$$H_{l-1}(\tilde{Z}(0)_{\mathbb{R}},\mathbb{Z}) = \mathbb{Z}_2.$$

- *for the case of type A_{2p-1} with $p =$ prime,*

$$H_{2p-2}(\tilde{Z}(0)_{\mathbb{R}},\mathbb{Z}) = \mathbb{Z}_{2p}.$$

6. Other Examples

Here we give a basic information on the generalized Toda lattices for the Lie algebras B_l, C_l and G_2. In particular, we provide the explicit structure of the compact varieties $\tilde{Z}(0)_{\mathbb{R}}$ for those of rank two cases.

6.1. **Toda lattice of type C_l.** This algebra is referred to as the real split algebra $\mathfrak{sp}(2l,\mathbb{R})$. The Lax matrix is then given by the $(2l)\times(2l)$ matrix,

$$L_C = \begin{pmatrix} b_1 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ a_1 & b_2 - b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{l-1} & b_l - b_{l-1} & 1 & \cdots & 0 \\ 0 & \cdots & 0 & a_l & -b_l + b_{l-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & a_1 & -b_1 \end{pmatrix}.$$

Following the same way as in the case of A_l, we obtain:

Proposition 6.1. *The solution $\{a_k(t), b_k(t)\}$ is given by*

$$a_k = a_k^0 \frac{D_{k+1}D_{k-1}}{D_k^2}, \quad b_k(t) = \frac{d}{dt}\ln D_k \quad \text{for} \quad 1 \le k \le l,$$

with the following constraints among the determinants $\{D_k\}$,

$$D_{2l-k} = D_k \quad \text{for} \quad 1 \le k \le l, \tag{6.1}$$

which implies $t_{2n} = 0$ for $n = 1, \cdots, l-1$. The determinants are also related to the τ-functions as

$$D_k\left[\exp\Big(\sum_{i=1}^{l} t_{2i-1}(L_C^0)^{2i-1}\Big)\right] = \tau_k(t_1, t_3, \ldots, t_{2l-1}), \quad \text{for} \quad 1 \le k \le l.$$

Proof. The expressions of a_k and b_k are easily obtained in the same way as in the case of A. The constraints (6.1) is a consequence of the structure of L_C, which gives $a_{2l-k} = a_k$ for $k = 1, ..., l$. We then show that constraints (6.1) imply $t_{2n} = 0$:

Recall the determinant $D_k = ||p_{2l}, \cdots, p_{2l+1-k}||$, which is the Schur polynomial with the rectangular Young diagram $Y = (2l+1-k, \cdots, 2l)$. Then the constraints (6.1) imply that the determinant $D_k = S_Y$ is equal to the Schur polynomial with the conjugate diagram (rectangular), denoted by Y', i.e. $S_Y = S_{Y'}$, which leads to the conditions $t_{2n} = 0$ for $n = 1, \cdots, l$. □

Example 6.1. C_2: The τ-functions are given by

$$\begin{cases} \tau_1(t_1,t_3) = p_3(t_1,0,t_3) = \dfrac{t_1^3}{6} + t_3\,, \\ \tau_2(t_1,t_3) = ||\,p_3,\,p_2\,||(t_1,0,t_3) = t_1\left(-\dfrac{t_1^3}{12} + t_3\right). \end{cases}$$

The Painlevé divisor $\mathcal{D}_{\{2\}}$ has two irreducible components, i.e. $t_1 = 0$ and $t_3 - t_1^3/12 = 0$. This implies that there are four subsystems which have the intersection with $\mathcal{D}_{\{2\}}$.

As shown in [8], the Γ_{--} polytope for the semisiple case of C_2 type is given by an octagon whose vertices are marked by the Weyl elemnts. In Figure 4, we describe the nilpotent limit of the Γ_{--} octagon. Four subsystems (boundaries) of the octagon intersecting with the Painlevé divisor $\mathcal{D}_{\{2\}}$ (the dashed curve) are identified as the subsystem $\langle\{\alpha_1\}\rangle = (0*)$ in the limit. Two subsystems intersecting with $\mathcal{D}_{\{1\}}$ (the solid curve) are also identified as $\langle\{\alpha_1\}\rangle = (*0)$ in the limit. Two other subsystems having no intersection with the Painlevé divisors are squeezed into the 0-cell $\langle\Pi\rangle$. In Figure 4, note that the Painlevé divisors are described by the solid and dashed curves in the (t_1,t_3)-coordinates.

Because of the identification of four boundaries corresponding to the subsystem $\langle\{\alpha_1\}\rangle = (0*)$, the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ has a singularity along this subsystem. This can be also seen from the Chevalley invariants $I_k(L_C)$, $k = 1, 2$, i.e. $\det(\lambda I - L_C) = \lambda^4 - I_1\lambda^2 + I_2$ with

$$I_1 = 2a_1 + a_2 + 2b_1^2 - 2b_1b_2 + b_2^2, \quad I_2 = a_2b_1^2 + (a_1 - b_1b_2 + b_1^2)\,.$$

Eliminating a_2 from those equations, we have an equation of the surface,

$$z^2 = x^4 + x^2y^2, \quad \text{with} \quad x = b_1,\; y = b_2,\; z = a_1 - b_1b_2.$$

which has the singularity along the y-axis, i.e. $a_1 = 0$. Notice that there are four different directions to the y-axis except $y = 0$, which are the four segments of the divisor $\mathcal{D}_{\{2\}}$ near the subsystem $\langle\{\alpha_1\}\rangle$, i.e. $a_1 = 0$.

6.2. **Toda lattice of type B_l.** This algebra is referred to as the orthogonal algebra $\mathfrak{so}(l, l+1)$. The Lax matrix L_B in (1.2) for the Toda lattice of type B_l is given by

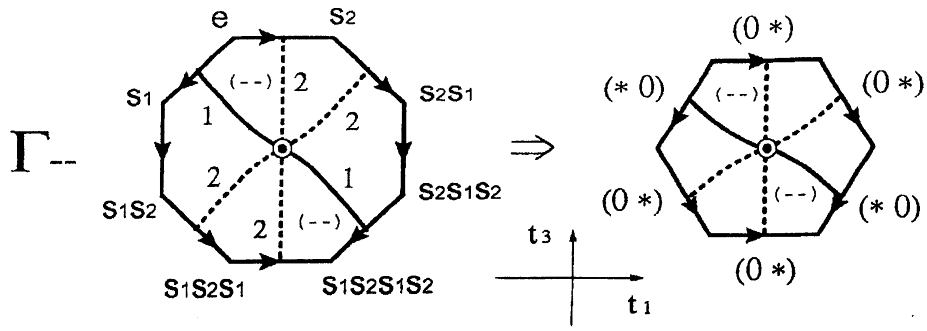

FIGURE 4. The C_2 Toda lattice in the nilpotent limit. The Painlevé divisors $\mathcal{D}_{\{1\}}$ and $\mathcal{D}_{\{2\}}$ are shown as the solid and the dashed curves, respectively. Four boundaries marked by $(0*)$ intersecting with $\mathcal{D}_{\{2\}}$ in the right hexagon should be identified with the orientation given by the direction of the flow. Two other boundaries marked by $(*0)$ are also identified. Then the compact variety $\tilde{Z}(0)_{\mathbb{R}}$is orientable and singular.

the $(2l+1)\times(2l+1)$ matrix,

$$L_B = \begin{pmatrix} b_1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_1 & b_2-b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{l-1} & 2b_l - b_{l-1} & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 2a_l & 0 & 1 & \cdots & \cdot 0 \\ 0 & \cdots & 0 & 0 & 2a_l & -2b_l + b_{l-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a_1 & -b_1 \end{pmatrix}.$$

As in the case of C_2, we obtain:

Proposition 6.2. *The solution* $\{a_k(t), b_k(t)\}$ *is given by*

$$a_k = a_k^0 \frac{D_{k+1}D_{k-1}}{D_k^2}, \quad b_k = \frac{d}{dt}\ln D_k, \quad \text{for} \quad 1 \le k \le l,$$

with the following constraints among the determinants $\{D_k \mid k = 1, \cdots, 2l\}$ *of (1.13) with* $D_k\left[\exp(\sum_{i=1}^{2l} t_i (L_B^0)^i)\right]$,

$$D_{2l+1-k} = D_k, \quad \text{for} \quad 1 \le k \le l, \tag{6.2}$$

which implies $t_{2i} = 0$ *for* $i = 1, \cdots, l$. *The determinants are also related to the* τ*-functions as*

$$\begin{cases} D_k\left[\exp\left(\sum_{i=1}^{l} t_{2i-1}(L_B^0)^{2i-1}\right)\right] & = \tau_k(t_1, t_3, \cdots, t_{2l-1}), \quad \text{for } 1 \le k \le l-1, \\ D_l\left[\exp\left(\sum_{i=1}^{l} t_{2i-1}(L_B^0)^{2i-1}\right)\right] & = -[\tau_l(t_1, t_3, \cdots, t_{2l-1})]^2. \end{cases}$$

Example 6.2. B_2: The determinants D_k, $k=1,2$, are expressed as

$$\begin{cases} D_1 &= p_4(t_1,0,t_3,0) &= \dfrac{t_1^4}{4!}+t_1t_3\,, \\ D_2 &= \|p_4,p_3\|(t_1,0,t_3,0) &= -\dfrac{t_1^6}{144}+\dfrac{t_1^3t_3}{6}-t_3^2\,. \end{cases}$$

Then the τ-functions are given by

$$\begin{cases} \tau_1(t_1,t_3) &= t_1\left(\dfrac{t_1^3}{4!}+t_3\right), \\ \tau_2(t_1,t_3) &= t_3-\dfrac{t_1^3}{12}\,. \end{cases}$$

Now the Painlevé divisor $\mathcal{D}_1$ has two irreducible component, i.e. $t_1=0$ and $t_3+t_1^3/4!=0$. The topological structure of the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ is the same as the case of C_2.

6.3. **Toda lattice of type G_2.** For the exceptional groups, we just give the case of G_2. The Lax matrix in this case can be given by the 7×7 matrix,

$$L_G=\begin{pmatrix} b_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ a_1 & b_2-b_1 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & a_2 & 2b_1-b_2 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 2a_1 & 0 & 1 & 0 & 0 \\ 0 & \cdot & 0 & 2a_1 & -2b_1+b_2 & 1 & 0 \\ 0 & \cdot & \cdot & 0 & a_2 & -b_2+b_1 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & a_1 & -b_1 \end{pmatrix}.$$

Similarly, we have:

Proposition 6.3. *The solution $\{b_k(t)\}$ is given by*

$$a_k=a_k^0\frac{D_{k+1}D_{k-1}}{D_k^2},\qquad b_k=\frac{d}{dt}\ln D_k\,,\quad \text{for } k=1,2.$$

with the following constraints among the determinants $\{D_k\}$,

$$D_{7-k}=D_k,\quad 4\le k\le 7\quad\text{and}\quad D_3=-D_1^2.$$

The determinants are also related to the tau-functions as

$$\begin{cases} D_k[\exp(tL_G^0)] &= \tau_k(t), \qquad k=1,2, \\ D_3[\exp(tL_G^0)] &= -[\tau_1(t)]^2. \end{cases}$$

Then the τ-functions are given by

$$\tau_1(t_1,t_3)=p_6(t_1,0,t_3,0,t_5(t_1,t_3),0)$$
$$\tau_2(t_1,t_3)=\|p_6,p_5\|(t_1,0,t_3,0,t_5(t_1,t_3),0)=p_6p_4-p_5^2\,.$$

Here t_5 is given by the condition $D_3=-D_1^2$, i.e.

$$p_6^2+p_6p_4p_2+2p_5p_4p_3-p_6p_3^2-p_5^2p_2-p_4^3=0\,,$$

which is the second degree polynomial for t_5. In [8] (Proposition 5.3), we have shown that there are two connected components in each Painlevé divisor, and this implies that we have two real roots of the polynomial. In Figure 5, we illustrate the nilpotent limit of the 12-gon of Γ_{--}. In the limit, four of the subsystems are squeezed to the 0-cell $\langle\Pi\rangle$. Taking account of the orientations of the subsystems,

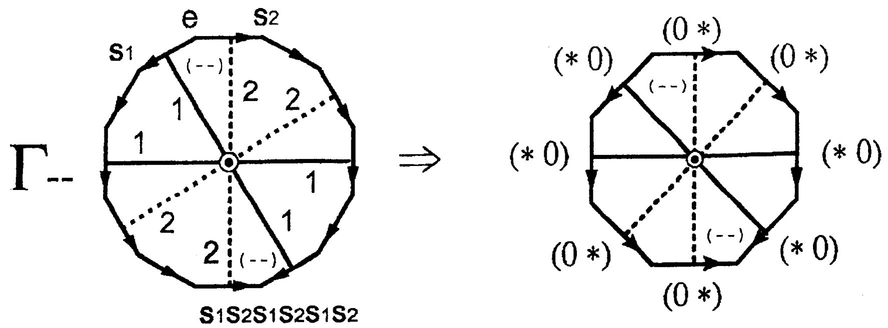

FIGURE 5. The G_2 Toda lattice in the nilpotent limit. The Painlevé divisors $\mathcal{D}_{\{1\}}$ and $\mathcal{D}_{\{2\}}$ are shown as the solid and the dashed curves, respectively.

we conclude that the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ is orientable and has singularity along both subsystems $\langle\{\alpha_k\}\rangle$ for $k = 1, 2$.

7. Homology and cohomology of the chain complex for type A

In this section we express the chain complex $(\mathcal{C}_*, \partial)$ (see (5.1)) and its counterpart $(\mathcal{C}^*, \delta)$ in abstract form. We then compute the corresponding homology or cohomology over the rational number $\mathbb{Q}$ in the case of a Lie algebra of type A.

7.1. The graphs $\mathcal{G}_{A_l}$ and $\mathcal{G}^{\mathcal{L}}_{A_l}$. For the purpose of finding the rational cohomology (or homology), we just need an oriented graph without specific weights. We then define the graph $\mathcal{G}_{A_l}$ based on Proposition 4.2:

Definition 7.1. An oriented graph $\mathcal{G}_{A_l}$ consists of the vertices $\langle J\rangle$ for $J \subset \Pi$ and the oriented edges $\Rightarrow$ between the cells $\langle J\rangle$ and $\langle J'\rangle$ with $|J'| = |J| + 1$. The oriented edges are defined as follows: Given J and $J' = J \cup \{\alpha_i\}$; $\alpha_i \notin J$, we write $\langle J\rangle = (\cdots 0[* \cdots \underset{i}{*} \cdots *]0 \cdots)$ so that $\langle J'\rangle = (\cdots 0[* \cdots * \, 0 \, * \cdots *]0 \cdots)$. Here one interval $I = [* \cdots *]$ in $\langle J\rangle$ indicating a connected Dynkin subdiagram containing α_i has been placed in the parenthesis for emphasis. Let us denote the interval I in $\langle J'\rangle$ as $[* \cdots * \, 0 \, * \cdots *] = [\overbrace{* \cdots *}^{n_1} \, 0 \, \overbrace{* \cdots *}^{n_2}]$. Then there is an edge $\langle J\rangle \Rightarrow \langle J'\rangle$ if and only if n_1 or n_2 is odd (i.e. the incidence number $[J; J'] \neq 0$).

This definition is extended to all real split semisimple Lie algebras in Proposition 8.1. Some of the orbit closures of G^{C_0} acting on the flag manifold are smooth Schubert varieties which are then circle bundles. The variety $\hat{Z}(0)_{\mathbb{R}}$ is not one of these orbit closures but its homology and cohomology over $\mathbb{Q}$ formally behaves as if a circle bundle structure were present. When one has a circle bundle, homology or cohomology with local coefficients become relevant in the computation of homology ore cohomology in terms of the Serre spectral sequence associated to the fiber bundle. We will proceed to abstractly construct a chain complex that formally plays the role of a chain complex for homology (respectively cohomology) with *local* coefficients. We then define below the graph $\mathcal{G}^{\mathcal{L}}$ and we are using the symbol

$\mathcal{L}$ here only as a label which reminds of this formal analogy with homology or cohomology with *local coefficients.*

Definition 7.2. A graph $\mathcal{G}^{\mathcal{L}}_{A_l}$ consists of the vertices $\langle J\rangle$ and the oriented edges $\stackrel{\mathcal{L}}{\Longrightarrow}$, where the edges are defined as follows: Let denote $\langle J\rangle = (\overbrace{\cdots * \cdots}^{l})$, and define $\langle J\rangle_1 := (\overbrace{\cdots * \cdots}^{l} *)$, i.e. add one more $*$ on the right and all the rest of $*$'s and 0's remain in the same positions. Then

$$\langle J\rangle \stackrel{\mathcal{L}}{\Longrightarrow} \langle J'\rangle \quad \text{if and only if} \quad \langle J\rangle_1 \Longrightarrow \langle J'\rangle_1.$$

With $\langle J\rangle = (\cdots 0\,[\overbrace{*\cdots*}^{n_1} \underset{i}{*} \overbrace{*\cdots*}^{n_2}]\,\overbrace{0\cdots}^{n_3})$, we have from Definition 7.1:

(A) If $n_3 = 0$ (i.e. the interval $[*\cdots*]$ includes $*$ in the last simple root α_l), then $\langle J\rangle \stackrel{\mathcal{L}}{\Longrightarrow} \langle J\cup\{\alpha_i\}\rangle$, if and only if n_1 is odd or n_2 is even.

(B) If $n_3 \neq 0$, then $\langle J\rangle \stackrel{\mathcal{L}}{\Longrightarrow} \langle J\cup\{\alpha_i\}\rangle$, if and only if n_1 is odd or n_2 is odd.

Example 7.3. We have $(**) \stackrel{\mathcal{L}}{\Longrightarrow} (*0)$, because in A_3 we have $(***) \Rightarrow (*0*)$ as in Definition 7.1. We also have $(0*) \stackrel{\mathcal{L}}{\Longrightarrow} (00)$, and these are the only arrows in the graph $\mathcal{G}^{\mathcal{L}}_{A_2}$.

For a given graph, we now define a square relation among the cells $\langle J_i\rangle$ for $i = 1, \ldots, 4$ as the boundaries of $\langle J_1\rangle$.

Definition 7.4. A quadruple $(\langle J_1\rangle, \langle J_2\rangle, \langle J_3\rangle, \langle J_4\rangle)$ is called a *square,* if $J_2, J_3 \supset J_1$ with $J_2 \neq J_3$, $J_4 = J_2 \cup J_3$ and $|J_2| = |J_3| = |J_1| + 1$ (note $|J_4| = |J_1| + 2$). We will represent this situation with the diagram:

$$\begin{array}{ccccc} & & \langle J_1\rangle & & \\ & \swarrow & & \searrow & \\ \langle J_2\rangle & & & & \langle J_3\rangle \\ & \searrow & & \swarrow & \\ & & \langle J_4\rangle & & \end{array}$$

If each $\rightarrow$ in this diagram can be replaced with $\Rightarrow$, then we call this a *square relative to* $\Rightarrow$.

Example 7.5. The following quadruple is a square which is also a square relative to $\Rightarrow$, (see Proposition 5.1):

$$\begin{array}{ccccc} & & (****) & & \\ & \swarrow & & \searrow & \\ (***0) & & & & (*0**) \\ & \searrow & & \swarrow & \\ & & (*0*0) & & \end{array}$$

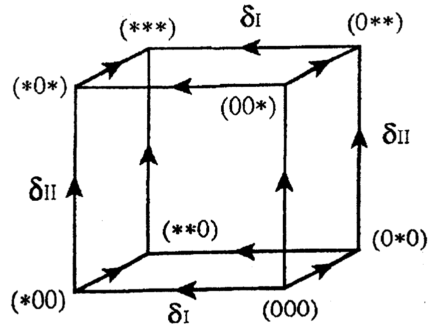

FIGURE 6. Double chain complex structure in $\mathcal{C}^*$ for A_3. Top of the cube corresponds to $\mathcal{C}^*(\mathcal{L})$ for A_2 and the bottom to a chain complex $\mathcal{C}^*$ for A_2.

7.2. **Two subcomplexes and a double chain complex structure.** We will work now with chain complex $\mathcal{C}^*$ which computes cohomology. The case of the homology chain complex $\mathcal{C}_*$ follows easily by reversing arrows in some of the arguments.

We start by pointing out two subgraphs of $\mathcal{G}_{A_l}$ that will play an important role and the associated subcomplexes of $\mathcal{C}^*$. First there is a subgraph consisting of all vertices $\langle J\rangle$ such that $\alpha_l \in J$, i.e. the cells ending to zero, denoted by $\langle J\rangle_0 := (\overbrace{\cdots * \cdots}^{l-1} 0)$. This is indicated as the bottom face in the cube of Figure 6 and give rise to a chain subcomplex associated to a Lie algebra of type A_{l-1}. This subgraph is the same as $\mathcal{G}_{A_{l-1}}$. Then there is another subgraph consisting of all $\langle J\rangle$ such that $\alpha_l \notin J$. These correspond to the cells ending in $*$, denoted by $\langle J\rangle_1 := (\overbrace{\cdots * \cdots}^{l-1} *)$, which is indicated as the top face in the cube of Figure 6. This subgraph gives $\mathcal{G}^{\mathcal{L}}_{A_{l-1}}$. We denote those subcomplexes as $K^{0,q}$ and $K^{1,q}$,

$$K^{0,q} := \bigoplus_{|J|=q+1} \mathbb{Z}\langle J\rangle_0, \qquad K^{1,q} := \bigoplus_{|J|=q} \mathbb{Z}\langle J\rangle_1,$$

which also define a filteration,

$$\mathcal{C}^* = K^0 \supset K^1 \supset 0, \quad \text{with} \quad K^r := \bigoplus_{p\geq 0,\, q\geq r} K^{p,q}.$$

Then we have a short exact sequence,

$$0 \longrightarrow K^1 \xrightarrow{i} K^0 \xrightarrow{j} K^0/K^1 \longrightarrow 0,$$

which provides a long exact sequence for the cohomology and induces a spectral sequence for the double chain complexes (see below). Note here that the graph $\mathcal{G}^{\mathcal{L}}_{A_{l-1}}$ is associated to K^1 and $\mathcal{G}_{A_{l-1}}$ to K^0/K^1. Figure 7 illustrates the case of A_3, and in which the direction of the arrows δ_{II} is indicated in the case of cohomology. Also, the differentials δ_{II} in the case of cohomology run opposite to the direction of the $\Rightarrow$.

We now note that $\mathcal{C}^*$ has a double chain complex structure (see for example [4]). Let δ_I be the differential of any of the two subcomplexes of $\mathcal{C}^*$ described above. These are the arrows along the horizontal faces of our cube in Figure 6 and let δ_{II}

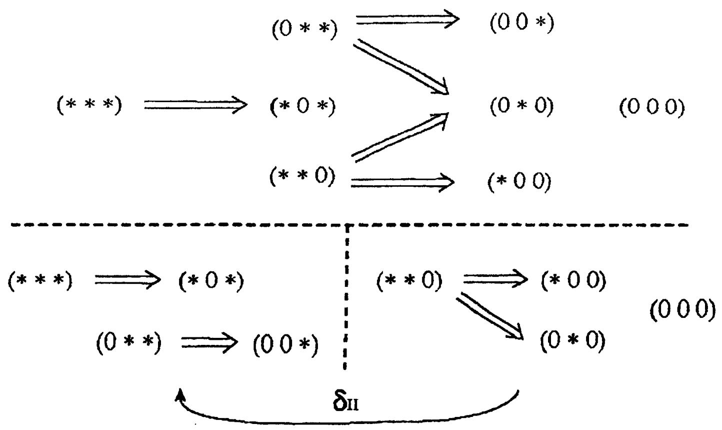

FIGURE 7. The graph $\mathcal{G}_{A_3}$ and its decomposition into two subgraphs giving rise to the double chain complex structure. One in the bottom right is a graph $\mathcal{G}_{A_2}$ for $\{\langle J\rangle = (\cdot\cdot\ 0)\}$. The bottom left is a graph $\mathcal{G}^{\mathcal{C}}_{A_2}$ for $\{\langle J\rangle = (\cdot\cdot\ *)\}$.

be given by $\delta_{II}\langle J\rangle = [J\setminus\{\alpha_l\}\,;\,J]\langle J\setminus\{\alpha_l\}\rangle$ if $\langle J\setminus\{\alpha_l\}\rangle \Rightarrow \langle J\rangle$ and 0 otherwise. Now we decompose the differential δ as $\delta = (-1)^q\delta_I + \delta_{II}$. Since $\delta_I^2 = 0, \delta_{II}^2 = 0$ and $\delta^2 = 0$, we note $\delta_I\delta_{II} = \delta_{II}\delta_I$.

The cohomology of $(\mathcal{C}^*,\delta)$ is then what is called the hypercohomology of the double chain complex, and we have a spectral sequence, $E_k^{p,q}$ for $k = 1, 2$,

$$\begin{cases} E_1^{0,q} = H_I^q(\mathcal{C}^*)\,, \quad E_1^{1,q} = H_I^q(\mathcal{C}^*(\mathcal{L})) \\ E_2^{p,q} = H_{II}^p(H_I^q(\mathcal{C}^*))\,. \end{cases}$$

Then we compute the cohomology with $H^k(\mathcal{C}^*) = \sum_{p+q=k} E_2^{p,q}$. Here the subindex I and II indicates which differential was used in computing cohomology. This spectral sequence replaces the Serre spectral sequence of a circle bundle which is not available in our case. The $H_I^q(\mathcal{C}^*)$ plays the role of the cohomology of the base and H_{II} plays the role of the cohomology along the fiber of a circle bundle.

The chain complex $\mathcal{C}^*(\mathcal{L})$ for A_l has a similar double complex structure. This time $\mathcal{C}^*(\mathcal{L})$ consists of two subcomplexes; each associated to a subgraph. Both subgraphs will be seen below to agree with the graph $\mathcal{G}_{A_{l-1}}$ obtained in the case of A_{l-1}. One subgraph consists of elements of the form $(\cdots \overset{r}{0} * \cdots * *)$ and the other with elements of the form $(\cdots \overset{r}{0} * \cdots 0 *)$. Let $l' = l - r$ Now all the maps δ_{II} are given by multiplication by $\pm l'$ if l' is even and multiplication by $l' + 1$ if l' is oddb.

First we note that the subgraph consisting of all J with $\alpha_l \in J$ is just $\mathcal{G}_{A_{l-1}}$. Then we show that the second subgraph consisting of vertices ending in $*$ (i.e. $\alpha_l \notin J$) is also $\mathcal{G}_{A_{l-1}}$. We refer to these two subgraphs as *bottom* and *top* subgraphs in reference to the cube in Figure 6.

Lemma 7.1. *The two oriented subgraphs bottom and top of $\mathcal{G}^{\mathcal{C}}_{A_l}$ are isomorphic.*

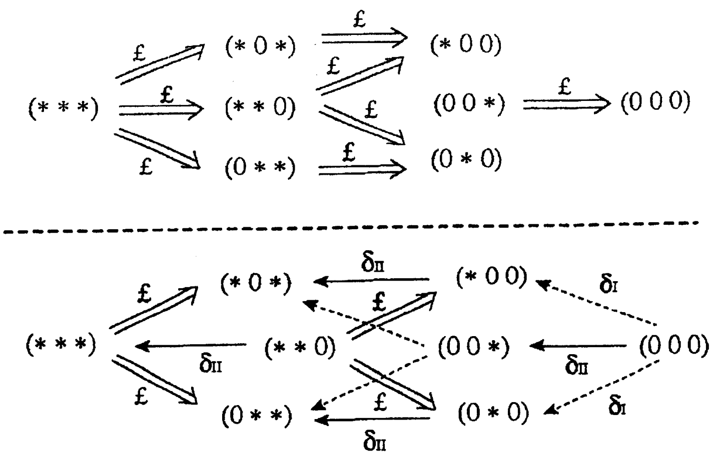

FIGURE 8. The graph $\mathcal{G}^{\mathcal{L}}_{A_3}$ and its decomposition into two identical subgraphs giving rise to the double chain complex structure. The direction of the arrows δ_{II} is as in Figure 7.

Proof. Just note that the bottom and top subgraphs consist of the vertices of the forms $(\overbrace{\cdots * \cdots}^{l-1} **)$ and $(\overbrace{\cdots * \cdots}^{l-1} 0*)$, respectively. Then by Definition 7.2, it is obvious that the parts $(**)$ and $(0*)$ do not affect the edges in those graphs, that is, they are identical. □

We also note that the isomorphism between the two oriented graphs is provided by the edges corresponding to δ_{II} and consisting of the edges in the cube of Figure 6 joining the bottom and top faces. Indeed, by Definition 7.2 we always have the edge in $(\overbrace{\cdots * \cdots}^{l-1} **) \Rightarrow (\overbrace{\cdots * \cdots}^{l-1} 0*)$ which implies $(\overbrace{\cdots * \cdots *}^{l}) \overset{\mathcal{L}}{\Longrightarrow} (\overbrace{\cdots * \cdots 0}^{l})$. From here one obtains that every time there is a $\overset{\mathcal{L}}{\Longrightarrow}$ in one of the two subgraphs, say the bottom subgraph, a square is produced with at least three $\Rightarrow$ (if we add an additional $*$ on the right as in Definition 7.2). This leads to the fourth $\Rightarrow$ and therefore to an oriented edge along the top subgraph.

In Figure 8, we illustrate the graph $\mathcal{G}^{\mathcal{L}}_{A_3}$, which is a subgraph of $\mathcal{G}_{A_4}$, and its decomposition into two identical subgraphs of $\mathcal{G}_{A_2}$ (referred to as *bottom* and *top*). Note that some of the arrows in the diagram which are labeled with a δ_{II} correspond to a $\overset{\mathcal{L}}{\Longrightarrow}$ in the top diagram and other (indicated with dashed arrows) do not correspond to an edge in the graph. For those (dashed arrows) the δ_{II} in the double chain complex is given by 0.

Now we have:

Theorem 7.6. *For the case of type A, the cohomology of C^* with rational coefficients satisfies:*

$$H^k(C^*;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for} \quad k = 0,1 \\ 0 & \text{for} \quad k \neq 0,1\,. \end{cases}$$

Proof. We will prove this by induction on the rank l and rely on Lemma 7.2 below. If we compute cohomology relative to δ_I, using Lemma 7.2 and the induction hypothesis what results is $E_1^{p,q}$ as indicated by

$$E_1^{p,q} = \begin{cases} 0 & \to & 0 & (q = l) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \to & 0 & (q = 2) \\ \mathbb{Q} & \to & 0 & (q = 1) \\ \mathbb{Q} & \to & 0 & (q = 0) \end{cases}$$

Now δ_{II} is necessarily trivial and we obtain a collapsed spectral sequence in which $E_1^{p,q} = E_2^{p,q}$. From here the statement of our theorem will follow. □

Lemma 7.2. *Assume that $H^k(C^*)$ is known for the case of A_{l-1} and all k. Then $H^k(C^*(\mathcal{L});\mathbb{Q}) = 0$ for all $k = 0, 1, \cdots$.*

Proof. This follows from the spectral sequence of the double chain complex. By our assumption we know the cohomology of the two subchain complexes that arise for Lie algebras of type A_{l-1}. Therefore we know, by assumption the cohomology relative to δ_I. We obtain $E_2^{p,q}$ by now computing cohomology relative to δ_{II}. This is shown in the array below in which all the horizontal arrows are given by multiplication by a non-zero scalar:

$$E_1^{p,q} = \begin{cases} 0 & \to & 0 & (q = l) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \to & 0 & (q = 2) \\ \mathbb{Q} & \to & \mathbb{Q} & (q = 1) \\ \mathbb{Q} & \to & \mathbb{Q} & (q = 0) \end{cases}$$

From here $E_2^{p,q} = 0$. □

Similarly we get for homology the following

Theorem 7.7. *The rational homology of $\mathcal{C}_*$ in the case of type A_l satisfies:*

$$H_k(\mathcal{C}_*;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for} & k = 0, 1 \\ 0 & \text{for} & k \neq 0, 1 . \end{cases}$$

Proof. This can be obtained using a spectral sequence argument associated to the double chain complex structure of $\mathcal{C}_*$. It also follows from the Universal Coefficients Theorem. □

Proposition 7.1. *The homology of $\mathcal{C}_*$ with $\mathbb{Z}_2$ coefficients satisfies:*

$$H_k(\mathcal{C}_*;\mathbb{Z}_2) = \binom{l}{k}\mathbb{Z}_2$$

Proof. This follows easily from the fact that all the incidence numbers are even. □

8. Graphs for arbitrary real split simple Lie algebras.

In this section, we determine the graphs for arbitrary $\mathbb{R}$-split simple Lie algebra which provide sufficient information for computing the (co)homology of the compact variety $\tilde{Z}(0)_{\mathbb{R}}$.

Let us first recall that *inductively* it suffices to compute all the edges $\Rightarrow$ from the top cell $\langle\emptyset\rangle = (*\cdots*)$. The results in this section will first show that *one* single edge from the top cell suffices to determine all the others uniquely in the case of type A. Hence a *nonorientability* condition (having at least one edge from the top) allows one to derive the graph $\mathcal{G}$ completely. For other Lie algebras the nonorientability condition fails but there is still information to proceed. For example in types B and C there are no $\Rightarrow$ arising from the top cell. Thus we will have instead *orientability.* In type D_l orientability depends on the parity of l.

8.1. Nonorientability and extremal simple roots.

We now define *orientability* and *nonorientability* for the compactified manifols $\tilde{Z}(0)_{\mathbb{R}}$. Recall that we have a cell decomposition for $\tilde{Z}(0)_{\mathbb{R}}$ with cells corresponding to subsystems labeled by subsets $J \subset \Pi$. From this it follows that there are only two possibilities for $H^l(\tilde{Z}(0)_{\mathbb{R}}, \mathbb{Z})$. Either it is $\mathbb{Z}$ or it is zero. Although $\tilde{Z}(0)_{\mathbb{R}}$ is not smooth we will refer to the first situation as *orientable* and to the second as *nonorientable.*

When $\tilde{Z}(0)_{\mathbb{R}}$ is nonorientable the graph $\mathcal{G}$ is such that there is at least one oriented edge $\Rightarrow$ from the top $\langle\emptyset\rangle$.

Definition 8.1. A simple root α_i is *extremal*, if there is exactly one simple root α_j such that α_i and α_j are joined in the Dynkin diagram. In addition we assume that α_i and α_j are joined by exactly one line. A vertex $\langle\{\alpha_j\}\rangle$ is called *extremal*, if α_j is the simple root connecting to an extremal root. The labeling for the simple roots are defined in the Appendix A.

Example 8.2. In the case of type A_l, the only extremal simple roots are those labeled by 1 and l, and the cells $\langle\{\alpha_k\}\rangle$ with $k = 2, l-1$ are the extremal vertices. From Proposition 4.2, we also have the edge from the top $\langle\emptyset\rangle$ to the extremal vertices $\langle\{\alpha_k\}\rangle$ for $k = 2, l-1$.

8.2. Determination of the graphs $\mathcal{G}$.

We now introduce the condition of *compatibility.* It is now assumed that all the arrows from the top cell are known. Compatibility is the precise condition which allows one to assemble the whole graph $\mathcal{G}$ of a semisimple Lie algebra inductively using the corresponding graphs for semisimple Lie algebras of smaller rank.

Definition 8.3. We say that $\mathcal{G}_{\mathfrak{g}}$ is *compatible* if we have $\mathcal{G}_{\mathfrak{g}} \cap \mathcal{G}_{\mathfrak{g}'} = \mathcal{G}_{\mathfrak{g}\cap\mathfrak{g}'} = \mathcal{G}_{\mathfrak{g}'}$ for any Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. In addition, as part of the compatibility condition, we assume that for the case of A_2 subdiagrams of the Dynkin diagram one obtains the graph already described in Figure 3. Any other subdiagrams associated to rank 2 semisimple Lie algebras (i.e. B_2, C_2 or G_2) are found to have no edges $\Rightarrow$ (see Section 6).

We now review a condition on squares of the graph $\mathcal{G}$ that is implied by $\partial^2 = 0$ (*the chain complex condition*). First we have the following obvious Lemma:

Lemma 8.1. *Let $(\langle J_1\rangle, \langle J_2\rangle, \langle J_3\rangle, \langle J_4\rangle)$ be a square relative to $\to$. Then we have the followings to satisfy the condition $\partial^2 = 0$;*

a) *If three of the $\to$ are $\Rightarrow$, then the fourth $\to$ in the square is also $\Rightarrow$.*
b) *If two arrows $\to$ along the left side of the square or along the right side of the square are both $\Rightarrow$, then all four must be $\Rightarrow$.*

In the case of a Lie algebra of type A, we will be looking for graphs associated to Dynkin diagrams of real split semisimple Lie algebras which satisfy the following three conditions:

C1: Nonorientability: $\exists \alpha_j \in \Pi$ such that there is an edge in $\langle\emptyset\rangle \Rightarrow \langle\{\alpha_j\}\rangle$
C2: Compatibility: $\mathcal{G}_{\mathfrak{g}} \cap \mathcal{G}_{\mathfrak{g}'} = \mathcal{G}_{\mathfrak{g}'}$ for any Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$
C3: Chain complex condition: $\partial^2 = 0$

Using these three conditions we will determine all α_i such that $\langle\emptyset\rangle \Rightarrow \langle\{\alpha_i\}\rangle$ in the case of a Lie algebra of type A. The nonorientability condition will be obtained from $[\emptyset, \{\alpha_2\}] \neq 0$. With the exception of D_l for l odd, and E_6 all other real split semisimple Lie algebras satisfy orientability.

Example 8.4. Type A_3: In this example, we illustrate the main arguments used in this section to compute the graph $\mathcal{G}$.

The compatibility condition **C2** implies that the graph for A_3 includes the cases of type A_2 and A_1. Then we have the following edges in the subgraphs corresponding to the case of A_2: $(**0) \Rightarrow (*00)$, $(**0) \Rightarrow (0*0)$, $(0**) \Rightarrow (0*0)$, $(0**) \Rightarrow (00*)$. Now from Proposition 4.2, that is, the nonorientability **C1**, we have the extremal edge from the top, $(***) \Rightarrow (*0*)$. Now using the chain complex condition **C3**, we can see that there is no additional edge, and we obtain the unique graph for A_3 as shown in Figure 3.

Our main result concerning the graphs $\mathcal{G}$ for arbitrary real split semisimple Lie algebras is Proposition 8.1 below which gives a complete list of the oriented edges $\Rightarrow$ from the top.

We label the simple roots as in see Appendix A so that and $\langle J\rangle$ can be denoted as alist of stars and zeros as in the case of type A.

Proposition 8.1. *In the graph $\mathcal{G}$, we have the following result on the edge from the top $(*\cdots*)$ to the vertex $(\overbrace{*\cdots*}^{n_1}\,0\,\overbrace{*\cdots*}^{n_2})$:*

- *For type A, there is an edge, iff n_1 or n_2 is odd.*
- *For type B, C, there are no edges (orientable case).*
- *For type D_l, there are no edges for l even, and for l odd, there is an edge iff $n_1 = 0$.*
- *For type E_6, there are only two edges for $n_1 = 0$ and $n_1 = 4$.*
- *For type E_7, E_8, there are no edges (orientable case).*
- *For type F_4, there are no edges (orientable case).*

We will give a proof of Proposition 8.1 in the case of a Lie algebra of type A by using the three conditions **C1**, **C2** and **C3**. For other Lie algebras we need to replace the nonorientability condition. Lemma 8.3 is useful in showing that incidence numbers are zero and ultimately leads to a proof of Proposition 8.1 in combination with **C2** and **C3**.

8.3. **Proof of Proposition 8.1.** We first state a Lemma leading to the proof of Proposition 8.1 for those types.

Lemma 8.2. *Assume that Proposition 8.1 is true for the Lie algebras of type A of rank smaller than l. Then for any real split simple Lie algebra of rank l and of type A, we have:* $(*\cdots*) \Rightarrow (\overbrace{*\cdots*}^{m_1}0\overbrace{*\cdots*}^{m_2})$ *with* $m_1 \equiv n_1 \pmod 2$, *if and only if* $(*\cdots*) \Rightarrow (\overbrace{*\cdots*}^{n_1}0\overbrace{*\cdots*}^{n_2})$.

Proof. Suppose $n_1 < m_1$ with $n_1 \equiv m_1 \pmod 2$. Hence $m_1 = n_1 + n_2' + 1$ with n_2' an odd number and $n_2 = m_2 + 1 + n_2'$ and $n_2 \equiv m_2 \pmod 2$.

We now have the following square in which the two bottom arrows are $\Rightarrow$, since n_2' is odd. Using the chain complex condition **C3** (Proposition 8.1), we have that one of the arrows from the top cannot be a $\Rightarrow$ without the other also being $\Rightarrow$:

$$\begin{tikzcd}[column sep=small] & (*\cdots*\cdots*\cdots*) \arrow[dl] \arrow[dr] & \\ (\overbrace{*\cdots*}^{m_1}0\overbrace{*\cdots*}^{m_2}) \arrow[dr] & & (\overbrace{*\cdots*}^{n_1}0\overbrace{*\cdots*}^{n_2}) \arrow[dl] \\ & (\overbrace{*\cdots*}^{n_1}0\overbrace{*\cdots*}^{n_2'}0\overbrace{*\cdots*}^{m_2}) & \end{tikzcd}$$

This completes the proof. Note here that one single edge $(*\cdots*) \Rightarrow (*0*\cdots*)$ (i.e. an extremal edge) determines *all* the edges in $(*\cdots*) \Rightarrow (\overbrace{*\cdots*}^{n_1}0\overbrace{*\cdots*}^{n_2})$ with $n_1 \equiv m_1 \pmod 2$. □

Now we can prove Proposition 8.1 in the cases of A:

Proof. First we use Lemma 8.2 to make m_1 smaller if necessary and then assume that $m_1 = 1$ or $m_1 = 2$. We have then the following square diagram,

$$\begin{tikzcd}[column sep=small] & (*\cdots*\cdots*) \arrow[dl] \arrow[dr] & \\ (**0\overbrace{*\cdots*}^{m_2}) \arrow[dr] & & (*0\overbrace{*\cdots*}^{m_2+1}) \arrow[dl] \\ & (*00\overbrace{*\cdots*}^{m_2}) & \end{tikzcd}$$

Now from Proposition 4.2, the top right arrow should be an edge, i.e. extremal edge, so that from Lemma 8.2 we have always the edge for the case with m_1 is odd. Also we have the edge on the bottom left by the A_2 case. Then from the condition **C3** (Proposition 8.1), we have the edge on the left side from the top if and only if we have the edge in the bottom right. However for type A in a smaller rank (m_2+1 in this case), the edge appears if only if m_2 is odd. Hence it tollows that at least one of m_1, m_2 must be odd. □

We now consider other cases of Lie algebras. Let us first state the following Lemma on an information for the incidence number $[\emptyset;\{\alpha_i\}]$ in terms of the length of the longest elements w_* and $w^{\{\alpha_i\}}$:

Lemma 8.3. *For $J=\{\alpha_i\}$, if the length $\ell(w_*w^J)$ is odd, then the incidence number $[\emptyset;J]=0$.*

Proof. Let $x\in W^-_{[J]}$. Then $w_*xw^J\in W^-_{[J]}$ (Lemma 4.1). Since $\ell(w_*xw^J)=\ell(w_*)-\ell(w^J)-\ell(x)$, if $\ell(w_*w^J)=\ell(w_*)-\ell(w^J)$ is odd, then x and w_*xw^J have different parity. Then from Definition 5.1, $[\emptyset;J]=0$. □

We also have:

Lemma 8.4. *Assume that*

a) $\Pi\setminus\{\alpha_r\}$ *is a Dynkin diagram with simple components of type A*

b) $\langle\{\alpha_{r'}\}\rangle$ *is a extremal vertex for a component of* $\Pi\setminus\{\alpha_r\}$.

Then $[\emptyset,\{\alpha_{r'}\}]=0$ *implies* $[\emptyset,\{\alpha_r\}]=0$.

Proof. Assume that $r'=2$ (by relabeling if necessary). We consider the square:

$$\begin{array}{ccccc} & & (*\cdots*) & & \\ & \swarrow & & \searrow & \\ (*0*\cdots*\cdots*) & & & & (*\cdots*\overset{r}{0}*\cdots*) \\ & \searrow & & \swarrow & \\ & & (*0*\cdots*\overset{r}{0}*\cdots*) & & \end{array}$$

By Proposition 4.2 and the compatibility condition of Definition 8.3, the bottom right-hand side $\to$ corresponds to a $\Rightarrow$. Therefore if the top right-hand side $\to$ corresponds to a $\Rightarrow$ $\partial^2\neq0$ because the two $\to$ in the left hand-side do not correspond to $\Rightarrow$. Therefore there is no arrow $\Rightarrow$ between $\langle\emptyset\rangle$ and $\langle\{\alpha_r\}\rangle$. □

Then we obtain the orientability for the cases of type B and C:

Proposition 8.2. *For type B or C, we have $[\emptyset;\{\alpha_i\}]=0$ for any i. Therefore $\tilde{Z}(0)_{\mathbb{R}}$ is orientable, i.e. $H_l=0$.*

Proof. First we note $\ell(w_*)=l^2$. For $J=\{\alpha_1\}$, we have $\ell(w^J)=(l-1)^2$. Therefore $\ell(w_*)-\ell(w^J)=l^2-(l-1)^2$ is odd. By Lemma 8.3 there is no arrow from $\langle\emptyset\rangle$ to $\langle\{\alpha_1\}\rangle$. In the case of $J=\{\alpha_2\}$, we have $\ell(w_*)-\ell(w^J)=l^2-(l-2)^2-1$. This is again an odd number and there is no arrow from the top $\langle\emptyset\rangle$ to $\langle\{\alpha_2\}\rangle$. We now show that no other $\Rightarrow$ are possible from the top cell. For $2<k\le l$ we apply Lemma 8.4 to conclude $[\emptyset;\{\alpha_k\}]=0$. Therefore there is no arrow from the top $\langle\emptyset\rangle$. □

The case of type D is given by the following Proposition:

Proposition 8.3. *For type D_l, we have $[\emptyset;\{\alpha_i\}]\neq0$ if and only if l is odd and $i=1$. In this case $[\emptyset;\{\alpha_1\}]=4$. Therefore $\tilde{Z}(0)_{\mathbb{R}}$ is orientable if and only if l is even.*

Proof. Note that for $J=\{\alpha_1\}$ we have $\ell(w_*)=l(l-1)$, $\ell(w^J)=(l-1)(l-2)$, so $\ell(w_*w^J)$ is even. We obtain the following elements in $W^-_{[\alpha_1]}$: e, $s_ls_{l-2}s_{l-3}\cdots s_1$,

$s_{l-1}s_{l-2}s_{l-3}\cdots s_1$, $w_*w^{\{\alpha_i\}}$. This gives a total of four. In the case of l even, $l-1$ is odd and there are two elements of even length and two of odd length. In the case when l is odd we have four elements of even length and $[\emptyset; J] = 4$.

We now show that for any other J , $[\emptyset; J] = 0$: In the case of $i = 2$, we have $\ell(w^J) = 1 + (l-2)(l-3)$ if $l \geq 6$, $\ell(w^J) = 7$, if $l = 5$ and $\ell(w^J) = 3$ for $l = 4$. In any case $\ell(w_*w^J) = l(l-1) - \ell(w^J)$ is odd and $[\emptyset; J] = 0$. For the cases $i > 2$, using Lemma 8.4, we get the result. □

The following Proposition is for the case of type F_4:

Proposition 8.4. *In the case of F_4 , $[\emptyset; \{\alpha_i\}] = 0$ for all $i = 1, 2, 3, 4$. Therefore $\tilde{Z}(0)_{\mathbb{R}}$ is orientable.*

Proof. We first note that $\ell(w_*) = 24$ and $\ell(w^{\{\alpha_1\}}) = 9$, so that $\ell(w_*w^{\{\alpha_1\}})$ is odd. Therefore $[\emptyset; \{\alpha_1\}] = 0$. Similarly $[\emptyset; \{\alpha_4\}] = 0$.

We now consider the case of $J = \{\alpha_2\}$. We have the following square:

$$\begin{tikzcd}[column sep=small] & (****) \arrow[dl] \arrow[dr] & \\ (*0**) \arrow[dr] & & (***0) \arrow[dl] \\ & (*0*0) & \end{tikzcd}$$

The two arrows on the right-hand side are not $\Rightarrow$. Hence $[\emptyset; \{\alpha_2\}] = 0$ follows from the $\partial^2 = 0$ condition. Similarly $[\emptyset; \{\alpha_3\}] = 0$. □

Proposition 8.5. *For type E_6 we have $[\emptyset, \{\alpha_i\}] = 0$ if and only if $i = 1, 5$. Moreover $[\emptyset; \{\alpha_i\}] = 6$ if $i = 1, 5$.*

Proof. For all $i \neq 1, 5$, $\ell(w_*w^{\{\alpha_i\}})$ is odd. We compute all the elements in $W^-_{[J]}$ for $J = \{\alpha_1\}$. The symmetrical case of $i = 5$ will then follow. We obtain the following list: e, $s_6s_3s_2s_1$, $s_4s_3s_6s_5s_4s_3s_2s_1$, $s_1s_2s_3s_6s_4s_3s_2s_1$, $w_*(s_6s_3s_2s_1)w^J$, w_*w^J. Therefore, since all have even lengths, $[\emptyset; \{\alpha_1\}] = 6$. □

Proposition 8.6. *For type E_7 or E_8 we have $[\emptyset, \{\alpha_i\}] = 0$ for all i. Therefore $\tilde{Z}(0)_{\mathbb{R}}$ is orientable.*

Proof. For E_7 and all $i \neq 3, 7$, $\ell(w_*w^{\{\alpha_i\}})$ is odd. For $i = 3, 7$ we can apply Lemma 8.4 because the subdiagrams that result by deleting α_3 or α_7 are of type A and $[\emptyset, \{\alpha_2\}] = 0$. We conclude $[\emptyset, \{\alpha_i\}] = 0$ for all i for the case of E_7.

In the case of E_8 $i \neq 4, 5, 6, 7$, $\ell(w_*w^{\{\alpha_i\}})$ is odd. For $i = 4, 5, 6$ we can use Lemma 8.4. For $i = 8$ we use the square:

$$\begin{tikzcd}[column sep=small] & (********) \arrow[dl] \arrow[dr] & \\ (0*******) \arrow[dr] & & (******0*) \arrow[dl] \\ & (0*****0*) & \end{tikzcd}$$

The top left hand side is not a $\Rightarrow$ and the bottom right hand side is a $\Rightarrow$ (by the D_l case where l is odd). Hence the top right hand side cannot correspond to a $\Rightarrow$ ($\partial^2 = 0$ would be violated). □

8.4. **Rational cohomology (Betti numbers) for other Lie simple algebras.** We first note that in the case of type A the cohomology of the compact variety $\tilde{Z}(0)_{\mathbb{R}}$ is closely related to that of certain Schubert variety. In particular we consider the Schubert varieties $V_l := \overline{N^+ s_1 \cdots s_l B^+/B^+}$. For example, if we fix a coordinate flag, $V_0^1 \subset \cdots \subset V_0^l \subset \mathbb{R}^{l+1}$ with $\dim V_0^k = k$, i.e. a flag corresponding to the 0 dimensional Schubert variety $V(e)$, then in type A we consider all complete flags,

$$V^1 \subset V^2 \subset \cdots \subset V^l \subset \mathbb{R}^{l+1}, \quad \text{with} \quad V^i \subset V_0^{i+1}, \quad i = 1, \cdots l-1 .$$

Then we have:

Proposition 8.7. *The cohomology $H^*(V_l, \mathbb{Z})$ of the Schubert variety V_l for type A is given as follows:*

$$H^k(V_l, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ \binom{l-1}{k-1}\mathbb{Z}_2 & k \neq 0, 1 \end{cases}$$

Proof. This follows from the main results in [5]. It can also be proved using the Serre spectral sequence associated to a circle bundle. □

We now recall that the compactified isospectral space $\tilde{Z}(0)_{\mathbb{R}}$ is a closure of a G^{C_0}-orbit of a generic element in the flag manifold. Among the G^{C_0}-orbits of different elements, there are some that form *Schubert varieties* , closures of an N^+-orbit. For example in type A the smooth manifolds V_l have a transitive action of G^{C_0} on their top cell and the remaining cells in the boundary are all G^{C_0}-orbits and are parametrized in the same way as the subsystems in the isospectral variety. The Schubert varieties can then be viewed as alternative compactifications of the isospectral variety of the nilpotent Toda lattice.

The space $\tilde{Z}(0)_{\mathbb{R}}$ and the corresponding Schubert variety then have the same number of cells given by G^{C_0}-orbits although there is an important difference: while $\tilde{Z}(0)_{\mathbb{R}}$ is singular the corresponding Schubert variety is a smooth manifold. Still, in the case of rank 2 the Schubert variety V_2 is just Klein bottle and it is is homeomorphic to the corresponding compactified isospectral variety.

We now describe the connection between the nilpotent Toda Lattice and the Schubert varieties V_l:

Theorem 8.5. *For type A, if all the incidence numbers along the edges $\Rightarrow$ in the graphs $\mathcal{G}_{A_l}$ are replaced with ± 2 then the chain complex that results computes the integral cohomology of the Schubert variety V_l.*

The proof is almost identical to the proof of Theorem 7.6 and is therefore omitted.

We now proceed to compute the rational homology and cohomology of $\tilde{Z}(0)_{\mathbb{R}}$. The computation depends only on the graph $\mathcal{G}$ and not on the actual incidence numbers associated to the $\Rightarrow$. In particular the rational cohomology is independent of the chosen sign function for the incidence number.

There are exactly three patterns which the rational cohomology obeys, one for the nonorientable cases and two for the orientable cases:

Theorem 8.6. *For type A_l, D_l with l odd and E_6, which give the nonorientable cases, we have*

$$H^k(\tilde{Z}(0)_{\mathbb{R}};\mathbb{Q}) = H^k(V_l;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for} \quad k=0,1 \\ 0 & \text{for} \quad k \neq 0,1\,. \end{cases}$$

For the orientable cases, we have:

- *For type D_l with l even, E_7 and E_8,*

$$H^k(\tilde{Z}(0)_{\mathbb{R}};\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for} \quad k=0,1,l-1,l \\ 0 & \text{for} \quad 1<k<l-1\,. \end{cases}$$

- *For type B_l, C_l, G_2 and F_4,*

$$H^k(\tilde{Z}(0)_{\mathbb{R}};\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for} \quad k=0,l \\ 2\mathbb{Q} & \text{for} \quad 0<k<l\,. \end{cases}$$

Remark 8.7. In the case of a Lie algebra of type A, several examples suggest that the connection given in Theorem 8.6 between $\tilde{Z}(0)_{\mathbb{R}}$ and Schubert varieties through their cohomology extends to integral coefficients with some modifications. For example, although $H^k(\tilde{Z}(0)_{\mathbb{R}};\mathbb{Z})$ and $H^k(V_l;\mathbb{Z})$ are not isomorphic, they still have the same *rank* as $\mathbb{Z}$-modules. In fact one observes from examples that the graphs obtained in this paper using the nilpotent Toda Lattice can be transformed into the graphs of [5] for Schubert varieties by making a change in the generators in the chain complex. The simplest example of this is the case of A_2. Here we must replace $\langle\{\alpha_2\}\rangle$ with $\langle\{\alpha_1\}\rangle + \langle\{\alpha_2\}\rangle$. In some sense this change of generators relates the structure of principal series representations for $SL(n,\mathbb{R})$ as encoded in the graphs of [5] with the nilpotent Toda lattice.

We will now prove Theorems 8.6 in several steps given by Propositions for the various types of Lie algebras. The proofs in the cases of a Lie algebra of type E are very similar to those for type D and details are omitted. The case of F_4 is also easy and is also omitted. The main ideas in all the proofs are already contained in the calculation of the cohomology for a Lie algebra of type A.

Definition 8.8. A graph $\mathcal{G}^{\mathcal{L}}_{X_l}$ for $X=A,B,C,D,E,F$ consists of the vertices $\langle J\rangle$ and the oriented edges $\xRightarrow{\mathcal{L}}$, where the edges in the same way as in Definition 7.2.

In particular $\mathcal{G}^{\mathcal{L}}_{X_l}$ agrees with $\mathcal{G}_{X_{l-1}}$ for $X=B,C$ and the incidence number corresponding to the edges in $\mathcal{G}^{\mathcal{L}}_{X_l}$ agree with the incidence numbers associated to the edges $\mathcal{G}_{X_{l-1}}$.

Notation 8.9. For any $X=A,B,C,D,E$, we consider a subgraph $\mathcal{G}_{X_l}[\overbrace{*\cdots*}^{k}]$ of $\mathcal{G}_{X_l}$ consisting of all vertices of the form $(\cdots\overbrace{*\cdots*}^{k})$ and the corresponding edges between them. Similarly we define $\mathcal{G}_{X_l}[\overbrace{0*\cdots*}^{k}]$. For example $\mathcal{G}_{A_l}[*]=\mathcal{G}^{\mathcal{L}}_{A_{l-1}}$ and $\mathcal{G}_{A_l}[0]=\mathcal{G}_{A_{l-1}}$. Also $\mathcal{G}_{A_l}[**]$ and $\mathcal{G}_{A_l}[0*]$ are *top* and *bottom* of $\mathcal{G}^{\mathcal{L}}_{A_{l-1}}$.

Each of these subgraphs gives rise to a chain complex. Within the context of a specific simple Lie algebra and concrete coefficients ($\mathbb{Z}$ or $\mathbb{Q}$) we will use the shorthand notation $H^q([\overbrace{*\cdots*}^{k}])$ or $H^q([\overbrace{0*\cdots*}^{k}])$ for its qth cohomology. There is a double chain complex structure and a corresponding spectral sequence expressing $H^q([\overbrace{*\cdots*}^{k-1}])$ in terms of $H^q([\overbrace{0*\cdots*}^{k}])$ and $H^q([\overbrace{*\cdots*}^{k}])$

$$E_1^{p,q} = \left\{ \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ H^q([\overbrace{0*\cdots*}^{k}]) & \to & H^q([\overbrace{**\cdots*}^{k}]) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right.$$

We now give a proof of Theorem 8.6 for type B or C:

Proof. We proceed by induction on the rank and use the same method that was used in the prof of Theorem 7.7 in the case of type A. We use a double chain complex strtucture corresponding to the two subgraphs $\mathcal{G}_{X_{l-1}}$ and $\mathcal{G}^{\mathcal{L}}_{X_{l-1}}$ for $X = B, C$. There are no $\Rightarrow$ involving these two subgraphs. This translates into $d_I = 0$ or $\delta_I = 0$ in the E_1 term. Hence we have a collapsed spectral sequence. We have

$$E_1^{p,q} = E_2^{p,q} = \left\{ \begin{array}{ccccc} 0 & \to & \mathbb{Q} & & (q=l) \\ 0 & \to & 2\mathbb{Q} & & (q=l-1) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & \to & 2\mathbb{Q} & & (q=2) \\ \mathbb{Q} & \to & 2\mathbb{Q} & & (q=1) \\ \mathbb{Q} & \to & \mathbb{Q} & & (q=0) \end{array} \right.$$

□

A proof of Theorem 8.6 for type D_l with l odd is as follows:

Proof. We summarize the steps of the proof which is analogous to the proof for the case of type B or C. If $p > 1$ is odd one can show that $H^k([\overbrace{*\cdots*}^{p}]) = \mathbb{Q}$ when $k = 0, 1$ and zero otherwise. If $p \neq 0$ is even then one can show that $H^k([\overbrace{*\cdots*}^{p}]) = 0$. Also that for $p \neq 1$ $H^k([0\overbrace{*\cdots*}^{p}]) = \mathbb{Q}$ when $k = 0, 1$ and zero otherwise. This pattern is broken with $H^k([0*]) = 0$ for all k. This is just a consequence of noticing that $\mathcal{G}_{D_l}[0*] = \mathcal{G}_{A_{l-1}}[*] = \mathcal{G}^{\mathcal{L}}_{A_{l-2}}$. By Lemma 7.2 it then follows that $H^k([0*]) = 0$ for all k.

Since $H^k([**]) = 0$ ($p = 2$ case above) , using the spectral sequence relating $H^k([0*])$ and $H^k([**])$, one obtains $H^k([*]) = 0$. This also breaks the previous patterns (which assumed $p > 1$). Finally since $H^k([0])$ is the cohomology associated to the graph $\mathcal{G}_{D_l}[0] = \mathcal{G}_{A_{l-1}}$ we obtain the desired result using again the spectral sequence involving $H^k([0])$and $H^k([*])$. □

Since the proof of Theorem 8.6 for type D_l with l even is analogous to that of the case D_l with l odd, we omit it.

Acknowledgement

One of the authors (Y.K) would like to thank the organizer for a financial support and an invitation to a RISM conference on "Integrable systems and related topics", July 30-Aug.1, 2003. Y.K also thanks to T. Ikeda and A. Nemethi for several useful discussions related on the paper.

Appendix A. Dynkin diagrams for real split simple Lie algebras

Here we list the Dynkin diagrams for real split simple Lie algebras. The simple roots for each algebra are labeled as in Figure 9.

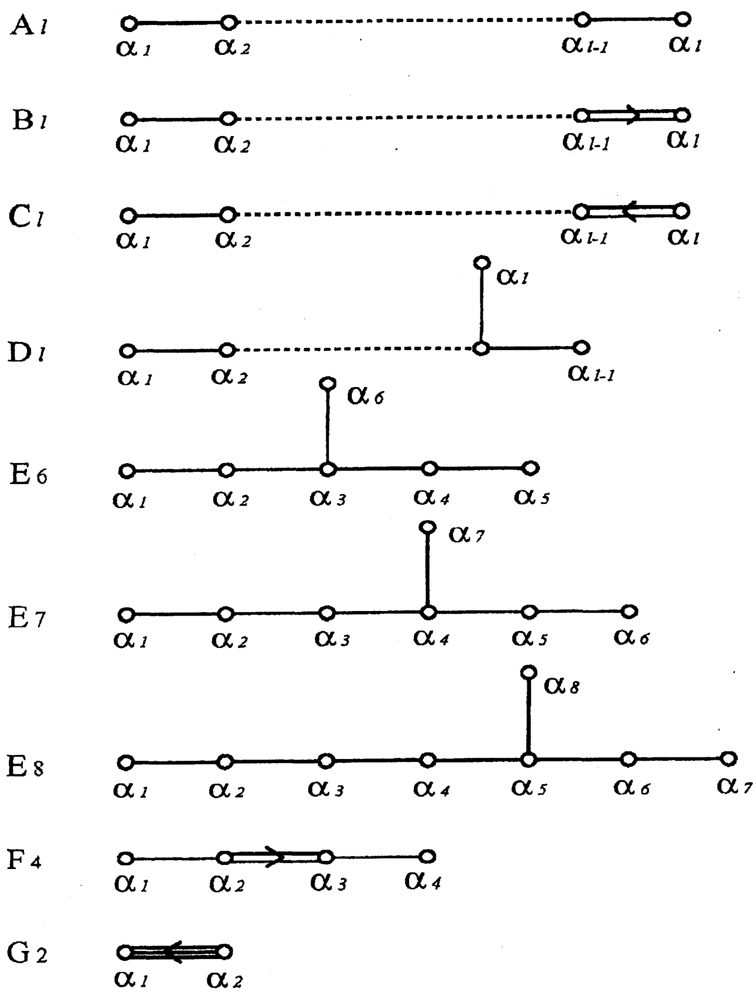

Figure 9. The Dynkin diagrams for simple Lie algebras and the labeling on the simple roots

References

[1] M. Adler and P. van Moerbeke, The Toda lattice, Dynkin diagrams, singularities and Abelian varieties, *Invent. math.* **103** (1991) 223-278.

[2] M. F. Atiyah, Convexity and commuting hamiltonians, *Bull. London Math. Soc.* **14** (1982) 1-15.

[3] O. I. Bogoyavlensky, On perturbations of the periodic Toda lattice, *Comm. math. Phys.*, **51** (1976) 201-209.

[4] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, (Graduate Texts in Mathematics 82), Splinger-Verlag, 1982. Chapter III.

[5] L. Casian and R. Stanton, Schubert cells and representation theory, *Invent. Math.* **137** (1999) 461-539.

[6] L. Casian and Y. Kodama, Topology of the isospectral real manifolds associated with the generalized Toda lattices on semisimple Lie algebras, *J. Phys. A: Math. Gen.* **34** (2001) 2353-2366.

[7] L. Casian and Y. Kodama, Toda lattice and toric varieties for real split semisimple Lie algebras. *Pacific J. of Math.* **207** (2002) 77-124.

[8] L. Casian and Y. Kodama, Blow-ups of the Toda lattices and their intersections with the Bruhat cells, *Contemp. Math.*, **301** (2002) 283-310.

[9] H. Flaschka and L. Haine, Varietes de drapeaux et reseaux de Toda. *Mathematische Zeitschrift*, **208** (1991), 545-556.

[10] P. Griffiths and J. Harris, *Principles of algebraic geometry* (Wiley, New York 1978).

[11] Y. Kodama and J. Ye, Toda Lattices with indefinite metric II: Topology of the iso-spectral manifolds. *Physica D*, **121** (1998), 89-108.

[12] B. M. Kostant, On Whittaker Vectors and Representation Theory, *Invent. Math.*, **48** (1978), 101-184.

[13] B. M. Kostant, The solution to a generalized Toda lattice and representation theory, *Adv. Math.* **34** (1979) 195-338.

[14] B. M. Kostant, Flag manifold, quantum cohomology, the Toda lattice and the representation with the highest weight ρ, *Selecta Math. New Series* **2** (1996) 43-91.

[15] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press (Oxford 1979).

[16] K. Rietsch, Quantum cohomology rings of Grassmannians and total positivity, *Duke Math. J.* **110** (2001) 523-553.

Department of Mathematics, Ohio State University, Columbus, OH 43210
E-mail address: casian@math.ohio-state.edu

Department of Mathematics, Ohio State University, Columbus, OH 43210
E-mail address: kodama@math.ohio-state.edu