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On global solutions for wave equations under the null condition in 3 space dimensions

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Abstract. Small global solutions for quasilinear wave equations are considered in three space dimensions in exterior domains. The obstacle is compact with smooth boundary and the local energy near the obstacle is assumed to decay exponentially with a possible loss of regularity. The null condition is needed to show global solutions for quadratic nonlinearity.

1 Introduction

This is a note on the joint work with Jason Metcalfe and Christopher D. Sogge [29]. The goal of this paper is to prove global existence of solutions to quadratic quasilinear Dirichlet-wave equations exterior to a class of compact obstacles. As in Metcalfe-Sogge [30], the main condition that we require for our class of obstacles is exponential local energy decay. Our result improves upon the earlier one of Metcalfe-Sogge [30] by allowing a more general null condition which only puts restrictions on the self-interaction of each wave family. In Minkowski space, such equations were studied and shown to have global solutions by Sideris-Tu [37], Agemi-Yokoyama [1], and Kubota-Yokoyama [25].

We use Klainerman's commuting vector fields method [20]:

$$\partial_0=\partial_t,\ \ \Omega_{ij}=x_i\partial_j-x_j\partial_i,\ \ 1\le i\neq j\le 3,\ \ L=t\partial_t+\sum_{1\le j\le 3}x_j\partial_j.$$

L is called the scaling operator. We denote $\{\partial_j\}_{0\le j\le 3}$ by ∂, $\{\Omega_{ij}\}_{1\le i\neq j\le 3}$ by Ω, $\{\partial,\Omega\}$ by Z, and $\{L,Z\}$ by Γ. For functions u, u' denotes ∂u. These operators have the commuting relations with d'Alembertian $\Box$:

$$\Box\Omega_{ij} = \Omega_{ij}\Box, \quad \Box L = (L+2)\Box, \quad L\Omega_{ij} = \Omega_{ij}L, \quad \partial_j L = (L+1)\partial_j. \tag{1.1}$$

Using Z, we can earn one weight by Klainerman-Sobolev inequality :

Lemma 1.1 *[20] [17, Lemma 2.4] [35, Lemma 3.3] Suppose that $h \in C^\infty(\mathbb{R}^3)$. Then, for $R > 2$,*

$$\|h\|_{L^\infty(R<|x|<R+1)} \le CR^{-1} \sum_{|\alpha|+|\beta|\le 2} \|\Omega^\alpha \partial_x^\beta h\|_{L^2(R-1<|x|<R+2)}. \tag{1.2}$$

We describe our assumptions on our obstacles $\mathcal{K} \subset \mathbb{R}^3$. We shall assume that $\mathcal{K}$ is smooth and compact, but not necessarily connected. By scaling, without loss of generality, we may assume

$$\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}, \ 0 \in \mathcal{K}\backslash\partial\mathcal{K}.$$

The only additional assumption states that there is exponential local energy decay with a possible loss of regularity. That is, if u is a solution to

$$\begin{cases} \Box u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K} \\ u(t,\cdot)|_{\partial\mathcal{K}} = 0 \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g, & \mathrm{supp}\, f \cup \mathrm{supp}\, g \subset \{\mathbb{R}^3\backslash\mathcal{K}, |x| \le 4\}, \end{cases} \tag{1.3}$$

then there must be constants $c, C > 0$ so that

$$\|u'(t,\cdot)\|_{L^2(x\in\mathbb{R}^3\backslash\mathcal{K}, |x|\le 4)} \le Ce^{-ct} \sum_{|\alpha|\le 1} \|\partial_x^\alpha u'(0,\cdot)\|_2. \tag{1.4}$$

Throughout this paper, we assume this local energy decay estimate for $\mathcal{K}$.

Lax, Morawetz and Phillips have shown (1.4) without a loss of regularity, namely $|\alpha| = 0$ in the RHS, when $\mathcal{K}$ is star-shaped in [26] (see also [27, Theorem 3.2]).

Morawetz, Ralston and Strauss have shown (1.4) without a loss of regularity ($|\alpha| = 0$) when $\mathcal{K}$ is bounded connected and nontrapping in [32, (3.1)]. Here if the lengths of all rays in $B_1(0)\backslash\mathcal{K}$ are bounded, then waves are not trapped and (1.4) holds without a loss of regularity. They also treat the multi-dimensional cases. See Melrose [28] for further results. Ralston [33] has shown that (1.4) could not hold without a loss of regularity when there are trapped rays..

Ikawa has shown (1.4) with an additional loss of regularity, namely $|\alpha| \le \ell$ with $\ell \ge 1$ in the RHS, when $\mathcal{K}$ is trapping. He has shown (1.4) with $\ell = 6$ when $\mathcal{K}$ consists of two disjoint strictly convex bodies in [12], and (1.4) with $\ell = 2$ when $\mathcal{K}$ consists of sufficiently separated several disjoint strictly convex bodies in [13]. Since we have the standard energy preservation

$$\|u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} = \|u'(0,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}$$

(see (3.3) with $\gamma = 0$), we can reduce the estimate (1.4) with an additional regularity, $\ell \ge 1$, to the estimate for $\ell = 1$ with different constants c and C by the interpolation. Therefore we can treat the above obstacles by the condition (1.4).

We note that we do not require exponential decay; in fact, $O((1+t)^{-1-\delta-m})$ with $\delta > 0$ and $m \ge 0$ may be sufficient with a tighter argument, where we need $1+\delta$ for the integral ability and m is the number of L we need in our argument (see the argument below (4.4) to bound $t^{\mu}e^{-ct/2}$). Currently, the authors are not aware of any 3-dimensional example that involves polynomial decay, but does not have exponential decay.

We consider quadratic, quasilinear systems of the form

$$\begin{cases} \Box u = F(\partial u, \partial^2 u), \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K} \\ u(t,\,\cdot\,)|_{\partial\mathcal{K}} = 0 \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{1.5}$$

Here $\Box$ denotes a vector-valued multiple speed d'Alembertian :

$$\Box u = (\Box_{c_1}u^1, \Box_{c_2}u^2, \dots, \Box_{c_D}u^D), \qquad F = (F^1, \cdots, F^D), \ \ D \ge 1, \tag{1.6}$$

where

$$\Box_{c_I} = \partial_t^2 - c_I^2\Delta, \qquad 1 \le I \le D.$$

We assume that the wave speeds c_I are positive and distinct:

$$0 < c_1 < \cdots < c_D.$$

Straightforward modifications of the argument give the more general case where the various components are allowed to have the same speed.

We shall assume that $F(\partial u, \partial^2 u)$ is of the form

$$F^I(\partial u, \partial^2 u) = \sum_{\substack{1\le J,K\le D\\ 0\le j,k\le 3}} A^{IJK}_{jk}\partial_j u^J \partial_k u^K + \sum_{\substack{0\le j,k,l\le 3\\ 1\le J,K\le D}} B^{IJK}_{jkl}\partial_j u^J \partial_k \partial_l u^K, \quad 1 \le I \le D. \tag{1.7}$$

For the energy estimates, we require the symmetry condition:

$$B^{IJK}_{jkl} = B^{KJI}_{jkl} = B^{IJK}_{jlk}.$$

To obtain global existence, we also require that the equations satisfy the following null condition which only involves the self-interactions of each wave family :

$$\sum_{0\le j,k\le 3} A^{III}_{jk}\xi_j\xi_k = 0 \quad \text{whenever} \quad \xi_0^2 = c_I^2(\xi_1^2+\xi_2^2+\xi_3^2), \quad I = 1,\dots,D, \tag{1.8}$$

$$\sum_{0\le j,k,l\le 3} B^{III}_{jkl}\xi_j\xi_k\xi_l = 0 \quad \text{whenever} \quad \xi_0^2 = c_I^2(\xi_1^2+\xi_2^2+\xi_3^2), \quad I = 1,\dots,D. \tag{1.9}$$

The terms which satisfy the above null conditions are treated by the following estimates :

Lemma 1.2 *[37, 40] If the semilinear null condition* (1.8) *holds, then*

$$\Big|\sum_{0\le j,k\le 3} A^{III}_{jk}\partial_j u \partial_k v\Big| \le C\frac{|\Gamma u||\partial v| + |\partial u||\Gamma v|}{\langle r\rangle} + C\frac{\langle c_I t - r\rangle}{\langle t+r\rangle}|\partial u||\partial v|. \tag{1.10}$$

Suppose that the quasilinear null condition (1.9) *holds. Then,*

$$\Big|\sum_{0\le j,k,l\le 3} B^{III}_{jkl}\partial_l u \partial_j\partial_k v\Big| \le C\frac{|\Gamma u||\partial^2 v| + |\partial u||\partial\Gamma v|}{\langle r\rangle} + C\frac{\langle c_I t - r\rangle}{\langle t+r\rangle}|\partial u||\partial^2 v|. \tag{1.11}$$

We briefly remark on the null condition for the boundaryless case in three space dimensions. John has shown in [14] that the nontrivial solution of single wave equation

$$\Box u = (\partial_t u)^2,$$

which data have the compact support, blows up in finite time. On the other hand, Christodoulou in [5] and Klainerman in [20] have shown independently the global solutions for small data when the nonlinear term satisfies the null condition. A typical example of such equation is given by

$$\Box u = a\{(\partial_t u)^2 - (\nabla u)^2\}, \quad a \in \mathbb{R}.$$

Alinhac has shown in [2] that the null condition is necessary to show the global solutions when the nonlinear term is quadratic quasilinear excluding u itself. Kovalyov pointed out in [22] that when we consider the systems of wave equations with different speeds, the situation become different and the the sytems tend to have global solutions for small data. A typical example which has global solutions for small data is given by

$$\begin{cases} (\partial_t^2 - c_1^2\Delta)u_1 = a(\partial_t u_2)^2 \\ (\partial_t^2 - c_2^2\Delta)u_2 = b(\partial_t u_1)^2 \end{cases} \qquad a, b \in \mathbb{R}, \quad c_1 \neq c_2 > 0.$$

For further historical sketch, we refer to the section 6 in [23] or [24].

We refer to compatibility conditions. For the solution u of (1.5), the functions $\{\partial_t^j u(0,x)\}_{j\geq 0}$ are called compatible functions. The compatible functions are functions of spatial variables and $\partial_t^j u(0,x)$ are expressed by $\{\partial_x^\alpha f\}_{|\alpha|\leq j}$ and $\{\partial_x^\alpha g\}_{|\alpha|\leq j-1}$. We say that the compatibility conditions of order s are satisfied if $\partial_t^j u(0,x)|_{\partial\mathcal{K}} = 0$ for all $0 \leq j \leq s$ (See [16, Definition 9.2]). Additionally, we say that $(f,g) \in C^\infty$ satisfies the compatibility conditions to infinite order if the compatibility conditions are satisfied to any order $s \geq 0$.

We can now state our main result:

Theorem 1.3 *Let $\mathcal{K}$ be a fixed compact obstacle with smooth boundary that satisfies* (1.4). *Assume that $F(\partial u, \partial^2 u)$ and $\Box$ are as above and that $(f,g) \in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfy the compatibility conditions to infinite order. Then there is a constant $\varepsilon_0 > 0$, and an integer $N > 0$ so that for all $\varepsilon < \varepsilon_0$, if*

$$\sum_{|\alpha|\leq N} \|\langle x\rangle^{|\alpha|}\partial_x^\alpha f\|_2 + \sum_{|\alpha|\leq N-1} \|\langle x\rangle^{1+|\alpha|}\partial_x^\alpha g\|_2 \leq \varepsilon \tag{1.12}$$

then (1.5) *has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^3\backslash\mathcal{K})$.*

This paper is organized as follows. In the next section, we will collect some preliminary results which are frequently used in this paper. We put several sections for energy estimates, L^2 estimates in space and time, and Sobolev embeddings, respectively. We will show the continuity argument in the last section to prove Theorem 1.3.

2 Preliminaries

We use the following Poincaré inequalities to bound u by u' near the obstacle:

$$\|u\|_{L^2(\mathbb{R}^3\setminus\mathcal{K},|x|<R)} \le C_R\|\nabla u\|_{L^2(\mathbb{R}^3\setminus\mathcal{K},|x|<R)} \text{ if } u|_{\partial\mathcal{K}} = 0, \tag{2.1}$$

where C_R is a constant dependent on $R \ge 1$ (cf. [7, (7.44)]).

We also use the following elliptic regularity : for any fixed $M \ge 0$

$$\sum_{2\le|\alpha|\le M+2} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3\setminus\mathcal{K},|x|<R)} \le C_R\Big(\sum_{|\alpha|\le M} \|\partial_x^\alpha \nabla u\|_{L^2(\mathbb{R}^3\setminus\mathcal{K},|x|<R+1)} + \sum_{|\alpha|\le M} \|\partial_x^\alpha \Delta u\|_{L^2(\mathbb{R}^3\setminus\mathcal{K},|x|<R+1)}\Big) \tag{2.2}$$

if $u|_{\partial\mathcal{K}} = 0$ (cf. [7, Theorem 8.13]).

Here we briefly sketch the elementary method to treat the nonlinearity.

Lemma 2.1 *Let $u \in C^\infty((0,\infty)\times\mathbb{R}^3\setminus\mathcal{K})$. Suppose u has the bound*

$$\sum_{|\alpha|\le M_0} \|Z^\alpha u'(t,x)\|_{L^\infty_x} \le \frac{C_0\varepsilon}{1+t} \tag{2.3}$$

for some constants $M_0 \ge 0$ and $C_0 \ge 0$. Then for any $M \ge 0$ and $\mu_0 \ge 0$, there exists a constant C such that we have

$$\begin{aligned}
\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha(u'u')(t)\|_{L^2_x} \le{}& \frac{C_0\varepsilon}{1+t}\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha u'(t)\|_{L^2_x}\\
&+C\sum_{M_0+1\le|\alpha|\le M-M_0+1} \|\langle x\rangle^{-1/2}Z^\alpha u'(t)\|_{L^2_x}\sum_{M_0+1\le|\alpha|\le M-M_0-1} \|\langle x\rangle^{-1/2}\partial^\alpha u'(t)\|_{L^2_x}\\
&+C\sum_{\substack{\mu+|\alpha|\le M-M_0+1\\ 1\le\mu\le\mu_0}} \|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'(t)\|_{L^2_x}\sum_{|\alpha|\le M-1} \|\langle x\rangle^{-1/2}\partial^\beta u'(t)\|_{L^2_x}\\
&+C\sum_{\substack{\mu+|\alpha|\le M/2+2\\ 1\le\mu\le\mu_0-1}} \|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'(t)\|_{L^2_x}\sum_{\substack{\mu+|\alpha|\le M-1\\ 1\le\mu\le\mu_0-1}} \|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'(t)\|_{L^2_x}.
\end{aligned} \tag{2.4}$$

Here ∂ can be replaced by Z in the above inequality.

Proof of Lemma 2.1 : We use the following estimates:

$$
\begin{aligned}
\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha(u'u')\|_2 &\lesssim \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\le M\\ \mu+\nu\le\mu_0}} \|L^\mu\partial^\alpha u' L^\nu\partial^\beta u'\|_2 \\
&\lesssim \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha u'\|_2 \sum_{|\beta|\le M_0} \|\partial^\beta u'\|_\infty + \sum_{\substack{M_0+1\le|\alpha|\le M-M_0-1\\ M_0+1\le|\beta|\le M-M_0-1}} \|\partial^\alpha u'\partial^\beta u'\|_2 \\
&\quad + \sum_{\substack{\mu+|\alpha|\le M-M_0-1\\ 1\le\mu\le\mu_0}} \sum_{M_0+1\le|\beta|\le M-1} \|L^\mu\partial^\alpha u'\partial^\beta u'\|_2 \\
&\quad + \sum_{\substack{\mu+|\alpha|\le M/2\\ 1\le\mu\le\mu_0-1}} \sum_{\substack{\nu+|\beta|\le M-1\\ 1\le\nu\le\mu_0-1}} \|L^\mu\partial^\alpha u' L^\nu\partial^\beta u'\|_2.
\end{aligned} \tag{2.5}
$$

Since we have by (1.2)

$$
\begin{aligned}
|L^\mu\partial^\alpha u'(t,x)| &\lesssim \langle x\rangle^{-1} \sum_{|\beta|\le 2} \|Z^\beta L^\mu\partial^\alpha u'(t,x)\|_{L^2(|x|-1\le|y|\le|x|+1)} \\
&\lesssim \langle x\rangle^{-1/2} \sum_{\nu+|\beta|\le\mu+|\alpha|+2} \|\langle x\rangle^{-1/2} L^\mu Z^\beta u'\|_2,
\end{aligned}
$$

we obtain the required result using (2.3). □

3 Energy Estimates

Since we are considering the quasilinear wave equation, we need associated energy estimates as follows. Let $\gamma = \{\gamma^{IJ,jk}\}_{1\le I,J\le D, 0\le j,k\le 3}$ be any smooth functions on $[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}$. We consider $\Box_\gamma$ which is defined by

$$
(\Box_\gamma u)^I(t,x) = (\partial_t^2 - c_I^2\Delta)u^I(t,x) + \sum_{J=1}^{D}\sum_{j,k=0}^{3} \gamma^{IJ,jk}(t,x)\partial_j\partial_k u^J(t,x), \quad 1\le I\le D.
$$

And we define the energy form associated with $\Box_\gamma$ as follows :

$$
e_0^I(u) = (\partial_0 u^I)^2 + \sum_{k=1}^{3} c_I^2(\partial_k u^I)^2 + 2\sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0 u^I\partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}\partial_j u^I\partial_k u^J \tag{3.1}
$$

$$
e_0 = e_0(u) = \sum_{I=1}^{D} e_0^I(u).
$$

We define the other components of the energy-momentum vector. For $I = 1, 2, \cdots, D$, and $k = 1, 2, 3$, let

$$e_k^I = e_k^I(u) = -2c_I^2\partial_0 u^I \partial_k u^I + 2\sum_{J=1}^{D}\sum_{j=0}^{3}\gamma^{IJ,jk}\partial_0 u^I \partial_j u^J$$

$$e_j = e_j(u) = \sum_{I=1}^{D} e_j^I, \quad j = 1, 2, 3$$

$$R_0^I(u) = 2\sum_{J=1}^{D}\sum_{k=0}^{3}(\partial_0\gamma^{IJ,0k})\partial_0 u^I \partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}(\partial_0\gamma^{IJ,jk})\partial_j u^I \partial_k u^J$$

$$R_k^I(u) = 2\sum_{J=1}^{D}\sum_{j=0}^{3}(\partial_k\gamma^{IJ,jk})\partial_0 u^I \partial_j u^J$$

$$R(u) = \sum_{I=1}^{D}\sum_{k=0}^{3} R_k^I(u).$$

Then we have the following most fundamental energy estimates (See [39], p13) :

Lemma 3.1 *Suppose that the functions $\gamma^{IJ,jk}$ satisfy the symmetry conditions*

$$\gamma^{IJ,jk} = \gamma^{JI,jk} = \gamma^{IJ,kj} \quad \textit{for} \quad 1 \le I, J \le D, \quad 0 \le j, k \le 3. \tag{3.2}$$

For any function u in $C^2((0,\infty)\times\mathbb{R}^3\backslash\mathcal{K})$, the following equation holds:

$$\partial_t e_0 + \mathrm{div}(e_1, e_2, e_3) = 2\partial_t u \cdot \Box_\gamma u + R(u). \tag{3.3}$$

Proof of Lemma 3.1: By direct computation, we have

$$\begin{aligned}\partial_0 e_0^I &= 2\partial_0 u^I \partial_0^2 u^I + 2\sum_{k=1}^{3} c_I^2 \partial_k u^I \partial_0\partial_k u^I + 2\partial_0 u^I \sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0\partial_k u^J \\ &+ 2\sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0^2 u^I \partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}(\partial_0\partial_j u^I \partial_k u^J + \partial_j u^I \partial_0\partial_k u^J) + R_0^I\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\sum_{k=1}^{3}\partial_k e_k^I &= -2\partial_0 u^I c_I^2 \Delta u^I - 2\sum_{k=1}^{3} c_I^2 \partial_k u^I \partial_0\partial_k u^I \\ &+ 2\partial_0 u^I \sum_{J=1}^{D}\sum_{j=0}^{3}\sum_{k=1}^{3}\gamma^{IJ,jk}\partial_j\partial_k u^J + 2\sum_{J=1}^{D}\sum_{j=0}^{3}\sum_{k=1}^{3}\gamma^{IJ,jk}\partial_0\partial_k u^I \partial_j u^J + \sum_{k=1}^{3} R_k^I.\end{aligned} \tag{3.5}$$

We obtain the required result using the symmetry condition (3.2). □

We use (3.3) to show the energy estimates for $L^\mu Z^\alpha u$. However, direct application causes derivative losses from $\mathrm{div}(e_1, e_2, e_3)$ since L, Ω, ∂_x don't preserve the Dirichlet condition. To avoid it, we cut L near the obstacle and construct the energy estimates for $\partial_t^j u$. Let $\eta \in C^\infty(\mathbb{R}^3)$ be a smooth function with $\eta(x) = 0$ for $|x| \le 1$ and $\eta(x) = 1$ for $|x| \ge 2$. We define $\tilde{L}$ by $\tilde{L} = t\partial_t + \eta r \partial_r$. By simple calculation, we have for any $\mu \ge 0$

$$\tilde{L}^\mu = L^\mu + \sum_{j+|\alpha|\le\mu-1} C_{\mu,j,\alpha}\chi_{\mu,j,\alpha}(x) L^j \partial_x^\alpha \partial_x, \quad \chi_{\mu,j,\alpha} \in C_0^\infty(\mathbb{R}^3), \quad \mathrm{supp}\,\chi_{\mu,j,\alpha} \subset B_2(0), \tag{3.6}$$

where $\{C_{\mu,j,\alpha}\}$ are constants dependent on lower indices.

Our first task is to show the energy estimates for $\tilde{L}^\mu \partial_t^j u$. We put

$$E_{M,\mu_0}(t) = E_{M,\mu_0}(u)(t) = \int \sum_{\substack{\mu+j\le M \\ \mu\le\mu_0}} e_0(\tilde{L}^\mu \partial_t^j u)(t,x)\, dx.$$

The estimate for $E_{M,\mu_0}(t)$ is given by the following lemma. And the energy estimates for $L^\mu \partial^\alpha u$ follows from it due to the elliptic regularity :

Lemma 3.2 *Assume that the perturbation terms $\gamma^{IJ,jk}$ satisfy* (3.2) *and the size condition*

$$\sum_{I,J=1}^{D} \sum_{j,k=0}^{3} \|\gamma^{IJ,jk}(t,x)\|_{L^\infty_{t,x\in\mathbb{R}^3\setminus\mathcal{K}}} \le \delta \tag{3.7}$$

for δ sufficiently small. Then for any $M \ge 0$ and $\mu_0 \ge 0$, there exists a constant $C = C(M, \mu_0, \mathcal{K})$ so that for any smooth function u in $[0,\infty) \times \mathbb{R}^3\backslash\mathcal{K}$ with $u(t,x)|_{x\in\partial\mathcal{K}} = 0$, the following estimates hold.

$$\sum_{\substack{\mu+|\alpha|\le M \\ \mu\le\mu_0}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2 \le C E_{M,\mu_0}^{1/2} + C \sum_{\substack{\mu+|\alpha|\le M-1 \\ \mu\le\mu_0}} \|L^\mu \partial^\alpha \Box u(t,\cdot)\|_2. \tag{3.8}$$

$$\begin{aligned}\partial_t E^{1/2}_{M,\mu_0}(t) &\le C \sum_{\substack{\mu+j\le M\\ \mu\le\mu_0}} \|\Box_\gamma \tilde{L}^\mu \partial_t^j u(t,\cdot)\|_2 + C\|\gamma'(t,\cdot)\|_\infty E^{1/2}_{M,\mu_0}(t) \\ &\le C \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha \Box_\gamma u(t,\cdot)\|_2 + C\|\gamma'(t,\cdot)\|_\infty E^{1/2}_{M,\mu_0}(t) \\ &\quad + C \sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\le M\\ \mu_1+\mu_2\le\mu_0\\ \mu_2+|\alpha_2|\le M-1}} \|(L^{\mu_1}\partial^{\alpha_1}\gamma(t,\cdot))(L^{\mu_2}Z^{\alpha_2}\partial^2 u(t,\cdot))\|_2 \\ &\quad + C \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0-1}} \|L^\mu\partial^\alpha u'(t,x)\|_{L^2(|x|<2)}. \end{aligned} \tag{3.9}$$

When we apply Gronwall's inequality to (3.9), we need the following lemma to bound the last term in (3.9).

Lemma 3.3 *For any $M\ge 0$ and μ_0, there exists a constant $C=C(M,\mu_0,\mathcal{K})$ such that for any smooth function u in $[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}$ with the Dirichlet condition $u(t,x)|_{x\in\partial\mathcal{K}}=0$ the following estimate holds.*

$$\begin{aligned}\sum_{\substack{\mu+j\le M\\ \mu\le\mu_0}} \int_0^t \|L^\mu\partial^\alpha u'(s,x)\|_{L^2(|x|<2)}ds &\le C \sum_{\substack{\mu+j\le M+2\\ \mu\le\mu_0}} \|\langle x\rangle (L^\mu\partial^\alpha u)(0,\cdot)\|_2 \\ &+ \sum_{\substack{\mu+|\alpha|\le M+1\\ \mu\le\mu_0}} \int_0^t\int_0^s \|L^\mu\partial^\alpha G(\tau,y)\|_{L^2(||y|-(s-\tau)|<10)}d\tau ds \\ &+ \sum_{\substack{\mu+|\alpha|\le M+1\\ \mu\le\mu_0}} \int_0^t \|L^\mu\partial^\alpha \Box u(s,y)\|_{L^2(|y|<4)}ds. \end{aligned} \tag{3.10}$$

For the energy estimates for $L^\mu Z^\alpha u$, we need the following estimates. Begin by setting

$$Y_{M,\mu_0}(t) = \int \sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}} e_0(L^\mu Z^\alpha u)(t,x)\,dx. \tag{3.11}$$

We, then, have the following lemma which shows how the energy estimates for $L^\mu Z^\alpha u$ can be obtained from the ones involving $L^\mu\partial^\alpha u$.

Lemma 3.4 *Assume* (3.2), (3.7) *and*

$$\|\gamma'(t,\cdot)\|_\infty \equiv \sum_{I,J=1}^{D}\sum_{j,k,l=0}^{3} \|\partial_l \gamma^{IJ,jk}(t,\cdot)\|_\infty \le \delta \tag{3.12}$$

for sufficiently small δ. Then,

$$
\begin{aligned}
\partial_t Y_{M,\mu_0} \leq\ & C Y_{M,\mu_0}^{1/2} \sum_{\substack{|\alpha|+\mu\leq M\\ \mu\leq\mu_0}} \|\Box_\gamma L^\mu Z^\alpha u(t,\,\cdot\,)\|_2 \\
& + C\|\gamma'(t,\,\cdot\,)\|_\infty Y_{M,\mu_0} + C \sum_{\substack{|\alpha|+\mu\leq M+1\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha u'(s,\,\cdot\,)\|^2_{L^2(|x|<2)} \\
\leq\ & C Y_{M,\mu_0}^{1/2} \Big\{ \sum_{\substack{|\alpha|+\mu\leq M\\ \mu\leq\mu_0}} \|L^\mu Z^\alpha \Box_\gamma u(t,\,\cdot\,)\|_2 \\
& \quad + \sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\leq M\\ \mu_1+\mu_2\leq\mu_0\\ \mu_2+|\alpha_2|\leq M-1}} \|(L^{\mu_1} Z^{\alpha_1}\gamma)(L^{\mu_2} Z^{\alpha_2}\partial^2 u)\|_2 \Big\} \\
& + C\|\gamma'(t,\,\cdot\,)\|_\infty Y_{M,\mu_0} + C \sum_{\substack{|\alpha|+\mu\leq M+1\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha u'(s,\,\cdot\,)\|^2_{L^2(|x|<2)}
\end{aligned}
\tag{3.13}
$$

4 Local energy estimates and L^2 estimates in space and time

First we derive local energy estimates for inhomogeneous wave equations near the obstacle.

Lemma 4.1 *Let $\mathcal{K}$ satisfy the local energy decay (1.4). Let u be the solution of*

$$
\begin{cases}
\Box u = F, \quad \mathrm{supp}_x F(t,x) \subset B_4(0) \\
u|_{\partial\mathcal{K}} = 0 \\
u(0) = f,\ \partial_t u(0) = g, \quad \mathrm{supp}\, f \cup \mathrm{supp}\, g \subset B_4(0).
\end{cases}
\tag{4.1}
$$

Then for any $M \geq 0$ and $\mu_0 \geq 0$, the following estimates holds :

$$
\begin{aligned}
\sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha u'(t,x)\|_{L^2(|x|<4)} \leq\ & C e^{-ct/2} \sum_{|\alpha|\leq M+1} \|\partial^\alpha u'(0,x)\|_{L^2(|x|<4)} \\
& + C \int_0^t e^{-c(t-s)/2} \sum_{\substack{\mu+|\alpha|\leq M+1\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha F(s,\cdot)\|_2 ds + \sum_{\substack{\mu+|\alpha|\leq M-1\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha F(t,\cdot)\|_2.
\end{aligned}
\tag{4.2}
$$

Proof of Lemma 4.1 : First we show (4.2) for $\mu_0 = 0$ using induction. The estimate for $M = 0$ follows from (1.4) and the Duhamel priciple. Let's assume that the estimate for

$M \geq 0$, and we consider the case $M+1$. We have

$$\sum_{|\alpha|\leq M+1} \|\partial^\alpha u'\|_{L^2(|x|<4)} \lesssim \sum_{|\alpha|\leq M} \|\partial^\alpha u'\|_{L^2(|x|<4)} + \sum_{\substack{j+|\alpha|\leq M+2\\ j\geq 1}} \|\partial_t^j \partial_x^\alpha u\|_{L^2(|x|<4)} + \sum_{|\alpha|=M+2} \|\partial_x^\alpha u\|_{L^2(|x|<4)}. \quad (4.3)$$

The first two terms in the RHS are treated by induction since $\partial_t u$ satisfies the Dirichlet condition. Applying (2.1) and (2.2) to the last term, we have

$$\sum_{|\alpha|=M+2} \|\partial_x^\alpha u(t)\|_{L^2(|x|<4)} \lesssim \|u'\|_{L^2(|x|<5)} + \sum_{|\alpha|\leq M} \|\partial_x^\alpha \partial_t^2 u\|_{L^2(|x|<5)} + \sum_{|\alpha|\leq M} \|\partial_x^\alpha \Box u\|_{L^2(|x|<5)}.$$

Again by induction, we obtain the required estimate for $M+1$. Here we can replace $c/2$ with c in (4.2) when $\mu_0 = 0$.

Next we show (4.2) for $\mu_0 \geq 1$ by induction. Let's assume that (4.2) holds for M and μ_0. We consider the case $\mu_0 + 1$. Since we have

$$\sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0+1}} \|L^\mu \partial^\alpha u'\|_{L^2(|x|<4)} \lesssim \sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha u'\|_{L^2(|x|<4)} + \sum_{\substack{\mu+|\alpha|\leq M\\ 1\leq\mu\leq\mu_0+1}} t^\mu \|\partial_t^\mu \partial^\alpha u'\|_{L^2(|x|<4)}, \quad (4.4)$$

it suffices by induction to show the last term in the RHS is bounded by the RHS in (4.2). If we use (4.2) for $\mu_0 = 0$ for $\partial_t^j u$ which satisfies the Dirichlet condition, and we use that $t^\mu e^{-ct/2}$ is bounded, then we obtain the required estimate. □

Remark. We abstractly remark on how we apply the local energy decay estimates to obtain the required estimates in the exterior domain case. Let u be the function defined in the exterior domain with the Dirichlet condition $u(t,\cdot)|_{\partial\mathcal{K}} = 0$. The estimates for u when $|x| \geq 2$ are obtained by those for boundaryless case by multiplying a smooth cutting function which vanishes near the obstacle. The estimates for u when $|x| < 2$ can be obtained by the following procedure. We decompose $\Box u$ by a smooth cutting function as

$$\Box u = F_1 + F_2, \quad \mathrm{supp} F_1 \subset \{|y| < 4\}, \quad \mathrm{supp} F_2 \subset \{|y| > 3\}.$$

And we consider two functions such as

$$\begin{cases} \Box u_j = F_j, \quad j = 1,2, \\ u_j|_{\partial\mathcal{K}} = 0 \\ u_j(0,\cdot) = f_j(\cdot), \quad \partial_t u_j(0,\cdot) = g_j(\cdot) \end{cases}$$

where f_j and g_j are decomposed functions of $u(0,\cdot)$ and $\partial_t u(0,\cdot)$ by the same cutting function to F_j. We have $u = u_1 + u_2$. And the estimates for u_1 can be obtained applying the local energy decay estimates. To obtain the estimates for u_2, we concider the function v which satisfies

$$\begin{cases} \Box v = F_2 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^3 \\ v(0,\cdot) = f_2(\cdot), \qquad \partial_t v(0,\cdot) = g_2(\cdot). \end{cases}$$

And we define w by $u_2 = v + w$. Since we are considering the case $|x| < 2$, using the smooth cutting function η which satisfies $\eta(y) = 1$ for $|y| \leq 2$ and $\eta(y) = 0$ for $|y| > 3$, we have $u_2(t,x) = \eta(x)v(t,x) + w(t,x)$. Especially we have

$$\Box u_2 = -2\nabla\eta \cdot \nabla v - \Delta\eta v.$$

Since the RHS of the above equation has its support near the obstacle, we obtain the estimates for u_2 by the local energy estimates again. Consequently we can obtain the required estimates for u.

We need weighted L^2 estimates. Put

$$S_T = \{[0,T] \times \mathbb{R}^3 \backslash \mathcal{K}\}$$

to denote the time strip of height T in $\mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}$.

Lemma 4.2 *(1) (Boundaryless case [17, Proposition 2.1]) There exists a constant $C > 0$ so that for any function u in $[0,\infty) \times \mathbb{R}^3$, the following estimate holds.*

$$(\log(2+T))^{-1/2} \|\langle x\rangle^{-1/2} u'\|_{L^2([0,T]\times\mathbb{R}^3)} \leq C \sum_{|\alpha|\leq 1} \|\partial^\alpha u(0,\cdot)\|_2 + C\int_0^T \|\Box u(t,\cdot)\|_2 dt. \quad (4.5)$$

(2) (Exterior domain case [18, (6.8), (6.9)]) There exists a constant C so that for any function u in $[0,\infty) \times \mathbb{R}^3 \backslash \mathcal{K}$ with the Dirichlet condition $u(t,x)|_{x\in\partial\mathcal{K}} = 0$, the following estimate holds. For any $M \geq 0$ and $\mu_0 \geq 0$

$$\begin{aligned}(\log(2+T))^{-1/2} \sum_{\substack{|\alpha|+\mu\leq M \\ \mu\leq\mu_0}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_T)} &\leq C \sum_{\substack{|\alpha|+\mu\leq M+2 \\ \mu\leq\mu_0}} \|(L^\mu\partial^\alpha u)(0,\cdot)\|_2 \\ &+ C\int_0^T \sum_{\substack{|\alpha|+\mu\leq M+1 \\ \mu\leq\mu_0}} \|L^\mu\partial^\alpha \Box u(t,\cdot)\|_2 \, dt + C \sum_{\substack{|\alpha|+\mu\leq M \\ \mu\leq\mu_0}} \|L^\mu\partial^\alpha\Box u\|_{L^2(S_T)} \quad (4.6)\end{aligned}$$

and

$$(\log(2+T))^{-1/2}\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}}\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\|_{L^2(S_T)}\le C\sum_{\substack{|\alpha|+\mu\le M+2\\ \mu\le\mu_0}}\|L^\mu Z^\alpha u(0,x)\|_{L^2_x}$$
$$+C\int_0^T\sum_{\substack{|\alpha|+\mu\le M+1\\ \mu\le\mu_0}}\|\Box L^\mu Z^\alpha u(t,\cdot)\|_2\,dt+C\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}}\|\Box L^\mu Z^\alpha u\|_{L^2(S_T)} \quad (4.7)$$

5 Pointwise Estimates

We consider pointwise estimates in this section.

Lemma 5.1 *Let F, f and g be any functions.*

(1) (Boundaryless case) Let u be a solution to

$$\begin{cases}(\partial_t^2-\Delta)u(t,x)=F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3\\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x).\end{cases}$$

Then

$$(1+t+|x|)|u(t,x)|\le C\sum_{\substack{\mu+|\alpha|\le 3\\ \mu\le 1, j\le 1}}\|(\langle x\rangle^j\partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L^2_x}$$
$$+C\int_0^t\int_{\mathbb{R}^3}\sum_{\substack{\mu+|\alpha|\le 3\\ \mu\le 1}}|L^\mu Z^\alpha F(s,y)|\,\frac{dy\,ds}{\langle y\rangle}. \quad (5.1)$$

(2) (Exterior domain case) Let u be a solution to

$$\begin{cases}(\partial_t^2-\Delta)u(t,x)=F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}\\ u(t,x)|_{x\in\partial\mathcal{K}}=0\\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x).\end{cases}$$

Then for any $M \geq 0$ *and* $\mu_0 \geq 0$

$$
\begin{aligned}
(1+t+|x|) \sum_{\substack{|\alpha|+\mu\leq M \\ \mu\leq\mu_0}} |L^\mu Z^\alpha u(t,x)| \leq C & \sum_{\substack{j+\mu+|\alpha|\leq M+8 \\ \mu\leq\mu_0+2,\ j\leq 1}} \|(\langle x\rangle^j \partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L^2_x} \\
& + C\int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\alpha|+\mu\leq M+7 \\ \mu\leq\mu_0+1}} |L^\mu Z^\alpha F(s,y)| \frac{dy\, ds}{|y|} \\
& + C\int_0^t \sum_{\substack{|\alpha|+\mu\leq M+4 \\ \mu\leq\mu_0+1}} \|L^\mu \partial^\alpha F(s,y)\|_{L^2(|y|<4)}\, ds. \qquad (5.2)
\end{aligned}
$$

Here and throughout $\{|y| < 4\}$ is understood to mean $\{y \in \mathbb{R}^3\backslash\mathcal{K} : |y| < 4\}$.

The proof of the above lemma for vanishing Cauchy data has been shown by Keel-Smith-Sogge in [18, (2.3), (2.4) and (4.2)] and Metcalfe-Sogge in [30, (3.2)].

The following estimates are the special version to treat the inhomogeneity F near the light cones, which follows from the Huygens principle.

Lemma 5.2 *Let* F *be any function.*

(1) (Boundaryless case) Let u *be a solution to*

$$
\begin{cases}
(\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), & (t,x) \in [0,\infty)\times\mathbb{R}^3 \\
u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = 0. &
\end{cases}
$$

Assume

$$
\mathrm{supp}F \subset \{(t,x); t\geq 1,\ \frac{c_1 t}{10} \leq |x| \leq 10 c_D t\}.
$$

Then

$$
\sup_{|x|\leq c_1 t/2} (1+t)|u(t,x)| \leq C \sup_{0\leq s\leq t} \int_{\mathbb{R}^3} \sum_{\substack{\mu+|\alpha|\leq 3 \\ \mu\leq 1}} |L^\mu Z^\alpha F(s,y)| dy. \qquad (5.3)
$$

(2) (Exterior domain case) Let u *be a solution to*

$$
\begin{cases}
(\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}^3\backslash\mathcal{K} \\
u(t,x)|_{x\in\partial\mathcal{K}} = 0 \\
u(t,\cdot) = 0 \ \ \textit{for} \ \ t\leq 0.
\end{cases}
$$

Assume

$$
\mathrm{supp}F \subset \{(t,x); t\geq 1\vee\frac{6}{c_1},\ \frac{c_1 t}{10} \leq |x| \leq 10 c_D t\}.
$$

Then for any $M \geq 0$ *and* $\mu_0 \geq 0$

$$\sup_{|x|\leq c_1 t/2}(1+t)\sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}}|L^\mu Z^\alpha u(t,x)| \leq C\sup_{0\leq s\leq t}\int_{\mathbb{R}^3\setminus\mathcal{K}}\sum_{\substack{|\alpha|+\mu\leq M+7\\ \mu\leq\mu_0+1}}|L^\mu Z^\alpha F(s,y)|dy$$
$$+\sup_{0\leq s\leq t}(1+s)\sum_{\substack{|\alpha|+\mu\leq M+3\\ \mu\leq\mu_0}}\|L^\mu\partial^\alpha F(s,y)\|_{L^2(|y|<4)}. \quad (5.4)$$

We also need the following $L^\infty - L^\infty$ estimates to treat the inhomogeneity away from the light cones, which are special (more elementary) version of Kubota-Yokoyama estimates (see Kubota-Yokoyama [25, Theorem 3.4] for the boundaryless case).

Lemma 5.3 *Let* F, f *and* g *be any functions.*

(1) (Boundaryless case) Let u *be a solution to*

$$\begin{cases} (\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3\\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x). \end{cases}$$

Assume

$$\mathrm{supp}F \subset \{(t,x); 0\leq t\leq 2,\ |x|\leq 2\}\cup\{(t,x); |x|\leq \frac{c_I t}{5}\ \ or\ \ |x|\geq 5c_I t\ \}. \quad (5.5)$$

Then for any $\theta > 0$, *there exists a constant* $C = C(\theta)$ *such that*

$$\sup_{|x|\leq c_I t/2}(1+t)|u(t,x)| \leq C\sum_{\substack{\mu+|\alpha|\leq 3\\ \mu\leq 1, j\leq 1}}\|(\langle x\rangle^j\partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L^2_x}$$
$$+C\sup_{\substack{s\geq 0\\ y\in\mathbb{R}^3}}\langle y\rangle^{2-\theta}(1+s+|y|)^{1+\theta}|F(s,y)|. \quad (5.6)$$

(2) (Exterior domain case) Let u *be a solution to*

$$\begin{cases} (\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3\setminus\mathcal{K}\\ u(t,x)|_{x\in\partial\mathcal{K}} = 0\\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x). \end{cases}$$

Assume (5.5). *Then for any* $\theta > 0$, $M \geq 0$ *and* $\mu_0 \geq 0$, *there exists a constant* $C =$

$C(\theta, M, \mu_0, \mathcal{K})$ *such that*

$$\sup_{|x|\le c_I t/2} (1+t) \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} |L^\mu Z^\alpha u(t,x)| \le C \sum_{\substack{j+\mu+|\alpha|\le M+8\\ \mu\le\mu_0+2,\ j\le 1}} \|(\langle x\rangle^j \partial^j_{t,x} L^\mu Z^\alpha u)(0,x)\|_{L^2_x}$$
$$+ C \sup_{\substack{s>0\\ y\in\mathbb{R}^3\setminus\mathcal{K}}} \langle y\rangle^{2-\theta}(1+s+|y|)^{1+\theta} \sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}} |L^\mu Z^\alpha F(s,y)|$$
$$+ C \sup_{\substack{s>0\\ y\in\mathbb{R}^3\setminus\mathcal{K}}} \langle y\rangle^{2-\theta}(1+s+|y|)^{1+\theta} \sum_{\substack{|\alpha|+\mu\le M+4\\ \mu\le\mu_0}} |L^\mu \partial^\alpha F(s,y)|. \quad (5.7)$$

6 Sobolev-type Estimates

We need the following Sobolev inequalities. The first inequality is due to Klainerman-Sideris [21], Sideris [35], and Hidano-Yokoyama [9]. The second one is the exterior domain analog of the first one.

Lemma 6.1 *Let $c > 0$, $0 \le \theta \le 1/2$ be any constants.*

(1) (Boundaryless case) For any function $u \in C_0^\infty((0,\infty)\times\mathbb{R}^3)$

$$\langle x\rangle^{1/2+\theta}\langle ct-|x|\rangle^{1-\theta}|u'(t,x)| \le C \sum_{\substack{\mu+|\alpha|\le 2\\ \mu\le 1}} \|L^\mu Z^\alpha u'(t,x)\|_{L^2_x} + C \sum_{|\alpha|\le 1} \|\langle t+|x|\rangle Z^\alpha \Box_c u(t,x)\|_{L^2_x}. \quad (6.1)$$

(2) (Exterior domain case) For any function $u \in C_0^\infty((0,\infty)\times\mathbb{R}^3\setminus\mathcal{K})$ with the Dirichlet condition $u|_{\partial\mathcal{K}} = 0$, and any $M \ge 0$, $\mu_0 \ge 0$

$$\langle x\rangle^{1/2+\theta}\langle ct-|x|\rangle^{1-\theta} \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} |L^\mu Z^\alpha u'(t,x)| \le C \sum_{\substack{\mu+|\alpha|\le M+2\\ \mu\le\mu_0+1}} \|L^\mu Z^\alpha u'(t,x)\|_{L^2_x}$$
$$+ C \sum_{\substack{\mu+|\alpha|\le M+1\\ \mu\le\mu_0}} \|\langle t+|x|\rangle L^\mu Z^\alpha \Box_c u(t,x)\|_{L^2_x}$$
$$+ C(1+t) \sum_{\mu\le\mu_0} \|L^\mu u'(t,x)\|_{L^\infty(|x|<2)}. \quad (6.2)$$

Proof of Lemma 6.1 : By (3.14c) in [35], and (4.2) in [25], we have

$$\langle x\rangle^{1/2+\theta}\langle ct-|x|\rangle^{1-\theta}|u'(t,x)| \le C\sum_{|\alpha|\le 2}\|Z^\alpha u'(t,x)\|_{L^2_x} + C\sum_{|\alpha|\le 1}\|\langle ct-|x|\rangle Z^\alpha \partial^2 u(t,x)\|_{L^2_x}$$

for any θ with $0\le\theta\le 1/2$. By (2.10) and (3.1) in [21], we have

$$\|\langle ct-|x|\rangle \partial^2 u(t,x)\|_{L^2_x} \le C\sum_{\substack{\mu+|\alpha|\le 1\\ \mu\le 1}}\|L^\mu Z^\alpha u'(t,x)\|_{L^2_x} + C\|\langle t+|x|\rangle \Box_c u(t,x)\|_{L^2_x}.$$

Combining the above two estimates, we obtain (6.1). The proof of (2) can be found as (4.7) in [29]. □

7 A sketch of the proof of Theorem 1.3

In this section, we show a sketch of the proof of Theorem 1.3. To prove our global existence theorem, we need a standard local existence theorem (See [10, Theorem 6.4.11] for the local existence theorem for the boundaryless case).

Theorem 7.1 *[16, Theorem 9.4] Let $s\ge 7$. Let $(f,g)\in H^s\oplus H^{s-1}$ satisfy the compatibility conditions of order $s-1$. Then* (1.5) *has a local solution $u\in C([0,T];H^s)$, where T depends on s and the norms of f and g. Moreover if $\|f\|_{H^s}+\|g\|_{H^{s-1}}$ is sufficiently small, then there exists C and T independent of f and g so that the solution of* (1.5) *exists for $0\le t\le T$ and satisfies*

$$\sup_{0\le t\le T}\sum_{j=0}^{s}\|\partial_t^j u(t,\cdot)\|_{H^{s-j}} \le C(\|f\|_{H^s}+\|g\|_{H^{s-1}}).$$

Let M_0 be sufficiently large number which is determined later so that the following all argument holds. We assume the smallness of the data (1.12) with $N=2M_0$. By the same argument for (10.2) in [18], we can show that there exists C independent of u such that

$$\sup_{t\ge 0}\sum_{|\alpha|\le N}\|\langle x\rangle^{|\alpha|}\partial^\alpha u(t)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|>5c_Dt)} \le C\varepsilon. \tag{7.1}$$

This inequality and the Klainerman-Sobolev inequality (1.2) yield

$$\sup_{\substack{t\ge 0,\ x\in\mathbb{R}^3\backslash\mathcal{K}\\ |x|\ge 6c_Dt}}\sum_{|\alpha|\le N-2}(1+|x|)^{1+|\alpha|}|\partial^\alpha u(t,x)| \le C'\varepsilon \tag{7.2}$$

for some constant $C' > 0$. Indeed, for x with $|x| > 6c_Dt$, if $|x| - 1 \geq 5c_Dt$, then the result follows from (7.1) and (1.2). If $|x| - 1 \leq 5c_Dt$, then the result follows from the standard embedding $H^2(\mathbb{R}^3 \backslash \mathcal{K}) \hookrightarrow L^\infty(\mathbb{R}^3 \backslash \mathcal{K})$ since such x is in a bounded set.

And we also have

$$\sum_{|\alpha| \leq N-2} \|\langle x \rangle^{-3/4+|\alpha|} \partial^\alpha u\|_{L^2(S_T, |x| \geq 6c_Dt)} \leq C\varepsilon(\log(1+T))^{1/2}. \tag{7.3}$$

Indeed, by (7.2), the square of the LHS is bounded by

$$C\varepsilon \sum_{|\alpha| \leq N-2} \int_0^T \int_{|x| \geq 6c_Dt} \langle x \rangle^{-5/2+|\alpha|} |\partial^\alpha u| dx dt,$$

so that by the Schwarz inequality and (7.1), we obtain (7.3).

Fix a cutoff function $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi(s) = 1$ if $s \leq 1/(12c_D)$ and $\chi(s) = 0$ if $s \geq 1/(6c_D)$, and set

$$u_0(t,x) \equiv \eta(t,x)u(t,x), \quad \eta(t,x) \equiv \chi(t/|x|).$$

Then by (7.1) and (7.2), we have

$$\sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial^\alpha u_0\|_2 + (1+t+|x|) \sum_{|\alpha| \leq N-2} |\langle x \rangle^{|\alpha|} \partial^\alpha u_0| \leq C\varepsilon. \tag{7.4}$$

And, by (7.3), we have

$$\sum_{|\alpha| \leq N-2} \|\langle x \rangle^{-3/4+|\alpha|} \partial^\alpha u_0\|_{L^2_{t,x}} \leq C\varepsilon(\log(1+T))^{1/2}.$$

We put $w \equiv u - u_0$. Then we have

$$\begin{cases} \Box w = (1-\eta)F(\partial u, \partial^2 u) - [\Box, \eta]u \\ w|_{\partial\mathcal{K}} = 0 \\ w(t,x) = 0, \quad t \leq 0 \end{cases} \tag{7.5}$$

for $0 < t < T$. Let v be the solution of

$$\begin{cases} \Box v = -[\Box, \eta]u \\ v|_{\partial\mathcal{K}} = 0 \\ v(t,x) = 0, \quad t \leq 0.. \end{cases} \tag{7.6}$$

Then we have $u = u_0 + v + (w - v)$, and

$$(1+t+|x|) \sum_{\mu+|\alpha|\le N-8} |L^\mu Z^\alpha v(t,x)| + \sum_{\mu+|\alpha|\le N-10} \|L^\mu Z^\alpha v'(t,\cdot)\|_2 \le C\varepsilon. \tag{7.7}$$

Indeed, by (5.2) and the fact $|L^\mu Z^\alpha \partial^\beta \eta| \le C|x|^{-|\beta|}$, the first term is bounded by

$$\int_0^t \int_{6c_D s\le|y|\le 12c_D s} (1+s)^{-3} \sum_{|\alpha|\le N} \langle y\rangle^{|\alpha|} |\partial^\alpha u| dyds,$$

which is bounded by the LHS of (7.1) by the Schwarz inequality. For the second term, we apply (3.3) with $\gamma = 0$. Then we have

$$\sum_{\mu+|\alpha|\le N-10} \partial_t \int e_0(L^\mu Z^\alpha v)dx \le C \sum_{\mu+|\alpha|\le N-9} \|L^\mu \partial^\alpha v'\|^2_{L^2(|x|\le 2)}$$
$$+ \sum_{\mu+|\alpha|\le N-10} \left|\int (\partial_t L^\mu Z^\alpha v)(\Box L^\mu Z^\alpha v)dx\right|. \tag{7.8}$$

The estimate for the first term and (1.1) show that the RHS is bounded by

$$\varepsilon^2\langle t\rangle^{-2} + \frac{\varepsilon}{1+t}\int_{6c_D t\le|y|\le 12c_D t} \sum_{\mu+|\alpha|\le N-10} |L^\mu Z^\alpha([\Box,\eta]u)|dy,$$

which is bounded by

$$\varepsilon^2\langle t\rangle^{-2} + \frac{\varepsilon}{(1+t)^3}\int_{6c_D t\le|y|\le 12c_D t} \sum_{|\alpha|\le N-9} |\langle x\rangle^{|\alpha|}\partial^\alpha u|dy.$$

So that (7.1) shows that

$$\sum_{\mu+|\alpha|\le N-10} \|L^\mu Z^\alpha v'\|_2^2 \le C \sum_{\mu+|\alpha|\le N-10} \int e_0(L^\mu Z^\alpha v)dx \le C\varepsilon^2,$$

which shows the estimate for the second term in (7.7) holds.

And we also have

$$\sum_{\mu+|\alpha|\le N-2} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha v'\|_{L^2(S_T)} \le C(\log(1+T))^{1/2}. \tag{7.9}$$

Indeed, by (4.7), and (1.1), the LHS is bounded by

$$C(\log(1+T))^{1/2}\int_0^T \sum_{\mu+|\alpha|\le N-1} \|L^\mu Z^\alpha[\Box,\eta]u(t,\cdot)\|_2 dt.$$

By the homogenuity of η, we have

$$\begin{aligned}\sum_{\mu+|\alpha|\le N-1}\|L^\mu Z^\alpha[\Box,\eta]u(t,\cdot)\|_2 &\le C\langle t\rangle^{-1}\sum_{\mu+|\alpha|\le N-1}\|L^\mu Z^\alpha u'(t,\cdot)\|_{L^2(6c_Dt\le|x|\le 12c_Dt)}\\ &\quad +C\langle t\rangle^{-2}\sum_{\mu+|\alpha|\le N-1}\|L^\mu Z^\alpha u(t,\cdot)\|_{L^2(6c_Dt\le|x|\le 12c_Dt)}\\ &\le C\varepsilon\langle t\rangle^{-2},\end{aligned}$$

where we have used (7.1). So that we obtain (7.9).

Especially, we have shown that there exists a constant $C_0>0$ such that

$$\sum_{|\alpha|\le N-10}\Big\{\|\Gamma^\alpha(u_0+v)'\|_2+(\log(2+t))^{-1/2}\|\langle x\rangle^{-1/2}\Gamma^\alpha(u_0+v)'\|_{L^2_{t,x}} + \sup_x(1+t+|x|)|\Gamma^\alpha(u_0+v)|\Big\}\le C_0\varepsilon. \quad (7.10)$$

The function $w-v$ satisfies the equation :

$$\begin{cases}\Box(w-v)=(1-\eta)F(\partial u,\partial^2 u)\\ (w-v)|_{\partial\mathcal{K}}=0\\ (w-v)(t,x)=0, \quad t\le 0.\end{cases} \quad (7.11)$$

Since $w-v$ has vanishing Cauchy data, it would be easy to handle when we apply the series of L^2 and pointwise estimates to $w-v$. We show the global existence of u by the continuity argument. Let us assume

$$(1+t+|x|)\sum_{|\alpha|\le M_0}|Z^\alpha(w-v)'|\le C_0\varepsilon. \quad (7.12)$$

Then we can show that for $0\le\mu_0\le 3$ and any constant $\sigma>0$, there exist positive constants A_{μ_0} and D_{μ_0} such that the following estimates hold :

$$\sum_{\substack{\mu+|\alpha|\le N-10-8\mu_0\\ \mu\le\mu_0}} \|(\tilde{L}^\mu\partial_t^j u)'(t,\,\cdot\,)\|_2 + \sum_{\substack{\mu+|\alpha|\le N-10-8\mu_0\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_2$$
$$+\,\varepsilon^{-1}(\log(2+t))^{-1/2} \sum_{\substack{\mu+|\alpha|\le N-10-8\mu_0-2\\ \mu\le\mu_0}} \|\langle x\rangle^{-1/2}L^\mu\partial^\alpha(w-v)'\|_{L^2(S_t)}$$
$$+ \sum_{\substack{\mu+|\alpha|\le N-10-8\mu_0-3\\ \mu\le\mu_0}} \|L^\mu Z^\alpha u'(t,\,\cdot\,)\|_2$$
$$+\varepsilon^{-1}(\log(2+t))^{-1/2} \sum_{\substack{\mu+|\alpha|\le N-10-8\mu_0-5\\ \mu\le\mu_0}} \|\langle x\rangle^{-1/2}L^\mu Z^\alpha(w-v)'\|_{L^2(S_t)} \le A_{\mu_0}\varepsilon(1+t)^{D_{\mu_0}(\varepsilon+\sigma)}. \tag{7.13}$$

The above estimates (7.13) lead to the pointwise and Sobolev type estimates of high order such as

$$\varepsilon^{-1}(1+t+|x|) \sum_{\substack{\mu+|\alpha|\le N-10-8\times3-13\\ \mu\le 2}} |L^\mu Z^\alpha(w-v)|$$
$$+ \sum_{1\le I\le D} |x|^{1/2+\theta}\langle c_I t-|x|\rangle^{1-\theta} \sum_{\substack{\mu+|\alpha|\le N-10-8\times3+3\\ \mu\le 2}} |L^\mu Z^\alpha u'| \le C\varepsilon(1+t)^{2D_3(\varepsilon+\sigma)} \tag{7.14}$$

for any $0\le\theta\le 1/2$. Using (7.14), we can show

$$\sum_{\substack{\mu+|\alpha|\le M_0+9\\ \mu\le 1}} \|L^\mu Z^\alpha(w-v)'\| + (1+t+|x|)\sum_{|\alpha|\le M_0} |Z^\alpha(w-v)'| \le C\varepsilon^{3/2} \tag{7.15}$$

for some constants $C>0$. The last estimate shows that if we take ε sufficiently small, then we can replace C_0 in (7.12) with $C_0/2$, which means the boundedness of pointwise estimate and moreover the energy of u such as

$$(1+t+|x|)\sum_{|\alpha|\le M_0} |Z^\alpha u'| + \sum_{\substack{\mu+|\alpha|\le M_0+9\\ \mu\le 1}} \|L^\mu Z^\alpha u'\| \le 2C_0\varepsilon.$$

Therefore we can conclude that the local solution is a global solution.

We give a sketch of the proof of the above estimates in the following. The new term which appears in the exterior domain case compared with the boundaryless case is the first term in the LHS of (7.13). By cutting L near the obstacle, we can avoid the derivative loss which comes from the boundary of the obstacle. We show (7.13) by an induction. We show for $\mu_0 \geq 0$ and $0 \leq M \leq N - 10 - 8\mu_0$

$$
\begin{aligned}
&\sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}} \|(\tilde{L}^\mu \partial_t^j u)'(t,\cdot)\|_2 + \sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha u'(t,\cdot)\|_2 \\
&\qquad + \varepsilon^{-1}(\log(2+t))^{-1/2} \sum_{\substack{\mu+|\alpha|\leq M-2\\ \mu\leq\mu_0}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha (w-v)'\|_{L^2(S_t)} \\
&\qquad\qquad + \sum_{\substack{\mu+|\alpha|\leq M-3\\ \mu\leq\mu_0}} \|L^\mu Z^\alpha u'(t,\cdot)\|_2 \\
&+ \varepsilon^{-1}(\log(2+t))^{-1/2} \sum_{\substack{\mu+|\alpha|\leq M-5\\ \mu\leq\mu_0}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha (w-v)'\|_{L^2(S_t)} \leq A_{M,\mu_0}\varepsilon(1+t)^{D_{M,\mu_0}(\varepsilon+\sigma)}
\end{aligned}
\tag{7.16}
$$

asuuming the estimates holds when M and μ_0 are replaced by $M-1$ or $\mu_0 - 1$, where A_{M,μ_0} and D_{M,μ_0} are positive constants. Let us focus on the first term in the LHS of (7.16). Let γ be set by

$$
\gamma^{IK,kl}(t,x) \equiv \sum_{1\leq J\leq D} \sum_{0\leq j\leq 3} B_{jkl}^{IJK} \partial_j u^J(t,x). \tag{7.17}
$$

By (3.3), we have

$$
\partial_t \sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}} \left\{\int e_0(\tilde{L}^\mu \partial_t^j u)dx\right\}^{1/2} \leq C \sum_{\substack{\mu+j\leq M\\ \mu\leq\mu_0}} \|\Box_\gamma \tilde{L}^\mu \partial_t^j u\|_2 + C\|\gamma'\|_\infty \sum_{\substack{\mu+j\leq M\\ \mu\leq\mu_0}} \left\{\int e_0(\tilde{L}^\mu \partial_t^j u)dx\right\}^{1/2}. \tag{7.18}
$$

Using the commuting property (1.1), the first term in the RHS of (7.18) is estimated by

$$
\sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}} \|L^\mu \partial^\alpha \Box_\gamma u\|_2 + \sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\leq M\\ \mu_1+\mu_2\leq\mu_0\\ \mu_2+|\alpha_2|\leq M-1}} \|(L^{\mu_1}\partial^{\alpha_1}\gamma)(L^{\mu_2}\partial^{\alpha_2}\partial^2 u)\|_2 + \sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0-1}} \|L^\mu \partial^\alpha u'\|_{L^2(|x|<2)}, \tag{7.19}
$$

where the last term is the additional term when Γ hits the cut-off function η in $\tilde{L}$. The first two terms in (7.19) are estimated by

$$\sum_{|\alpha|\le M_0}\|\partial^\alpha u'\|_\infty\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'\|_2$$
$$+\sum_{M_0+1\le|\alpha|\le M}\|\langle x\rangle^{-1/2}\partial^\alpha u'\|_2\sum_{\substack{\mu+|\alpha|\le M-M_0+2\\ 1\le\mu\le\mu_0}}\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\|_2$$
$$+\sum_{\substack{\mu+|\alpha|\le(M+1)/2+2\\ 1\le\mu\le\mu_0-1}}\|\langle x\rangle^{-1/2}L^\mu Z^\alpha u'\|_2\sum_{\substack{\mu+|\alpha|\le M\\ 1\le\mu\le\mu_0-1}}\|\langle x\rangle^{-1/2}L^\mu\partial^\alpha u'\|_2, \quad (7.20)$$

where we have used (1.2) for the lower order regularity terms. The first term in (7.20) can be estimated by (3.8) such as

$$\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'\|_2\le C\sum_{\substack{\mu+j\le M\\ \mu\le\mu_0}}\|(\tilde L^\mu\partial_t^j u)'\|_2+\frac{C\varepsilon}{1+t}\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'\|_2$$
$$+\sum_{M_0+1\le|\alpha|\le M}\|\partial^\alpha u'\|_2\sum_{\substack{\mu+|\alpha|\le M-M_0+1\\ 1\le\mu\le\mu_0}}\|L^\mu\partial^\alpha u'\|_2$$
$$+\sum_{\substack{\mu+|\alpha|\le M/2+2\\ 1\le\mu\le\mu_0-1}}\|L^\mu\partial^\alpha u'\|_2\sum_{\substack{\mu+|\alpha|\le M\\ 1\le\mu\le\mu_0-1}}\|L^\mu\partial^\alpha u'\|_2, \quad (7.21)$$

where we have used the standard Sobolev embedding $H^2\hookrightarrow L^\infty$ instead of (1.2). With the second term in the RHS in (7.21) moved to the LHS for sufficiently small ε, we also have the estimate to bound the second term in (7.16) by the first term. Using the above estimates (7.18), (7.19), (7.20) and (7.21), and applying the Gronwall inequality to (7.18), and (3.10) to the last term in (7.19) similarly, we can consequently conclude that the term

$$\sum_{\substack{\mu+j\le M\\ \mu\le\mu_0}}\{\int e_0(\tilde L^\mu\partial_t^j u)dx\}^{1/2},$$

which bounds the first term in (7.16), is deduced from the induction on (7.16).

The most technically important improvement in [29] is the estimate for the first term in (7.15). By (3.3) with $\gamma = 0$, we have that the first term in (7.15) is bounded by

$$C \sum_{1\le I\le D} \sum_{\substack{|\alpha|+\nu\le M_0+9\\ \nu\le 1}} \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \Big|\Big\langle \partial_0 L^\nu Z^\alpha (w-v)^I, \Box L^\nu Z^\alpha (w-v)^I \Big\rangle\Big|\, dy\, ds$$
$$+ C \sum_{\substack{|\alpha|+\nu\le M_0+9\\ \nu\le 1}} \Big| \int_0^t \sum_{a=1}^{3} \int_{\partial\mathcal{K}} \partial_0 L^\nu Z^\alpha (w-v) \partial_a L^\nu Z^\alpha (w-v)\, n_a\, d\sigma\, ds \Big| \quad (7.22)$$

where $n = (n_1, n_2, n_3)$ is the outward normal at a given point on $\partial\mathcal{K}$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the standard Euclidean inner product on $\mathbb{R}^D$. Since $\mathcal{K} \subset \{|x| < 1\}$, we have that the last term is bounded by

$$C \int_0^t \int_{\{x\in\mathbb{R}^3\setminus\mathcal{K}, |x|<1\}} \sum_{\substack{|\alpha|+\nu\le M_0+10\\ \nu\le 1}} |L^\nu \partial^\alpha (w-v)'(s,y)|^2\, dy\, ds.$$

Since we also have that $[\Box, L] = 2\Box$ and $[\Box, Z] = 0$ and that $\Box(w-v) = (1-\eta)\Box u$, we see that (7.22) is controlled by

$$C \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{\substack{|\alpha|+\nu\le M_0+9\\ \nu\le 1}} |L^\nu Z^\alpha \partial (w-v)^I| \sum_{\substack{|\alpha|+\nu\le M_0+9\\ \nu\le 1}} |L^\nu Z^\alpha \Box u^I|\, dy\, ds$$
$$+ C \int_0^t \int_{\{x\in\mathbb{R}^3\setminus\mathcal{K}, |x|<1\}} \sum_{\substack{|\alpha|+\nu\le M_0+10\\ \nu\le 1}} |L^\nu \partial^\alpha (w-v)'(s,y)|^2\, dy\, ds. \quad (7.23)$$

Since we have the bound

$$\sum_{\substack{\mu+|\alpha|\le M_0+9\\ \mu\le 1}} |L^\mu Z^\alpha \Box u^I| \le \langle y\rangle^{-1} \sum_{\substack{|\alpha|+\mu\le M_0+11\\ \mu\le 2}} |L^\mu Z^\alpha u^I| \sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1}} |L^\mu Z^\alpha \partial(u^I)|$$
$$+ \frac{\langle c_I s - |y|\rangle}{\langle s + |y|\rangle} \sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}} |L^\mu Z^\alpha \partial(u^I)| \sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1}} |L^\mu Z^\alpha \partial(u^I)|$$
$$+ \sum_{(J,K)\ne(I,I)} \sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}} |L^\mu Z^\alpha \partial(u^J)| \sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1}} |L^\mu Z^\alpha \partial(u^K)|, \quad (7.24)$$

where we have used that the null condition has the commuting property with Γ (see [37, Lemma 4.1]) and the estimates (1.10) and (1.11), the first term in (7.23) is bounded by

$$C\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}}\Big(\langle y\rangle^{-1}\sum_{\substack{|\alpha|+\nu\le M_0+11\\ \nu\le 2}}|L^\mu Z^\alpha u|+\frac{\langle c_I s-|y|\rangle}{\langle s+|y|\rangle}\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}|L^\mu Z^\alpha\partial(w-v)^I|\Big)$$
$$\Big(\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}|L^\mu Z^\alpha(w-v)'|^2+\sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1}}|L^\mu Z^\alpha u'|^2\Big)$$
$$+\sum_{1\le J,K\le D}\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}|L^\mu Z^\alpha\partial(w-v)^I|\sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1\\ J\ne I}}|L^\mu Z^\alpha\partial(u^J)|\sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1}}|L^\mu Z^\alpha\partial(u^K)|dyds \tag{7.25}$$

Applying (7.14) to the integral of the first term in (7.25), we have it is bounded by

$$C\varepsilon\int_0^t(1+s)^{-1/2+}\Big(\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}\|\langle y\rangle^{-1/2}L^\mu Z^\alpha(w-v)'\|_2^2+\sum_{\substack{|\alpha|+\mu\le M_0+10\\ \mu\le 1}}\|\langle y\rangle^{-1/2}L^\mu Z^\alpha u'\|_2^2\Big)ds, \tag{7.26}$$

which is $O(\varepsilon^3)$ by (7.10) and (7.13). For the second integral in (7.25), we split $\mathbb{R}^3\backslash\mathcal{K}$ into two sets Λ_I^c and Λ_J^c, and apply the second estimate in (7.14) for each cases, then we have the same bounds of (7.26) for it. Here we note that $1+t+|x|\sim\langle c_I t-|x|\rangle$ when $(t,x)\in\Lambda_I^c$. This completes the proof of (7.15) for the the first term. Here we note that this estimate yields

$$|x|^{1/2+\theta}\langle c_I t-|x|\rangle^{1-\theta}\sum_{|\alpha|\le M_0+7}|Z^\alpha u'| \tag{7.27}$$
$$\le C\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}\|L^\mu Z^\alpha u'\|_2+\sum_{|\alpha|\le M_0+8}\|(1+t+|x|)Z^\alpha\Box u\|_2+(1+t)\|u'\|_{L^\infty(|x|<2)}$$
$$\le C\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}\|L^\mu Z^\alpha u'\|_2+C\varepsilon,$$

where we have used (6.2), (7.10) and (7.12).

For the estimate for the second term in (7.15), we use the smooth functions ρ, $\beta\in C^\infty(\mathbb{R})$ which satisfies $\rho(r)=1$ for $c_1t/5\le r\le 5c_D$, and $\rho(r)=0$ for $r\le c_1/10$ or $r\ge 10c_D$, $\beta(r)=1$ for $r\ge 2\vee(12/c_1)$, and $\rho(r)=0$ for $r\le 1\vee(6/c_1)$. And we put

$$\phi(t,x)\equiv\beta(t)\rho(|x|/t).$$

The function ϕ has its support near the light cones. Applying (5.4) and (5.7) to the second term in (7.15), we have

$$\sup_{|x|\le c_1 t/2}(1+t+|x|)\sum_{|\alpha|\le M_0}|Z^\alpha(w-v)'|\le C\sum_{\substack{|\alpha|+\mu\le M_0+8\\ \mu\le 1}}\int|L^\mu Z^\alpha(\phi\Box(w-v))|dy$$
$$+C\sup_{\substack{0\le s\le t\\ y\in\mathbb{R}^3\setminus\mathcal{K}}}|y|^{2-\theta}(1+s+|y|)^{1+\theta}\sum_{|\alpha|\le M_0+5}|L^\mu Z^\alpha((1-\phi)\Box(w-v))|, \quad (7.28)$$

which is bounded by

$$C\sum_{\substack{|\alpha|+\mu\le M_0+9\\ \mu\le 1}}\|L^\mu Z^\alpha u'\|_2+C\varepsilon^2$$

where we have used (7.27). Since we have by (1.2)

$$\sup_{|x|\ge c_1 t/2}(1+t+|x|)\sum_{|\alpha|\le M_0}|Z^\alpha(w-v)'|\le C\sum_{|\alpha|\le M_0+2}\|Z^\alpha(w-v)'\|_2, \quad (7.29)$$

the estimate for the second term in (7.15) follows from that for the first term.

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