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L^p estimates for some integral operators

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Abstract

We will introduce a method to obtain L^p boundedness for some integral operators from L^{p_0}-boundedness, $p > p_0$, and also mention some applications of this method.

1. Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. We will consider spaces of functions defined on $\mathbf{R}^n$. In the classical Calderón-Zygmund theory, Calderón-Zygmund singular integral operators were defined as follows:

$$T(f)(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} K(x-y)f(y)d(y).$$

Here $K(x)$ satisfies that

$$|K(x)| \leq \frac{B}{|x|^n} \qquad |x| > 0, \tag{1}$$

$$|K(-y) - K(x-y)| \leq B\frac{|x|^\gamma}{|y|^{n+\gamma}} \qquad |y| > 2|x|,\ \gamma \in (0,1] \tag{2}$$

T is bounded on $L^2(\mathbf{R}^n)$ if, for example, $\hat{K}(\xi)$ is bounded. And by the well known argument of the Calderón-Zygmund decomposition, we get that T is of weak type (1,1), that is

$$|\{x \in \mathbf{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} |f(x)|dx$$

for any $\lambda > 0$ and any $f \in L^1(\mathbf{R}^n)$.
(The smoothness condition (2) may be replaced by other weaker conditions.)

If T is L^2 bounded, by Marcinkiewicz interpolation theorme we have L^p boundedness of T for any $p \in (1,2)$. And because $K(-x)$ satisfies (1) and (2) if we replace $K(x)$ by $K(-x)$, we have also L^p boundedness for any $p \in (2,\infty)$ by duality. This is the classical argument. (See for example [S]).

Obviously it does not necessarily require L^2 boundedness for T to be of weak type (1,1). In order to get that it suffices that T is of type (p_0, p_0) for some $p_0 \in (1,\infty)$, namely T is bounded on $L^{p_0}(\mathbf{R}^n)$.

In this note we would like to show that the operator T which is defined by the kernel $K(x)$ satisfying the conditions (1) and (2) is L^p bounded for any $p \in (1,\infty)$ if T is of weak type (1,1) without using duality argument.

Let T be a linear or sublinear integral transformation from a class $\mathcal{A}$ of measurable functions on $\mathbf{R}^n$ to the space of all the measurable functions on $\mathbf{R}^n$ with the kernel $K(x,y)$:

$$Tf(x) = \int K(x,y)f(y)dx \quad (x \notin \text{the support of } f),$$

We assume that if f is in the class $\mathcal{A}$, then $f\chi_I$ and $f - f\chi_I = f\chi_{\mathbf{R}^n\setminus I}$ are also in the class $\mathcal{A}$. Here $\chi_I(x)$ is the characteristic function of a cube I. Then $T(f\chi_I)(x)$ and $Tf\chi_{\mathbf{R}^n\setminus I}(x)$ are well defined for any $f \in \mathcal{A}$ and any cube I.

Theorem 1. Let the kernel $K(x,y)$ of an operator T satisfy that if $(x,y) \in \Omega = \{(x,y);\, x \neq y\}$ and $2|x-x'| \leq |x-y|$,

$$|K(x,y) - K(x',y)| \leq B\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \qquad \gamma \in (0,1]. \tag{3}$$

If T is of weak type (p_0,p_0) for some $p_0, 0 < p_0 < \infty$, that is, there exists a postive constant C which satisfies that

$$|\{x \in \mathbf{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda^{p_0}}\int_{\mathbf{R}^n} |f(x)|^{p_0}dx$$

for any $\lambda > 0$ and any $f \in \mathcal{A}$, then T satisfies that for any p, $\max\{1,p_0\} < p < \infty$,

$$\int_{\mathbf{R}^n} |Tf(x)|^p dx \leq C_p \int_{\mathbf{R}^n} |f(x)|^p dx$$

for all $f \in \mathcal{A} \cap L^{p_0}(\mathbf{R}^n)$.

We notice that if T is of type (p_0,p_0), namely,

$$\int_{\mathbf{R}^n} |Tf(x)|^{p_0} dx \leq C \int_{\mathbf{R}^n} |f(x)|^{p_0} dx$$

then T is also of weak type (p_0,p_0).

So if T is L^2 bounded and $K(x,y)$ satisfies the smoothness condition (3), we may claim that T is L^p bounded for any $p \in (2,\infty)$. Then we can get L^p boundedness of T without duality argument and there is no need for $K(x,y)$ to satisfy that the smoothness condtion with respect to y, that is,

$$|K(x,y)-K(x,y')|\leq B\frac{|y-y'|^\gamma}{|x-y|^{n+\gamma}}, \qquad |x-y|\geq 2|y-y'|.$$

2. A version of Calderón-Zygmund decomposition

Let μ be a non-negative Borel measure of $\mathbf{R}^n$, which satifies the doubling condition:

$$\mu(2I)\leq C_0\mu(I)<\infty$$

for any cube I of $\mathbf{R}^n$. Here $2I$ is the cube having the same center as I, but expanded two times. Then, it holds that

$$\mu(\mathbf{R}^n)=\infty \qquad \text{or} \qquad 0.$$

Let $\mathcal{M}f$ be the Hardy-Littlewood maximal function of f.

$$\mathcal{M}f(x)=\sup_{I\ni x}\frac{1}{\mu(I)}\int_I|f(y)|d\mu$$

As is well known, if μ satisfies the doubling condition, the maximal theorem holds:

$$\begin{aligned}\mu(\{x;\,\mathcal{M}f(x)>\lambda\}) &\leq \frac{C_1}{\lambda}\int_{\mathbf{R}^n}|f(x)|d\mu \qquad \forall\lambda>0,\\ \left(\int_{\mathbf{R}^n}(\mathcal{M}f(x))^p\mu\right)^{\frac{1}{p}} &\leq C_p\left(\int_{\mathbf{R}^n}|f(x)|^pd\mu\right)^{\frac{1}{p}} \qquad (1<p\leq\infty).\end{aligned}$$

Lebesgue's differentiation theorem shows that

$$|f(x)|\leq\mathcal{M}f(x) \qquad \text{for a.e.}\mu\ x.$$

That is, $\mathcal{M}f$ is a majorant of $|f|$. Then we have immediately

$$\int|f(x)|^pd\mu\leq\int(\mathcal{M}f(x))^pd\mu$$

for every p, $0<p<\infty$.

We can take the same majorant in the following case:

Lemma 2. Let families $\mathcal{F}_k = \{I_k^{(j)}\}_j$ $(k = 1, 2, ...)$ of subucues of a cube Q satisfy that there exists a sufficient large numuber A depending only on μ and p, $0 < p < \infty$ such that

$$\begin{aligned} I_k^{(i)} \cap I_k^{(j)} &= \phi \qquad i \neq j,\ I_k^{(i)}, I_k^{(j)} \in \mathcal{F}_k && (4) \\ \forall I_k^{(j)} \in \mathcal{F}_k,\ \exists I_{k-1}^{(i)} \in \mathcal{F}_{k-1};\quad I_k^{(j)} &\subset I_{k-1}^{(i)} && (5) \\ \sum_{I_{k+1}^{(j)} \subset I_k^{(\nu)}} \mu(I_{k+1}^{(j)}) &\leq \frac{1}{A}\mu(I_k^{(\nu)}). && (6) \end{aligned}$$

Then $\exists C = C(A, p, \mu)$ such that

$$\int_Q \left(\sum_{k=1}^{\infty} \sum_j |a_k^{(j)}| \chi_{I_k^{(j)}}(x) \right)^p d\mu \leq C \int_Q \left(\sup_k \sum_j |a_k^{(j)}| \chi_{I_k^{(j)}}(x) \right)^p d\mu.$$

for any sequence $\{a_k^{(j)}\}_{I_k^{(j)}}$.

This lemma claims that if

$$|F(x)| \leq \sum_{k=1}^{\infty} \sum_j |a_k^{(j)}| \chi_{I_k^{(j)}}(x)$$

then $\sup_k \sum_j |a_k^{(j)}| \chi_{I_k^{(j)}}(x)$ plays the same role as a pointwise majornat of $F(x)$ though the supports of $\{a_k^{(j)}\}_{k,j}$ are overlapping, because of the packing condition (6). Therefore

$$\int_Q |F(x)|^p d\mu \leq C \int_Q \left(\sup_k \sum_j |a_k^{(j)}| \chi_{I_k^{(j)}}(x) \right)^p d\mu$$

In particular, if

$$|a_k^{(j)}| \leq C \left(\sup_{J \supset I_k^{(j)}} \frac{1}{\mu(J)} \int_J |f(x)|^{p_0} d\mu \right)^{\frac{1}{p_0}}, \qquad p_0 > 0$$

then for every $p \in (0, \infty)$

$$\int_Q |F(x)|^p d\mu \leq C \int_Q (\mathcal{M}(|f|^{p_0})(x))^{\frac{p}{p_0}} d\mu.$$

Let T be a linear or sublinear integral transformation from a class $\mathcal{A}$ of (Borel) measurable functions on $\mathbf{R}^n$ to the space of all the measurable functions on $\mathbf{R}^n$ with the kernel

$K(x,y)$, namely

$$Tf(x) = \int K(x,y)f(y)d\mu(y) \quad (x \notin \text{the support of } f),$$

or

$$Tf(x) = \sup_{I \ni x} \int |K_I(x,y)f(y)|d\mu(y).$$

We can obtain a version of Calderón-Zygmund decomposition for Tf.

Lemma 3. Let T be of weak type (p_0,p_0), $0 < p_0 < \infty$, $f(x)$ be a function in $\mathcal{A}$ and let $\{B_I\}_I$ be any sequence of numbers corresponding to all subcubes I of a cube Q. For $A > 1$ there exist a constant $m_{Tf}(Q)$, a function $g(x)$ and families $\mathcal{F}_k = \{I_k^{(j)}\}_j$ $(k = 1,2,\dots)$ of subcubes of Q such that $\mathcal{F}_k = \{I_k^{(j)}\}_j$ which satisfy (2), (3) and (4) in Lemma 2 and

$$\begin{aligned}
Tf(x) &= g(x) + m_{Tf}(Q) + \sum_{k=1}^{\infty}\sum_j a_k^{(j)}\chi_{I_k^{(j)}}(x) \\
|m_{Tf}(Q)| &\le C\left(\frac{1}{\mu(Q)}\int_{\mathbf{R}^n}|f(x)|^{p_0}d\mu\right)^{\frac{1}{p_0}}, \\
|g(x)| &\le C\left((\mathcal{M}(|f|^{p_0})(x))^{\frac{1}{p_0}} + \sup_{I\ni x}\sup_{y\in I}|T(f\chi_{\mathbf{R}^n\setminus 2I})(y) - B_I|\right) \qquad \text{for a.e.} x \\
|a_k^{(j)}| &\le C\left(\left(\sup_{J\supset I_k^{(j)}}\frac{1}{\mu(J)}\int_J|f(x)|^{p_0}d\mu\right)^{\frac{1}{p_0}} + \sup_{y\in I_k^{(j)}}|T(f\chi_{\mathbf{R}^n\setminus I_k^{(j)}})(y) - B_{I_k^{(j)}}|\right).
\end{aligned}$$

Here C depends only on n, μ and A.

This is our version of Calderón-Zygmund decomposition. However it's not a decomposition of f, but a decomposition of Tf. We notice that there is no need of integrability for Tf.

We may decompose the maximal function $\mathcal{M}f(x)$ as follows:

Lemma 4. Let $\mathcal{M}f(x) < \infty$ for a.e. x in $\mathbf{R}^n$ and let $\{B_I\}_I$ be any sequence of numbers correspending to all subcubes I of a cube Q. For $A > 1$ there exist families $\mathcal{F}_k = \{I_k^{(j)}\}_j$ $(k = 1,2,\dots)$ of subcubes of Q such that $\mathcal{F}_k = \{I_k^{(j)}\}_j$ which satisfy (2), (3) and (4) in Lemma 2 and

$$\begin{aligned}
\mathcal{M}f(x) &= g(x) + m_{\mathcal{M}f}(Q) + \sum_{k=1}^{\infty}\sum_{j} a_k^{(j)} \chi_{I_k^{(j)}}(x) \\
|m_{\mathcal{M}f}(Q)| &\le C\frac{1}{\mu(Q)}\int_{\mathbf{R}^n} |f(x)|d\mu, \\
|g(x)| &\le C\sup_{x\in I}\frac{1}{\mu(I)}\int_I |f(y) - B_I|d\mu \qquad \text{for a.e.}x \\
|a_k^{(j)}| &\le C\sup_{J\supset I_k^{(j)}}\frac{1}{\mu(J)}\int_J |f(x) - B_J|d\mu.
\end{aligned}$$

From this decomposition we can also get the estimate of $\mathcal{M}f(x)$ by the sharp maximal funtion $\mathcal{M}^\sharp f(x)$ where

$$\mathcal{M}^\sharp f(x) = \sup_{x\in I}\frac{1}{\mu(I)}\int_I |f(y) - f_I|d\mu, \qquad f_I = \frac{1}{\mu(I)}\int_I f(x)d\mu.$$

Theorem 5. Let $f(x)$ satisfy that there exist positive numbers A and q such that for any $\lambda > 0$

$$|\{x;\ \mathcal{M}f(x) > \lambda\}| \le A\lambda^{-q}.$$

Then it holds that

$$\int_{\mathbf{R}^n} (\mathcal{M}f(x))^p d\mu \le C_p\int_{\mathbf{R}^n} (\mathcal{M}^\sharp f(x))^p d\mu \qquad (0 < p < \infty).$$

3. Main result

In Lemma 3, if it holds that there exists a number B_I for every cube I and $y \in I$

$$|T(f\chi_{\mathbf{R}^n\setminus 2I})(y) - B_I| \le C\sup_{J\supset I}\left(\frac{1}{\mu(J)}\int_J |f(x)|^{q_0}d\mu\right)^{\frac{1}{q_0}},$$

Then

$$|a_k^{(j)}| \le C\left(\sup_{J\supset I_k^{(j)}}\frac{1}{\mu(J)}\int_J |f(x)|^{r_0}d\mu\right)^{\frac{1}{r_0}}$$

where $r_0 = \max\{p_0, q_0\}$. And also we have

$$|g(x)| \le C\,(\mathcal{M}(|f|^{r_0})(x))^{\frac{1}{r_0}}.$$

Using the maximal theorem we get for $p \in (r_0, \infty)$

$$\int_{\mathbf{R}^n} (\mathcal{M}(|f|^{r_0})(x))^{\frac{p}{r_0}}\, d\mu \le C \int_{\mathbf{R}^n} |f(x)|^p d\mu.$$

And if $f \in L^{p_0}(\mathbf{R}^n)$ then

$$\lim_{|Q|\to\infty} \frac{1}{\mu(Q)} \int_{\mathbf{R}^n} |f(x)|^{p_0} d\mu = 0.$$

Therefore we have:

Theorem 6. If T is of weak type (p_0, p_0) with respect to μ and there exist numbers B_I for all cubes I and $y \in I$ such that

$$|T(f\chi_{\mathbf{R}^n\setminus 2I})(y) - B_I| \le C \sup_{J\supset I} \left(\frac{1}{\mu(J)} \int_J |f(x)|^{q_0} d\mu\right)^{\frac{1}{q_0}}, \tag{7}$$

then it holds that for every p, $r_0 = \max\{p_0, q_0\} < p < \infty$

$$\int_{\mathbf{R}^n} |Tf(x)|^p d\mu \le C \int_{\mathbf{R}^n} |f(x)|^p d\mu$$

for $f \in \mathcal{A} \cap L^{p_0}(\mathbf{R}^n)$. Here C is independent of f.

In case of that $K(x,y)$ satisfies the smoothness condition (3), let

$$B_I = T(f\chi_{\mathbf{R}^n\setminus 2I})(x_0)$$

where x_0 is the center of I, then (7) is satisfied with respect to Lebesgue measure for $q_0 = 1$. Thus we have Theorem 1.

4. A weighted result

If a postive loccally integrable function $u(x)$ satisfies A_∞ condtion: $\exists C > 0$, $\exists \delta > 0$

$$\int_E u(x)d\mu \le C \left(\frac{\mu(E)}{\mu(I)}\right)^\delta \int_I u(x)d\mu$$

for any subset E of a cube I, then we can easily see that if

$$\sum_{I_{k+1}^{(j)} \subset I_k^{(\nu)}} \mu(I_{k+1}^{(j)}) \le \frac{1}{A}\mu(I_k^{(\nu)})$$

then $u(x)$ satisfies that

$$\sum_{I_{k+1}^{(j)} \subset I_k^{(\nu)}} \int_{I_{k+1}^{(j)}} u(x)d\mu \le \frac{1}{A^\delta} \int_{I_k^{(\nu)}} u(x)d\mu.$$

Thus the packing condition (6) of Lemma 2 holds for an A_∞ weight $u(x)$ if the families $\mathcal{F}_k = \{I_k^{(j)}\}_j$ of subcubes of Q satisfy (4), (5) and (6).

Theorem 7. Let $u(x)$ satisfy A_∞ condition. If T is of weak type (p_0, p_0) with respect to μ and $K(x,y)$ satisfies (7) then it holds that for every p, $0 < p < \infty$

$$\int_{\mathbf{R}^n} |Tf(x)|^p u(x)d\mu \le C \int_{\mathbf{R}^n} (\mathcal{M}(|f|^{r_0})(x))^{\frac{p}{r_0}} u(x)d\mu$$

if $f \in \mathcal{A} \cap L^{p_0}(\mathbf{R}^n)$ and $r_0 = \max\{p_0, q_0\}$. Here C is independent of f.

Let $u(x)$ satisfy A_p condition, $1 < p < \infty$:

$$\frac{1}{\mu(I)} \int_I u(x)d\mu \left(\frac{1}{\mu(I)} \int_I u^{1-p'}(x)d\mu\right)^{p-1} \le C, \quad (1-p)(1-p') = 1,$$

and $\sigma(x) = u^{1-p'}(x)$, then

$$\frac{1}{(\mu(I))^p} \int_I u(x)d\mu \le C \left(\int_I \sigma(x)d\mu\right)^{1-p}.$$

Therefore if $u(x)$ satisfies A_p then $u(x)$ is an A_∞ weight and the following argument follows:

$$\begin{aligned}
&\int_E \left(\frac{1}{\mu(I)} \int_I |f(x)|d\mu\right)^p u(x)d\mu \\
\le\ & C \left(\frac{\mu(E)}{\mu(I)}\right)^\delta \int_I u(x)d\mu \left(\frac{1}{\mu(I)} \int_I |f(x)|d\mu\right)^p \\
=\ & C \left(\frac{\mu(E)}{\mu(I)}\right)^\delta \frac{1}{(\mu(I))^p} \int_I u(x)d\mu \left(\int_I \frac{|f(x)|}{\sigma(x)} \sigma(x)d\mu\right)^p \\
\le\ & C \left(\frac{\mu(E)}{\mu(I)}\right)^\delta \int_I \sigma(x)d\mu \left(\frac{1}{\int_I \sigma(x)d\mu} \int_I \frac{|f(x)|}{\sigma(x)} \sigma(x)d\mu\right)^p.
\end{aligned}$$

Set

$$\mathcal{M}_\sigma^* f(x) = \sup_{I \ni x, I:\text{dyadic}} \frac{1}{\int_I \sigma(x)d\mu} \int_I |f(x)|\sigma(x)d\mu,$$

then (roughly speaking)

$$\int_E \left(\frac{1}{\mu(I)} \int_I |f(x)|d\mu\right)^p u(x)d\mu \le C \left(\frac{\mu(E)}{\mu(I)}\right)^\delta \int_I \left(\mathcal{M}_\sigma^* \left(\frac{f}{\sigma}\right)(x)\right)^p \sigma(x)d\mu.$$

Using the fact

$$\begin{aligned}\int \left(\mathcal{M}_\sigma^* \left(\frac{f}{\sigma}\right)(x)\right)^p \sigma(x)d\mu &\leq C\int \left|\left(\frac{f}{\sigma}\right)(x)\right|^p \sigma(x)d\mu \\ &= \int |f(x)|^p u(x)d\mu\end{aligned}$$

we have the following theorem (Coifman and Fefferman):

Theorem 8. Let the kernel $K(x, y)$ of an operator T satisfy the smoothness condition (4) and T be of weak-type (1,1) with respect to Lebesgue measure dx. Then if $u(x)$ satisfy A_p, $1 < p < \infty$,

$$\int_{\mathbf{R}^n} |Tf(x)|^p u(x)dx \leq C\int_{\mathbf{R}^n} |f(x)|^p u(x)dx$$

for all $f \in \mathcal{A} \cap L^1(\mathbf{R}^n)$.

This argument is due to E. T. Sawyer and C. Fefferman. It might seem to be more complicated than other argument of weighted norm inequalities. However we think it may be useful in cases of two-weight inequalities and the ordinary L^p argument. Our main ideas can be seen in [F1] and [F2].

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