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Weighted H^p spaces on a domain and singular integrals

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This note gives a survey of some recent results on weighted Hardy spaces and singular integral operators on a bounded domain.

In Section 1, we shall review the works by Der-Chen Chang, S. Krantz, E. M. Stein, and G. Dafni concerning the Hardy spaces on a bounded domain of $\mathbb{R}^n$ and the estimates of certain singular integral operators in those spaces. In Section 2, we review definitions of weighted Hardy spaces on a domain and some of their properties; here the results will be given in the most general setting, i.e., the results will be given for arbitrary open subsets of $\mathbb{R}^n$, $n \geqq 1$, which is equipped with a Borel measure satisfying certain doubling conditions. In Section 3, we consider the 1 dimensional case with a special weight and give some results on the estimates of singular integral operators in the weighted Hardy spaces on an interval. In the last section, Section 4, we give some remarks on the proofs of the results of Sections 2 and 3.

We use the following notation. A *ball* is the subset of $\mathbb{R}^n$ defined by

$$B(x,t) = \{y \in \mathbb{R}^n \mid |x-y| < t\} \quad (x \in \mathbb{R}^n, \quad 0 < t < \infty).$$

If $B = B(x,t)$ and $0 < a < \infty$, then we write $aB = B(x, at)$. We write $\mathbf{1}_E$ to denote the defining function of a set E.

§1. Review of some previous works

For arbitrary open subset $\Omega \subset \mathbb{R}^n$, we can define the space $H^p(\Omega)$ as follows.

Definition 1([Mi1]). Let Ω be an open subset of $\mathbb{R}^n$ and $0 < p \leqq 1$. Take a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B(0,1)$ and $\int \varphi(x)dx = 1$. We write $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ for $t > 0$. For distributions f on Ω, we define

$$f^+(x) = \sup\{|\langle f, \varphi_t(x-\cdot)\rangle| \mid 0 < t < 4^{-1}d_\Omega(x)\},$$

where $d_\Omega(x) = \operatorname{dis}(x, \Omega^c)$, and define

$$\|f\|_{H^p(\Omega)} = \|f^+\|_{L^p(\Omega)} = \left(\int_\Omega f^+(x)^p dx\right)^{1/p}.$$

We define $H^p(\Omega)$ as the set of all those distributions f on Ω such that $\|f\|_{H^p(\Omega)} < \infty$.

It is known that the equivalence class of the quasinorm $\|\cdot\|_{H^p(\Omega)}$ and the space $H^p(\Omega)$ do not depend on the choice of φ. It is also known that if Ω is a sufficiently nice domain (for example, if Ω is a bounded Lipschitz domain), then $H^p(\Omega)$ coincides with the set of the restrictions to Ω of the elements of the ordinary H^p space on $\mathbb{R}^n$. For details, see [Mi1].

If $\Omega \neq \mathbb{R}^n$, the space $H^p(\Omega)$ is not suitable to consider singular integrals, as the following simple example shows.

Example. Let $\Omega = (0, \pi) \subset \mathbb{R}$ and consider

$$(Tf)(x) = \text{p.v.} \int_0^{\pi} \frac{f(y)}{x - y} dy \qquad (0 < x < \pi).$$

For $0 < \xi < \pi/10$, set

$$f_\xi = \xi^{-1} \mathbf{1}_{(19\xi/20, 21\xi/20)}.$$

Then $\|f_\xi\|_{H^1((0,\pi))} \approx 1$. But, since

$$(Tf_\xi)(x) \approx x^{-1} \quad \text{for} \quad 2\xi < x < \pi,$$

we have

$$\int_{2\xi}^{\pi} |(Tf_\xi)(x)| dx \approx \int_{2\xi}^{\pi} x^{-1} dx \approx \log \xi^{-1},$$

which tends to ∞ as $\xi \to 0$.

Der-Chen Chang, G. Dafni, and E. M. Stein [CDS] considered two kinds of H^p spaces on a bounded domain of $\mathbb{R}^n$ which work well for singular integrals. (Closely related works is also given by Der-Chen Chang, S. Krantz, and E. M. Stein [CKS].) We shall review their results.

Definition 2. Let φ_t be the same as in Definition 1. For distributions f on $\mathbb{R}^n$, we define

$$f^{+,1}(x) = \sup\{|\langle f, \varphi_t(x - \cdot)\rangle| \mid 0 < t < 1\}$$

and

$$\|f\|_{h^p(\mathbb{R}^n)} = \|f^{+,1}\|_{L^p(\mathbb{R}^n)}.$$

We define $h^p(\mathbb{R}^n)$ as the set of all those distributions f on $\mathbb{R}^n$ such that $\|f\|_{h^p(\mathbb{R}^n)} < \infty$. If Ω is a bounded domain in $\mathbb{R}^n$, we define

$$h^p_z(\overline{\Omega}) = \{f \in h^p(\mathbb{R}^n) \mid \operatorname{supp} f \subset \overline{\Omega}\}$$

and define $\|f\|_{h^p_z(\overline{\Omega})} = \|f\|_{h^p(\mathbb{R}^n)}$ for $f \in h^p_z(\overline{\Omega})$.

Remark. The definition of $h^p(\mathbb{R}^n)$ goes back to Goldberg [G]. The space $h^p_z(\overline{\Omega})$ was studied by Jonsson, Sjögren, Wallin, and other people; see [JW].

Definition 3([CDS]). Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary and let $0 < p \leqq 1$. Set

$$C^\infty_d(\overline{\Omega}) = \{\varphi \in C^\infty(\overline{\Omega}) \mid \varphi|_{\partial\Omega} = 0\}.$$

For $x \in \Omega$, we define $\mathcal{T}(x)$ as the set of all those $\varphi \in C_d^\infty(\overline{\Omega})$ for which there exists an extension $\widetilde{\varphi} \in C^\infty(\mathbb{R}^n)$ of φ and a ball B such that $B \ni x$, $R =$ (the radius of B) $\leqq 1$, $\operatorname{supp}\widetilde{\varphi} \subset B$, $|\partial^\alpha \widetilde{\varphi}(x)| \leqq R^{-n-|\alpha|}$ for $|\alpha| \leqq [n/p - n]$. For f in the dual space $(C_d^\infty(\overline{\Omega}))'$, we define

$$m_d(f)(x) = \sup\{|\langle f, \varphi\rangle| \mid \varphi \in \mathcal{T}(x)\}$$

and

$$\|f\|_{h_d^p(\overline{\Omega})} = \|m_d(f)\|_{L^p(\Omega)}.$$

We define $h_d^p(\overline{\Omega})$ as the set of all those $f \in (C_d^\infty(\overline{\Omega}))'$ such that $\|f\|_{h_d^p(\overline{\Omega})} < \infty$.

Consider the following boundary value problems.
Dirichlet boundary value problem for Laplacian:

$$(\mathrm{D}) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $f \in C^\infty(\overline{\Omega})$ is a given function. We write the solution of (D) as $u = G_{\mathrm{D}}(f)$.
Neumann boundary value problem for Laplacian:

$$(\mathrm{N}) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega u(x)dx = 0, \end{cases}$$

where $f \in C^\infty(\overline{\Omega})$ is a given function satisfying $\int_\Omega f(x)dx = 0$. We write the solution of (N) as $u = G_{\mathrm{N}}(f)$.
Der-Chen Chang, Dafni, and Stein proved the following theorem.

Theoprem A([CDS]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary. Then:*

(1) *The mapping* $f \mapsto \dfrac{\partial^2 G_{\mathrm{D}}(f)}{\partial x_i \partial x_j}$ *is a bounded operator of* $h_d^p(\overline{\Omega})$ *to itself;*

(2) *The mapping* $f \mapsto \dfrac{\partial^2 G_{\mathrm{D}}(f)}{\partial x_i \partial x_j}$ *is a bounded operator of* $h_z^p(\overline{\Omega})$ *to itself;*

(3) *The mapping* $f \mapsto \dfrac{\partial^2 G_{\mathrm{N}}(f)}{\partial x_i \partial x_j}$ *is a bounded operator of* $h_z^p(\overline{\Omega})$ *to itself.*

Remark. Recently S. Mayboroda and M. Mitrea extended these results to domains Ω with Lipschitz boundary.

The purpose of this note is to show that the H^p space on Ω which is of the same type as that of Definition 1 but is considered with certain weight also works well for singular integrals. We shall define the H^p spaces for arbitrary domains of arbitrary dimension with general doubling weights but as for singular integrals we shall give only partial results restricting to the case of 1 dimensional interval with special weight.

§2. Weighted H^p spaces on a domain

Let Ω be an open subset of $\mathbb{R}^n$ and let $n \leqq \sigma < \infty$. We define the set $\mathrm{Double}(\Omega, \sigma)$ as the set of all those Borel measures λ on Ω for which there exist $A = A_\lambda$ such that whenever B is a ball with $2B \subset \Omega$ then

$$0 < \lambda(B) \leqq At^\sigma \lambda(t^{-1}B) < \infty$$

for all $t \geqq 1$. We say that λ is a *doubling measure* on Ω if λ belongs to $\mathrm{Double}(\Omega, \sigma)$ for some $\sigma \geqq n$.

Definition 4([Mi2], [Mi3]). Let Ω be an open subset of $\mathbb{R}^n$, let λ be a doubling measure on Ω, and let $0 < p \leqq 1$. For distributions f on Ω, we define f^+ in the same way as in Definition 1 and define

$$\|f\|_{H^p(\Omega,\lambda)} = \|f^+\|_{L^p(\Omega,\lambda)} = \left(\int_\Omega f^+(x)^p d\lambda(x) \right)^{1/p}.$$

We define $H^p(\Omega, \lambda)$ as the set of all those distributions f on Ω such that $\|f\|_{H^p(\Omega,\lambda)} < \infty$.

The space $H^p(\Omega)$ of Definition 1 coincides with $H^p(\Omega, \lambda)$ with λ =(the Lebesgue measure).

We shall give two properties of these spaces.

The first property concerns with the change of variables. Let Ω and $\tilde{\Omega}$ are two open subsets of $\mathbb{R}^n$ and let $\Phi : \Omega \to \tilde{\Omega}$ be a C^∞ diffeomorphism. We make the correspondence between $f \in \mathcal{D}'(\Omega)$ and $\tilde{f} \in \mathcal{D}'(\tilde{\Omega})$, by $f(y) = \tilde{f}(\Phi(y))$ in the sense of distribution or by

$$\langle \tilde{f}(x), \psi(x) \rangle = \langle f(y), \psi(\Phi(y)) |J_\Phi(y)| \rangle$$

for all $\psi \in C_0^\infty(\tilde{\Omega})$, where J_Φ denotes the Jacobian determinant of Φ. We make the correspondence between a Borel measure λ on Ω and a Borel measure λ^* on $\tilde{\Omega}$, by $\lambda^*(E) = \lambda(\Phi^{-1}(E))$ for $E \subset \tilde{\Omega}$ or by

$$\int_{\tilde{\Omega}} g(x) d\lambda^*(x) = \int_\Omega g(\Phi(y)) d\lambda(y)$$

for all Borel functions g on $\tilde{\Omega}$.

Then we have the following Proposition.

Proposition 1. *Suppose Φ satisfies the following estimates:*
(i) $|J_\Phi(y)| \geqq A^{-1} d_{\tilde{\Omega}}(\Phi(y))^n d_\Omega(y)^{-n}$,
(ii) $|\partial_y^\alpha \Phi(y)| \leqq A_\alpha d_{\tilde{\Omega}}(\Phi(y)) d_\Omega(y)^{-|\alpha|}$ *for each* $\alpha \neq 0$.
Then:

(1) $\lambda \in \mathrm{Double}(\Omega, \sigma)$ *if and only if* $\lambda^* \in \mathrm{Double}(\widetilde{\Omega}, \sigma)$*;*
(2) *The mapping* $f \mapsto \widetilde{f}$ *is an isomorphism of* $H^p(\Omega, \lambda)$ *onto* $H^p(\widetilde{\Omega}, \lambda^*)$*.*

Remark. It can be shown that in the case $n = 2$ every conformal mapping Φ satisfies the conditions of this proposition with universal absolute constants A and A_α.

The next property concerns with the multiplication of functions.

Proposition 2. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and let* $0 < p \leqq 1$*. Let* w *be a strictly positive* C^∞ *function on* Ω *satisfying the estimate:*

$$|\partial_y^\alpha w(y)| \leqq A_\alpha w(y) d_\Omega(y)^{-|\alpha|}.$$

Then for each doubling measure λ *on* Ω*,* $w^{-p}\lambda$ *(where* $(w^{-p}\lambda)(E) = \int_E w(y)^{-p} d\lambda(y)$*) is also a doubling measure on* Ω *and the mapping* $f \mapsto wf$ *is an isomorphism of* $H^p(\Omega, \lambda)$ *onto* $H^p(\Omega, w^{-p}\lambda)$*.*

§3. Singular integrals on weighted H^p spaces on a domain; 1 dimensional case

In this section, we shall consider the case where

$$\Omega = (0, \pi) \quad \text{(the open interval of } \mathbb{R})$$

and consider the Borel measures $\lambda_{a,b}$ on $(0, \pi)$ defined by

$$\lambda_{a,b}(E) = \int_E \theta^a (\pi - \theta)^b d\theta \qquad (E \subset (0, \pi)),$$

where a, $b \in \mathbb{R}$. We shall simply write

$$L^p_{a,b} = L^p((0, \pi), \lambda_{a,b}), \quad H^p_{a,b} = H^p((0, \pi), \lambda_{a,b}).$$

We shall show that many singular integral operators work in $H^p_{a,b}$ if a and b are chosen appropriately.

We make first an observation by considering the simplest singular integral

$$(Tf)(\theta) = \text{p.v.} \int_0^\pi \frac{f(\phi)}{\theta - \phi} d\phi \qquad (0 < \theta < \pi).$$

For $0 < \xi < \pi/10$, we set

$$f_\xi = \xi^{-a-1} \mathbf{1}_{(19\xi/20, 21\xi/20)}.$$

Then $\|f_\xi\|_{H^1_{a,b}} \approx 1$. We have

$$(Tf_\xi)(\theta) \approx \xi^{-a}\theta^{-1} \quad \text{and} \quad 2\xi < \theta < \pi.$$

Thus if $a < 0$ and $b > -1$ we have

$$\int_{2\xi}^\pi |(Tf_\xi)(\theta)| d\lambda_{a,b}(\theta) \approx \int_{2\xi}^{\pi/2} \xi^{-a}\theta^{-1+a} d\theta + \int_{\pi/2}^\pi \xi^{-a}\theta^b d\theta \approx 1.$$

With a little more calculations we can check that $\|Tf_\xi\|_{H^1_{a,b}} \approx 1$ if $a < 0$ and $b > -1$. Notice that the last estimate does not hold if $a = 0$ (unweighted case) or if $a > 0$.

This suggests that we may obtain the boundedness of singular integrals in $H^p_{a,b}$ if we choose a and b appropriately. In fact we have the following theorems.

Theorem 1 (Multiplier theorem for cosine series, [Mi4, Theorem 3.1]). *Suppose F is a C^∞ function on $[0,\infty)$ which satisfies the estimate*

$$\left|\frac{d^m F(x)}{dx^m}\right| \leqq A_m(1+x)^{-m}$$

for each $m \in \{0, 1, 2, \cdots\}$. Suppose $0 < p \leqq 1$ and $-1 < a, b < p-1$. Then we have

$$\left\|\sum_{n=0}^{\infty} F(n)a_n \cos n\theta\right\|_{H^p_{a,b}} \leqq c\left\|\sum_{n=0}^{\infty} a_n \cos n\theta\right\|_{H^p_{a,b}}$$

for all $\{a_n\} \in l^2$.

The next theorem is a special case of the transplantation theorem for Jacobi series.

Theorem 2 (Transplantation theorem between cosine and sine series, [Mi5]). *Let $0 < p \leqq 1$ and $a, b \in \mathbb{R}$.*
(1) *Suppose $a > -1-p$, $b > -1-p$, and suppose $A = (a+1)/2p + 1/2$ and $B = (b+1)/2p + 1/2$ are not positive integers. Set $N = \max\{0, [A]\}$ and $M = \max\{0, [B]\}$. Then there exists a finite constant C such that the estimate*

$$\left\|\sum_{n=0}^{\infty} a_n \sin(n+1)\theta\right\|_{H^p_{a,b}} \leqq C\left\|\sum_{n=0}^{\infty} a_n \cos n\theta\right\|_{H^p_{a,b}}$$

holds for all those $\{a_n\} \in l^2$ satisfying

$$a_n = 0 \quad \text{for} \quad 0 \leqq n \leqq N+M-1. \tag{$\star$}$$

(2) *Suppose $a > -1$, $b > -1$, and suppose $A = (a+1)/2p$ and $B = (b+1)/2p$ are not positive integers. Set $N = \max\{0, [A]\}$ and $M = \max\{0, [B]\}$. Then there exists a finite constant C such that the estimate*

$$\left\|\sum_{n=0}^{\infty} a_n \cos n\theta\right\|_{H^p_{a,b}} \leqq C\left\|\sum_{n=0}^{\infty} a_n \sin(n+1)\theta\right\|_{H^p_{a,b}}$$

holds for all those $\{a_n\} \in l^2$ satisfying ($\star$).

Notice that the moment condition ($\star$) is vacuous if $N = M = 0$ which is the case $-1-p < a, b < p-1$ in (1) and $-1 < a, b < 2p-1$ in (2).

Here are some remarks concerning Theorems 1 and 2.

Remark.
(a) Theorem 2 is an extension of the transplantation theorem of Muckenhoupt [Mu], where

the case $1 < p < \infty$ is treated.
(b) The cases where A or B is a positive integer is not covered by Theorem 2 nor by Muckenhoupt's theorem in [Mu]. It may be an interesting problem to extend the theorems to these exceptional cases.
(c) In Theorem 2, the set of all those $f \in L^2_{0,0} \cap H^p_{a,b}$ which can be written as $f = \sum_{n=0}^{\infty} a_n \cos n\theta$ (in the case (1)) or $f = \sum_{n=0}^{\infty} a_n \sin(n+1)\theta$ (in the case (2)) with the moment condition ($\star$) is dense in $H^p_{a,b}$.
(d) It is quite believable that one can extend Theorems 1 and 2 to the case of higher dimensions. For example, one might well obtain results similar to Theorem A by using weighted H^p spaces of Section 2.

§4. On the proofs

The proofs of Propositions 1 and 2 and Theorems 1 and 2 are based on grand maximal function characterization of $H^p(\Omega, \lambda)$.

Definition 5. For $x \in \mathbb{R}^n$, m a positive integer, and for $r > 0$, we define the set $\mathcal{T}_m(x,r)$ as the set of all those $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B(x,r)$ and $|\partial_y^\alpha \varphi(y)| \leqq r^{-n-|\alpha|}$ for $|\alpha| \leqq m$. If Ω is an open subset of $\mathbb{R}^n$ and $x \in \Omega$, then we define $\mathcal{T}_m(x)$ as the union of $\mathcal{T}_m(x,r)$ over $0 < r < 4^{-1} d_\Omega(x)$. If Ω is an open subset of $\mathbb{R}^n$ and f is a distribution on Ω, then we define

$$f_m^*(x) = \sup\{|\langle f, \varphi\rangle| \mid \varphi \in \mathcal{T}_m(x)\}.$$

Proposition 3([Mi3, Theorem 1.1]). *Let Ω be an open subset of $\mathbb{R}^n$, $0 < p \leqq 1$, $n \leqq \sigma < \infty$, and let $\lambda \in$ Double(Ω, σ). Let m be a positive integer such that $m > \sigma/p - n$. Then*

$$\|f\|_{H^p(\Omega,\lambda)} \approx \|f_m^*\|_{L^p(\Omega,\lambda)}.$$

Propositions 1 and 2 can be easily proved by the use of Proposition 3.

From Proposition 3, we can also deduce the atomic decomposition for $H^p(\Omega, \lambda)$, which reads as follows.

We introduce the atoms of $H^p(\Omega, \lambda)$, $0 < p \leqq 1$, as follows.

Definition 6([Mi3]). Let Ω be an open subset of $\mathbb{R}^n$, $0 < p \leqq 1$, and λ be a doubling measure on Ω. Let L be a positive integer and $0 < s < 1/20$.
(1) A function $f \in L^\infty(\Omega)$ is called an $(H^p(\Omega,\lambda), 0)$ *atom* if there exists a $\xi \in \Omega$ such that $\operatorname{supp} f \subset B_\xi = B(\xi, s d_\Omega(\xi))$ and $\|f\|_{L^\infty} \leqq \lambda(B_\xi)^{-1/p}$.
(2) A function $f \in L^\infty(\Omega)$ is called an $(H^p(\Omega,\lambda), L)$ *atom* if there exist an $x \in \Omega$ and an r such that $0 < r \leqq s d_\Omega(x)$, $\operatorname{supp} f \subset B(x,r)$, $\|f\|_{L^\infty} \leqq \lambda(B(x,r))^{-1/p}$, and $\int f(y)P(y)dy = 0$ for all polynomials P of degree $\leqq L-1$.

The atomic decomposition for $H^p(\Omega, \lambda)$ reads as follows.

Proposition 4([Mi3, Theorems 1.2 and 1.3]). *Let* Ω *be an open subset of* $\mathbb{R}^n$, $0 < p \leqq 1$, $n \leqq \sigma < \infty$, $\lambda \in \mathrm{Double}(\Omega, \sigma)$, *and let* $0 < s < 1/20$. *Let* L *be a positive integer such that* $L > \sigma/p - n$. *Then a distribution* f *on* Ω *belongs to* $H^p(\Omega, \lambda)$ *if and only if it can be written as follows:*
(i) $f = \sum_i \lambda_i \varphi_i + \sum_j \mu_j \psi_j$ *with the series converging unconditionally in* $H^p(\Omega, \lambda)$;
(ii) φ_i *are* $(H^p(\Omega, \lambda), 0)$ *atoms and* ψ_j *are* $(H^p(\Omega, \lambda), L)$ *atoms;*
(iii) λ_i *and* μ_j *are nonnegative real numbers and* $\left(\sum_i \lambda_i^p + \sum_j \mu_j^p\right)^{1/p} < \infty$.
Moreover, for $f \in H^p(\Omega, \lambda)$, *we have*

$$\|f\|_{H^p(\Omega,\lambda)} \approx \inf\left\{\left(\sum_i \lambda_i^p + \sum_j \mu_j^p\right)^{1/p}\right\},$$

where the inf is taken over all those λ_i, μ_j, φ_i, *and* ψ_j *satisfying* (i), (ii), *and* (iii).

Notice that under the isomorphisms of Propositions 1 and 2 (i.e., change of variables and multiplication by a fixed function) the $(H^p(\Omega, \lambda), 0)$ atoms are essentially preserved but the moment conditions of $(H^p(\Omega, \lambda), L)$ atoms change their form. In other words, for a given $f \in H^p(\Omega, \lambda)$, we can obtain, by combining Propositions 1, 2, and 3, atomic decomposition of f in many different forms using atoms which satisfy different moment conditions. In the proof of Theorem 2, atoms which are orthogonal to polynomials of $\cos\theta$ are useful. For details, see [Mi5].

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