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L^q estimates for the Stokes equations around a rotating body

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1 Introduction

Consider the motion of a viscous incompressible fluid around a compact rigid body $\mathcal{B}=\mathbb{R}^3\setminus D$ (with smooth boundary ∂D), that is formulated as the exterior problem for the Navier-Stokes equations. The case that the body $\mathcal{B}$ is rotating with a prescribed angular velocity, say $\omega=(0,0,1)^T$, is of particular interest. The problem is then the Navier-Stokes equations in the domain $D(t)=O(t)D$, that depends on the time-variable unless the body $\mathcal{B}$ is axisymmetric, subject to the inhomogeneous nonslip boundary condition, where

$$O(t)=\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is reasonable to reduce the problem to an equivalent one in the exterior domain D by using the coordinate system attached to the body $\mathcal{B}$ and by making a change of unknown functions. The reduced problem is ([1], [4], [9], [14])

$$\begin{aligned} \partial_t u+u\cdot\nabla u &=\Delta u+(\omega\wedge x)\cdot\nabla u-\omega\wedge u-\nabla p, && \text{in } D\times(0,\infty), \\ \nabla\cdot u &=0, && \text{in } D\times[0,\infty), \end{aligned}$$

subject to

$$u|_{\partial D}=\omega\wedge x, \qquad u\to 0 \text{ as } |x|\to\infty, \qquad u|_{t=0}=a,$$

where $u=(u_1,u_2,u_3)^T$ and p are unknown velocity and pressure, respectively, and $\wedge$ stands for the usual exterior product of three-dimensional vectors; so,

$\omega \wedge x = (-x_2, x_1, 0)^T$. The most interesting (and difficult as well) feature is that the drift term $(\omega \wedge x) \cdot \nabla u$ is not subordinate to the viscous term Δu. In fact, the fundamental solution of the linear operator

$$L = -\Delta - (\omega \wedge x) \cdot \nabla + \omega \wedge \tag{1.1}$$

cannot be estimated from above by $C/|x-y|$ unlike the Laplace operator, see Proposition 3.1. Furthermore, the generated semigroup (superscript ^{T} denotes the transpose)

$$e^{-tL} f(x) = O(t)^T [e^{t\Delta} f](O(t)x)$$

on $L^2(\mathbb{R}^3)^3$ is not analytic unlike the heat semigroup $e^{t\Delta}$, although it possesses some smoothing properties (the associated semigroup for the exterior problem enjoys such properties as well [13], [14], [15], [16]).

There are some studies on the problem above within the framework of L^2 space; weak solution [1], local unique solution [14], stationary solution (time-periodic solution of the original problem) [10], [19], local and global strong solution [11]. In particular, Galdi [10] has derived some pointwise estimates such as $|u(x)| \leq C/|x|$ for stationary solutions provided ω is sufficiently small.

In the present paper, toward an analysis of the problem above in general L^q spaces, we are concerned with the fundamental estimate

$$\|\nabla u\|_q + \|p\|_q \leq C\|f\|_{-1,q}$$

for the linearized stationary problem

$$Lu + \nabla p = f, \qquad \nabla \cdot u = 0. \tag{1.2}$$

See Theorem 2.1 (whole space problem) and Theorem 2.2 (exterior problem) in the next section. In section 3, we first consider the whole space problem by real analytic method based on dyadic decomposition, square function and maximal function to derive the estimate above for $1 < q < \infty$. The argument is a development of the previous study [6], in which an L^q estimate of $\{\nabla^2 u, \nabla p\}$ for (1.2) in the whole space $\mathbb{R}^3$ was provided. See also Farwig [5], in which the translation of the body as well as the rotation has been taken into account. The final section is devoted to the analysis of the exterior problem by a localization procedure, which was developed in [2], [17] and [18]. Unlike the whole space problem, there is the restriction $n/(n-1) = 3/2 < q < 3 = n$. For the classical Stokes problem (the case $\omega = 0$) in general space dimensions $n \geq 3$, Theorem 2.2 is due to Borchers and Miyakawa [2], Galdi and Simader [12], Kozono and Sohr [17], [18], where the restriction above is optimal; that is, $q > n/(n-1)$ is necessary for the solvability in the class $\{u,p\} \in \widehat{W}_0^{1,q}(D)^n \times L^q(D)$ for all $f \in \widehat{W}^{-1,q}(D)^n$, and so is $q < n$ for the uniqueness in that class. For the function spaces, see the next section. Indeed the behavior of the fundamental solution of (1.1) is worse than that of the Laplace operator, but Theorem 2.2 tells us that the same result as in the case $\omega = 0$ holds true.

2 Results

To begin with, we introduce notation. Given a domain $\Omega\,(=\mathbb{R}^3, D, \cdots)$, the class $C_0^\infty(\Omega)$ consists of C^∞ functions with compact supports contained in Ω. By $L^q(\Omega)$ we denote the usual Lebesgue space with norm $\|\cdot\|_{q,\Omega}$. For $\Omega = \mathbb{R}^3, D$ and $1 < q < \infty$, we need the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_{q,\mathbb{R}^3}} = \{v \in L^q_{loc}(\mathbb{R}^3); \nabla v \in L^q(\mathbb{R}^3)^3\}/\mathbb{R},$$

$$\widehat{W}_0^{1,q}(D) = \overline{C_0^\infty(D)}^{\|\nabla\cdot\|_{q,D}}$$
$$= \begin{cases} \{v \in L^{3q/(3-q)}(D); \nabla v \in L^q(D)^3, v|_{\partial D} = 0\} & \text{for } 1 < q < 3\,(=n), \\ \{v \in L^q_{loc}(\overline{D}); \nabla v \in L^q(D)^3, v|_{\partial D} = 0\} & \text{for } 3 \le q < \infty, \end{cases} \tag{2.1}$$

and their dual spaces

$$\widehat{W}^{-1,q}(\mathbb{R}^3) = \widehat{W}^{1,q/(q-1)}(\mathbb{R}^3)^*, \qquad \widehat{W}^{-1,q}(D) = \widehat{W}_0^{1,q/(q-1)}(D)^*,$$

with norms $\|\cdot\|_{-1,q,\mathbb{R}^3}$ and $\|\cdot\|_{-1,q,D}$, respectively. The characterization above of the space $\widehat{W}_0^{1,q}(D)$ is due to Galdi and Simader [12] (see also Kozono and Sohr [17]). For a bounded domain Ω, we use the usual Sobolev spaces $W_0^{1,q}(\Omega)$ and $W^{-1,q}(\Omega) = W_0^{1,q/(q-1)}(\Omega)^*$ with norm $\|\cdot\|_{-1,q,\Omega}$. For simplicity, we use the abbreviations $\|\cdot\|_q = \|\cdot\|_{q,D}$ and $\|\cdot\|_{-1,q} = \|\cdot\|_{-1,q,D}$ for the exterior domain D.

Let $B_r(x)$ be the ball centered at x with radius $r > 0$. For sufficiently large $r > 0$, we set $D_r = D \cap B_r$ as well as $B_r = B_r(0)$.

Let us consider the boundary value problem for the linearized equation

$$\begin{cases} -\Delta u - (\omega \wedge x)\cdot\nabla u + \omega \wedge u + \nabla p = f & \text{in } D, \\ \nabla \cdot u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{2.2}$$

Let $1 < q < \infty$. Given $f \in \widehat{W}^{-1,q}(D)^3$, we call $\{u,p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ *weak solution* to (2.2) if

1. $\nabla \cdot u = 0$ in $L^q(D)$;
2. $(\omega \wedge x) \cdot \nabla u - \omega \wedge u \in \widehat{W}^{-1,q}(D)^3$;
3. $\{u,p\}$ satisfies $(2.2)_1$ in the sense of distributions, that is,

$$\langle \nabla u, \nabla \varphi \rangle - \langle (\omega \wedge x) \cdot \nabla u - \omega \wedge u, \varphi \rangle - \langle p, \nabla \cdot \varphi \rangle = \langle f, \varphi \rangle \tag{2.3}$$

holds for all $\varphi \in C_0^\infty(D)^3$, where $\langle \cdot, \cdot \rangle$ stands for various duality pairings; by density, $\{u,p\}$ satisfies (2.3) for all $\varphi \in \widehat{W}_0^{1,q/(q-1)}(D)^3$.

Since we make use of a cut-off technique, we first consider the whole space problem with the inhomogeneous divergence condition

$$-\Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u + \nabla p = f, \quad \nabla \cdot u = g \qquad \text{in } \mathbb{R}^3, \tag{2.4}$$

a weak solution of which is defined in the same way as above.

The results on the existence, uniqueness and L^q estimates of weak solutions to (2.4) and to (2.2) are, respectively, as follows.

Theorem 2.1 *Let $1 < q < \infty$ and suppose that*

$$f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3, \quad g \in L^q(\mathbb{R}^3), \quad (\omega \wedge x) g \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3.$$

Then the problem (2.4) possesses a weak solution $\{u,p\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ subject to the estimate

$$\begin{aligned} &\|\nabla u\|_{q,\mathbb{R}^3} + \|p\|_{q,\mathbb{R}^3} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q,\mathbb{R}^3} \\ &\le C \left(\|f\|_{-1,q,\mathbb{R}^3} + \|g\|_{q,\mathbb{R}^3} + \|(\omega \wedge x) g\|_{-1,q,\mathbb{R}^3} \right), \end{aligned} \tag{2.5}$$

with some $C > 0$. The solution is unique in the class above up to a constant multiple of ω for u.

Theorem 2.2 *Let $3/2 < q < 3$. For every $f \in \widehat{W}^{-1,q}(D)^3$, there exists a unique weak solution $\{u,p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ of the problem (2.2) subject to the estimate*

$$\|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q} \le C \|f\|_{-1,q}, \tag{2.6}$$

with some $C > 0$.

Remark 2.1. In Theorem 2.2, we have the embedding relation $\widehat{W}_0^{1,q}(D) \subset L^{3q/(3-q)}(D)$ by (2.1). In this sense, the condition $u \to 0$ is satisfied at infinity.

3 Whole space problem

This section is devoted to the analysis of the whole space problem (2.4). Theorem 2.1 is implied by the following.

Theorem 3.1 *Let* $1 < q < \infty$ *and* $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$. *Then the equation*

$$Lu \equiv -\Delta u - (\omega \wedge x) \cdot \nabla u + \omega \wedge u = f \quad \text{in } \mathbb{R}^3 \tag{3.1}$$

possesses a weak solution $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ *subject to the estimate*

$$\|\nabla u\|_{q,\mathbb{R}^3} + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q,\mathbb{R}^3} \le C\|f\|_{-1,q,\mathbb{R}^3}, \tag{3.2}$$

with some $C > 0$. *The solution is unique in* $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ *up to a constant multiple of* ω *for* u.

For $f \in \mathcal{S}(\mathbb{R}^3)^3$, the equation (3.1) admits a solution of the form [6]

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x,y) f(y)dy = \int_0^\infty O(t)^T [e^{t\Delta} f](O(t)x)dt \tag{3.3}$$

with the kernel

$$\Gamma(x,y) = \int_0^\infty O(t)^T E^t(O(t)x - y)dt, \tag{3.4}$$

where $e^{t\Delta} = E^t *$ is the heat semigroup and

$$E^t(x) = t^{-3/2} E(x/\sqrt{t}), \qquad E(x) = (4\pi)^{-3/2} e^{-|x|^2/4}.$$

On the Fourier side, the solution (3.3) is written as

$$\begin{aligned} \widehat{u}(\xi) &\equiv (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)dx \\ &= \int_0^\infty O(t)^T e^{-|\xi|^2 t} \widehat{f}(O(t)\xi)dt, \end{aligned} \tag{3.5}$$

where $i = \sqrt{-1}$. As mentioned in section 1, we have the following negative assertion on a pointwise estimate of $\Gamma(x,y)$, which shows that the operator $(\omega \wedge x) \cdot \nabla$ is not subordinate to the Laplacian.

Proposition 3.1 *There is* no *constant* $C > 0$ *such that*

$$|x-y||\Gamma(x,y)| \le C, \quad \forall (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Proof. This was shown in [6], but we give the proof for completeness. We intend to estimate the RHS of

$$|\Gamma(x,y)| > \Gamma_{33}(x,y) = \int_0^\infty E^t(O(t)x - y)dt > 0.$$

We take, for instance, $x_\rho = (\rho, 0, 0)$ and $y_\rho = (0, \rho, 0)$ to get

$$\Gamma_{33}(x_\rho, y_\rho) = \int_0^\infty (4\pi t)^{-3/2} e^{-\rho^2(1-\sin t)/2t} dt \ge \frac{C \log \rho}{\rho},$$

for all $\rho > 1$ with $C > 0$ independent of α. In fact, we have

$$\Gamma_{33}(x_\rho, y_\rho) \ge \sum_{k=0}^{\infty} J_k(\rho) \ge \sum_{k=1}^{[\rho^2]} J_k(\rho)$$

with

$$J_k(\rho) = \int_{-\pi/6}^{\pi/6} \{4\pi(t + \pi/2 + 2k\pi)\}^{-3/2} e^{-\rho^2(1-\cos t)/2(t+\pi/2+2k\pi)} dt,$$

which is estimated from below as

$$\begin{aligned} J_k(\rho) &\ge (12k\pi^2)^{-3/2} \int_{-\pi/6}^{\pi/6} e^{-\rho^2(1-\cos t)/4k\pi} dt \\ &\ge 2(12k\pi^2)^{-3/2} \int_0^{\pi/6} e^{-\rho^2 t^2/8k\pi} dt = \frac{C}{k\rho} \int_0^{\sqrt{\pi}\rho/12\sqrt{2k}} e^{-t^2} dt. \end{aligned}$$

for $k \ge 1$. If in particular $k \le \rho^2$, we then find

$$J_k(\rho) \ge \frac{C}{k\rho} \int_0^{\sqrt{\pi}/12\sqrt{2}} e^{-t^2} dt = \frac{C}{k\rho}.$$

As a consequence,

$$\Gamma_{33}(x_\rho, y_\rho) \ge \frac{C}{\rho} \sum_{k=1}^{[\rho^2]} \frac{1}{k} \ge \frac{C}{\rho} \int_1^{\rho^2} \frac{ds}{s} = \frac{C \log \rho}{\rho},$$

which completes the proof. □

For the proof of Theorem 3.1, an essential step is to show

$$\|\nabla u\|_{q,\mathbb{R}^3} \leq C\|F\|_{q,\mathbb{R}^3}, \tag{3.6}$$

for the force of the form $f = \nabla \cdot F$ with $F \in C_0^\infty(\mathbb{R}^3)^9$ on account of the following density property.

Lemma 3.1 (Kozono and Sohr [17, Corollary 2.3]) *Let Ω be any domain and let $1 < q < \infty$. Then the space $\{\nabla \cdot F; F \in C_0^\infty(\Omega)^9\}$ is dense in $\widehat{W}^{-1,q}(\Omega)^3$.*

Let us thus derive the L^q estimate of the operator T defined by

$$TF(x) = \nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma(x,y) : F(y)dy \tag{3.7}$$

to show (3.6), where

$$(\nabla_y \Gamma(x,y) : F(y))_\ell = \sum_{1\leq\mu,\nu\leq 3} \partial_{y_\nu} \Gamma_{\ell\mu}(x,y) F_{\mu\nu}(y) \qquad (1 \leq \ell \leq 3).$$

As in Proposition 3.1, the kernel of (3.7) does *not* seem to enjoy the pointwise estimate $|\nabla_x \nabla_y \Gamma(x,y)| \leq C/|x-y|^3$; that is, the operator T does *not* seem to be of Calderón-Zygmund type. Nevertheless, the L^2 estimate is quite easy.

Proposition 3.2 *The operator T enjoys*

$$\|TF\|_{2,\mathbb{R}^3} \leq \|F\|_{2,\mathbb{R}^3},$$

for all $F \in C_0^\infty(\mathbb{R}^3)^9$.

Proof. By the solution formula (3.5) we have

$$(\widehat{TF})(\xi) = -\xi \otimes \int_0^\infty O(t)^T e^{-|\xi|^2 t}(O(t)\xi) \cdot \widehat{F}(O(t)\xi)dt.$$

The Planchrel theorem thus leads us to

$$\begin{aligned}
\|TF\|_{2,\mathbb{R}^3}^2 = \|\widehat{TF}\|_{2,\mathbb{R}^3}^2 &\leq \int_{\mathbb{R}^3} |\xi|^4 \left\{\int_0^\infty e^{-|\xi|^2 t}|\widehat{F}(O(t)\xi)|dt\right\}^2 d\xi \\
&\leq \int_{\mathbb{R}^3} |\xi|^2 \int_0^\infty e^{-|\xi|^2 t}|\widehat{F}(O(t)\xi)|^2 dt d\xi \\
&= \int_0^\infty \int_{\mathbb{R}^3} |\xi|^2 e^{-|\xi|^2 t}|\widehat{F}(\xi)|^2 d\xi dt = \|\widehat{F}\|_{2,\mathbb{R}^3}^2 = \|F\|_{2,\mathbb{R}^3}^2,
\end{aligned}$$

which completes the proof. □

We rewrite (3.7) as the form

$$TF = \left(T^{(\ell,m)}F\right)_{1\le \ell,m\le 3} \quad \text{for} \quad F = (F_{\mu\nu})_{1\le\mu,\nu\le 3}$$

with

$$\begin{aligned} T^{(\ell,m)}F(x) &= \partial_m u_\ell(x) \\ &= \sum_{\mu,\nu,k} \int_0^\infty O(t)^T_{\ell\mu} O(t)_{km} (H^t_{k\nu} * F_{\mu\nu})(O(t)x)\frac{dt}{t}, \end{aligned} \tag{3.8}$$

where $H = (H_{k\nu})_{1\le k,\nu\le 3}$ is the Hessian matrix of E, that is,

$$H_{k\nu}(x) = \partial_{x_\nu}\partial_{x_k} E(x), \qquad H^t_{k\nu}(x) = t^{-3/2} H_{k\nu}(x/\sqrt{t}). \tag{3.9}$$

We need also the adjoint operator

$$T^*G = \left(T^{*(\mu,\nu)}G\right)_{1\le\mu,\nu\le 3} \quad \text{for} \quad G = (G_{\ell m})_{1\le \ell,m\le 3}$$

with

$$T^{*(\mu,\nu)}G(y) = \sum_{k,\ell,m} \int_0^\infty O(t)^T_{\ell\mu} O(t)_{km} \int_{\mathbb{R}^3} H^t_{k\nu}(O(t)x - y) G_{\ell m}(x)\,dx \frac{dt}{t}, \tag{3.10}$$

for which the argument will be parallel to that for the operator T.

We now introduce the Littlewood-Paley dyadic decomposition

$$\sum_{j=-\infty}^{\infty} \widehat{\eta}_j(\xi) = 1 \qquad (\xi \in \mathbb{R}^3 \setminus \{0\})$$

with

$$\widehat{\eta}_j(\xi) = \beta(2^{-j}|\xi|) - \beta(2^{-j+1}|\xi|),$$

where $\beta \in C^\infty((0,\infty);[0,1])$ is a fixed function so that $\beta \equiv 1$ on $(0,1]$ and $\beta \equiv 0$ on $[2,\infty)$. Note that

$$\operatorname{supp} \widehat{\eta}_j \subset \{\xi;\ 2^{j-1} \le |\xi| \le 2^{j+1}\}. \tag{3.11}$$

By use of η_j, we decompose the function H in (3.9) as

$$H_{k\nu} = \sum_{j=-\infty}^{\infty} H_{k\nu,j}, \qquad H_{k\nu,j} = (2\pi)^{-3/2}\eta_j * H_{k\nu} \quad \left(\widehat{H_{k\nu,j}} = \widehat{\eta_j}\widehat{H_{k\nu}}\right).$$

In (3.8) and (3.10), respectively, we replace H by $H_j = (H_{k\nu,j})_{1\le k,\nu\le 3}$ to define the decomposed operator

$$T_j = \left(T_j^{(\ell,m)}\right)_{1\le \ell,m\le 3}, \qquad T_j^* = \left(T_j^{*(\mu,\nu)}\right)_{1\le \mu,\nu\le 3},$$

with

$$T_j^{(\ell,m)}F(x) = \sum_{\mu,\nu,k}\int_0^\infty O(t)^T_{\ell\mu}O(t)_{km}(H^t_{k\nu,j} * F_{\mu\nu})(O(t)x)\frac{dt}{t}, \tag{3.12}$$

$$T_j^{*(\mu,\nu)}G(y) = \sum_{k,\ell,m}\int_0^\infty O(t)^T_{\ell\mu}O(t)_{km}\int_{\mathbb{R}^3} H^t_{k\nu,j}(O(t)x-y)G_{\ell m}(x)dx\frac{dt}{t}, \tag{3.13}$$

where

$$H^t_{k\nu,j}(x) = t^{-3/2}H_{k\nu,j}(x/\sqrt{t}),$$

namely,

$$\widehat{H^t_{k\nu,j}}(\xi) = \widehat{H_{k\nu,j}}(\sqrt{t}\xi) = \widehat{\eta_j}(\sqrt{t}\xi)\widehat{H_{k\nu}}(\sqrt{t}\xi),$$

so that (3.11) leads to

$$\operatorname{supp}\widehat{H^t_{k\nu,j}} \subset \left\{\xi;\ \tfrac{2^{j-1}}{\sqrt{t}} \le |\xi| \le \tfrac{2^{j+1}}{\sqrt{t}}\right\}. \tag{3.14}$$

In order to estimate $T_j^{(\ell,m)}F$ and $T_j^{*(\mu,\nu)}G$ defined by (3.12) and (3.13), respectively, we make use of the square function (see Stein [22])

$$Sv(x) = \left(\int_0^\infty |(\phi^s * v)(x)|^2\frac{ds}{s}\right)^{1/2},$$

where $\{\phi^s\}_{s>0} \subset \mathcal{S}(\mathbb{R}^3)$ is a fixed family of radially symmetric functions constructed in the following way: we take $\gamma \in C_0^\infty(1/2, 2)$ so that

$$\int_{1/2}^{2} \gamma(\sigma)^2 \frac{d\sigma}{\sigma} = \frac{1}{2},$$

define $\phi(x)$ by $\widehat{\phi}(\xi) = \gamma(|\xi|)$ and set

$$\phi^s(x) = s^{-3/2}\phi(x/\sqrt{s}) \qquad \left(\widehat{\phi^s}(\xi) = \gamma(\sqrt{s}|\xi|)\right)$$

for $s > 0$. Then we have

$$\int_{\mathbb{R}^3} \phi^s(x)dx = 0; \qquad \int_0^\infty \widehat{\phi^s}(\xi)^2 \frac{ds}{s} = 1 \qquad (\xi \in \mathbb{R}^3 \setminus \{0\}), \tag{3.15}$$

and

$$\operatorname{supp} \widehat{\phi^s} \subset \left\{\xi;\ \tfrac{1}{2\sqrt{s}} < |\xi| < \tfrac{2}{\sqrt{s}}\right\}. \tag{3.16}$$

It is known that $\|S \cdot \|_{q,\mathbb{R}^3}$ is equivalent to $\| \cdot \|_{q,\mathbb{R}^3}$ on the space $L^q(\mathbb{R}^3), 1 < q < \infty$ [22, Chapter I, 8.23]. Hence,

$$\|T_j^{(\ell,m)}F\|_{q,\mathbb{R}^3}^2 \leq C\|ST_j^{(\ell,m)}F\|_{q,\mathbb{R}^3}^2 = C\|(ST_j^{(\ell,m)}F)^2\|_{q/2,\mathbb{R}^3}. \tag{3.17}$$

Assume now that $1 < q/2 < \infty$. Then we will estimate

$$\langle (ST_j^{(\ell,m)}F)^2, w\rangle \equiv \int_{\mathbb{R}^3} w(x) \int_0^\infty |(\phi^s * T_j^{(\ell,m)}F)(x)|^2 \frac{ds}{s} dx \tag{3.18}$$

for $w \in L^{q/(q-2)}(\mathbb{R}^3)$. By (3.12) we see

$$\begin{aligned}&(\phi^s * T_j^{(\ell,m)}F)(x)\\ &= \sum_{\mu,\nu,k} \int_{I(s,j)} O(t)^T_{\ell\mu} O(t)_{km} (\phi^s * H^t_{k\nu,j} * F_{\mu\nu})(O(t)x)\frac{dt}{t}\end{aligned}$$

with

$$I(s,j) = [2^{2j-4}s,\ 2^{2j+4}s]$$

because (3.14) and (3.16) imply

$$\widehat{\phi^s}(\xi)\widehat{H^t_{k\nu,j}}(\xi) = 0, \qquad t \notin I(s,j)$$

and because ϕ^s is radially symmetric. We use the Schwarz inequality twice to obtain

$$\begin{aligned}
&|(\phi^s * T_j^{(\ell,m)}F)(x)|^2 \\
&\le C \sum_{\mu,\nu,k} \int_{I(s,j)} \frac{dt}{t} \int_{I(s,j)} \left\{ \int_{\mathbb{R}^3} |H^t_{k\nu,j}(O(t)x-y)||(\phi^s * F_{\mu\nu})(y)|dy \right\}^2 \frac{dt}{t} \\
&\le 8(\log 2)\, C \sum_{\mu,\nu,k} \|H_{k\nu,j}\|_{1,\mathbb{R}^3} \int_{I(s,j)} \left(|H^t_{k\nu,j}| * |\phi^s * F_{\mu\nu}|^2\right)(O(t)x)\frac{dt}{t}.
\end{aligned}$$

Therefore, (3.18) is estimated as

$$\begin{aligned}
&|\langle (ST_j^{(\ell,m)}F)^2, w\rangle| \\
&\le C \sum_{\mu,\nu,k} \|H_{k\nu,j}\|_{1,\mathbb{R}^3} \int_0^\infty \int_{I(s,j)} \int_{\mathbb{R}^3} |w(O(t)^T x)| \left(|H^t_{k\nu,j}| * |\phi^s * F_{\mu\nu}|^2\right)(x)dx \frac{dt}{t}\frac{ds}{s} \\
&= C \sum_{\mu,\nu,k} \|H_{k\nu,j}\|_{1,\mathbb{R}^3} \int_0^\infty \int_{\mathbb{R}^3} |(\phi^s * F_{\mu\nu})(x)|^2 \\
&\qquad \int_{I(s,j)} \left(|\widetilde{H^t_{k\nu,j}}| * |w(O(t)^T \cdot)|\right)(x)\frac{dt}{t}dx\frac{ds}{s},
\end{aligned}$$

where $\widetilde{H^t_{k\nu,j}}$ is the reflection of $H^t_{k\nu,j}$, that is, $\widetilde{H^t_{k\nu,j}}(x) = H^t_{k\nu,j}(-x)$. Set

$$M_j^{(k,\nu)}w(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} \left(|\widetilde{H^t_{k\nu,j}}| * |w(O(t)^T \cdot)|\right)(x)\frac{dt}{t}. \tag{3.19}$$

Then we have

$$|\langle (ST_j^{(\ell,m)}F)^2, w\rangle| \le C \sum_{\mu,\nu,k} \|H_{k\nu,j}\|_{1,\mathbb{R}^3} \int_{\mathbb{R}^3} M_j^{(k,\nu)}w(x) SF_{\mu\nu}(x)^2 dx. \tag{3.20}$$

Similarly,

$$|\langle (ST_j^{*(\mu,\nu)}G)^2, w\rangle| \le C \sum_{k,\ell,m} \|H_{k\nu,j}\|_{1,\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{M}_j^{(k,\nu)} w(y) SG_{\ell m}(y)^2 dy, \quad (3.21)$$

where

$$\mathcal{M}_j^{(k,\nu)} w(y) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} \left(|H_{k\nu,j}^t| * |w|\right)(O(t)y)\frac{dt}{t}. \quad (3.22)$$

To proceed with the estimates, it is necessary to find the behavior of the following for $j \to \pm\infty$: $M_j^{(k,\nu)} w$ and $\mathcal{M}_j^{(k,\nu)} w$ as well as $\|H_{k\nu,j}\|_{1,\mathbb{R}^3}$. For this aim, the following lemma plays an important role (see [6]); that is, we derive a pointwise estimate of $H_{k\nu,j}(x)$, independently of (k,ν), in terms of $\psi(x) = (1+|x|^2)^{-2}$.

Lemma 3.2 *Let ψ be as above. Then there is a constant $C > 0$, independent of $x \in \mathbb{R}^3$, $j \in \mathbb{Z}$ and $1 \le k, \nu \le 3$, such that*

$$|H_{k\nu,j}(x)| \le C 2^{-2|j|} \psi^{2^{-2j}}(x), \quad (3.23)$$

where $\psi^t(x) = t^{-3/2}\psi(x/\sqrt{t})$.

Proof. The proof is the same as in [6], but we give it for completeness. Since $\widehat{H_{k\nu}}(\xi) = -(2\pi)^{-3/2}\xi_\nu\xi_k e^{-|\xi|^2}$, we have

$$|\partial_\xi^\alpha \widehat{H_{k\nu}}(\xi)| \le \begin{cases} C_\alpha |\xi|^{\max\{2-|\alpha|,0\}}, & |\xi| < 1, \\ C_\alpha |\xi|^{-6}, & |\xi| \ge 1, \end{cases} \quad (3.24)$$

for multi-index α. On the other hand, there is a nonnegative function $\zeta \in C_0^\infty(1/4, 4)$ such that, for $|\alpha| \le 4$,

$$|\partial_\xi^\alpha \widehat{\eta_j}(\xi)| \le C_\alpha 2^{-j|\alpha|}\zeta(2^{-j}|\xi|). \quad (3.25)$$

In fact, setting $b(r) = \beta(r) - \beta(2r) \in C_0^\infty(1/4, 4)$ and writing $\widehat{\eta_j}(\xi) = b(2^{-j}|\xi|)$, we have only to choose ζ so that $\max\{|(d/dr)^n b(r)|; 0 \le n \le 4\} \le \zeta(r)$, when we take $|\xi| \sim 2^j$ by (3.11) into account. From (3.25) it follows that

$$2^{j|\alpha|}|\partial_\xi^\alpha \widehat{H_{k\nu,j}}(\xi)| = 2^{j|\alpha|}\left|\sum_{0\le\alpha'\le\alpha}\binom{\alpha}{\alpha'}\partial_\xi^{\alpha-\alpha'}\widehat{\eta_j}(\xi)\,\partial_\xi^{\alpha'}\widehat{H_{k\nu}}(\xi)\right|$$
$$\le C_\alpha \sum_{0\le\alpha'\le\alpha} 2^{j|\alpha'|}\zeta(2^{-j}|\xi|)|\partial^{\alpha'}\widehat{H_{k\nu}}(\xi)|,$$

for $|\alpha| \le 4$. Since $|\xi| \ge 2^{j-1} \ge 1/2$ for $j \ge 0$ and $|\xi| \le 2^{j+1} \le 1$ for $j < 0$, we use (3.24) and note $|\xi| \sim 2^j$ again to see

$$|\partial_\xi^\alpha \widehat{H_{k\nu,j}}(\xi)| \le C2^{-j|\alpha|-2|j|}\zeta(2^{-j}|\xi|).$$

As a consequence,

$$|(1-2^{2j}\Delta_\xi)^2\widehat{H_{k\nu,j}}(\xi)| \le C2^{-2|j|}\zeta(2^{-j}|\xi|).$$

This together with

$$\frac{H_{k\nu,j}(x)}{\psi^{2-2j}(x)} = 2^{-3j}(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ix\cdot\xi}(1-2^{2j}\Delta_\xi)^2\,\widehat{H_{k\nu,j}}(\xi)d\xi \tag{3.26}$$

imply that

$$\frac{|H_{k\nu,j}(x)|}{\psi^{2-2j}(x)} \le C2^{-3j-2|j|}\int_{\mathbb{R}^3}\zeta(2^{-j}|\xi|)d\xi = C2^{-2|j|},$$

which completes the proof. □

To investigate $M_j^{(k,\nu)}w$ and $\mathcal{M}_j^{(k,\nu)}w$, see Proposition 3.3 below after the following lemma, we introduce the Hardy-Littlewood maximal function

$$Mg(x) = \sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|g(y)|dy \tag{3.27}$$

and need a variant of its L^p-boundedness.

Lemma 3.3 *Consider (3.27) in one dimension and let* $1 < p \le \infty$. *Then there is a constant* $C = C(p) > 0$ *such that*

$$\|Mg\|_{p,I} \le C\|g\|_{p,I}$$

for all 2π*-periodic function* g *on* $\mathbb{R}$ *with* $g \in L^p(I)$, *where* $I = (0, 2\pi)$.

Proof. We first note that Mg is also 2π-periodic. Since

$$\|Mg\|_{\infty,I} = \|Mg\|_{\infty,\mathbb{R}} \le C\|g\|_{\infty,\mathbb{R}} = C\|g\|_{\infty,I}$$

for all 2π-periodic $g \in L^\infty(I)$, it suffices to show the weak $(1,1)$ estimate; then the Marcinkiewicz interpolation theorem implies the assertion. For 2π-periodic $g \in L^1(I)$ and $\lambda > 0$, we set $E_\lambda = \{\theta \in I;\ Mg(\theta) > \lambda\}$ and

$$A_\theta(r) = \frac{1}{2r}\int_{B_r(\theta)} |g(t)|dt, \qquad r > 0,$$

where $B_r(\theta) = (\theta - r, \theta + r)$. We then find

$$Mg(\theta) \equiv \sup_{r>0} A_\theta(r) = \sup_{0<r<2\pi} A_\theta(r), \qquad \theta \in I.$$

In fact, for $2\pi < r < 4\pi$, 2π-periodicity of g yields

$$\begin{aligned} A_\theta(r) &= \frac{1}{2r}\left(\int_{\theta-r}^{\theta-2\pi} + \int_{\theta-2\pi}^{\theta+2\pi} + \int_{\theta+2\pi}^{\theta+r}\right) \\ &= \frac{1}{2r}\left\{4\pi A_\theta(2\pi) + 2(r-2\pi)A_\theta(r-2\pi)\right\}, \end{aligned}$$

from which together with $A_\theta(2\pi) = A_\theta(\pi)$ it follows that

$$A_\theta(r) \le \sup_{0<\rho<2\pi} A_\theta(\rho).$$

Therefore, $\sup_{0<r<4\pi} A_\theta(r) = \sup_{0<r<2\pi} A_\theta(r)$. The same procedure implies that for all $n \in \mathbb{N}$

$$\sup_{0<r<2^n\pi} A_\theta(r) = \sup_{0<r<2\pi} A_\theta(r).$$

Fix $\lambda > 0$ arbitrarily, and for $\theta \in E_\lambda$ we choose $r \in I$ so that $A_\theta(r) > \lambda$; then, we have

$$\int_{B_r(\theta)} |g(t)|dt > \lambda|B_r(\theta)| = 2\lambda r. \tag{3.28}$$

Since the length of the members of the family $\{B_r(\theta)\}_{\theta\in E_\lambda}$, which is a covering of E_λ, is bounded, the Vitali covering lemma ([21, Chapter I, 1.6]) implies that there is at most countable sub-family $\{B^{(k)}\}_k$, whose members are disjoint each other, such that $\sum_k |B^{(k)}| \geq |E_\lambda|/5$. This combined with (3.28) yields

$$|E_\lambda| \leq 5\sum_k |B^{(k)}| < \frac{5}{\lambda}\sum_k \int_{B^{(k)}} |g(t)|dt \leq \frac{5}{\lambda}\int_{-2\pi}^{4\pi} |g(t)|dt = \frac{15}{\lambda}\|g\|_{1,I}.$$

We thus get

$$\sup_{\lambda>0} \lambda|E_\lambda| \leq 15\|g\|_{1,I},$$

which is the desired weak $(1,1)$ estimate. □

Proposition 3.3 *Let $1 < p < \infty$. Then the sublinear operators defined by (3.19) and (3.22), respectively, enjoys*

$$\|M_j^{(k,\nu)}w\|_p \leq C2^{-2|j|}\|w\|_p, \qquad \|\mathcal{M}_j^{(k,\nu)}w\|_p \leq C2^{-2|j|}\|w\|_p,$$

with some $C = C(p) > 0$ independent of $j \in \mathbb{Z}$ and $1 \leq k, \nu \leq 3$.

Proof. The reflection $\widetilde{H_{k\nu,j}}(x)$ also satisfies (3.23) on account of $\psi(-x) = \psi(x)$. Note that $\psi^{2^{-2j}t}(x) \leq C\psi^{2^{-2j}r}(x)$ for $2^{-4}r \leq t \leq 2^4 r$. Thus, we have

$$\begin{aligned} 0 \leq M_j^{(k,\nu)}w(x) &\leq C2^{-2|j|}\sup_{r>0}\int_{2^{-4}r}^{2^4 r}\int_{\mathbb{R}^3}\psi^{2^{-2j}t}(x-y)|w(O(t)^T y)|dy\frac{dt}{t} \\ &\leq C2^{-2|j|}\sup_{r>0}\int_{\mathbb{R}^3}\psi^{2^{-2j}r}(x-y)\int_{2^{-4}r}^{2^4 r}|w(O(t)^T y)|\frac{dt}{t}dy. \end{aligned}$$

Set

$$Rw(x) = \sup_{r>0}\int_{2^{-4}r}^{2^4 r}|w(O(t)^T x)|\frac{dt}{t}. \tag{3.29}$$

In terms of this together with (3.27), we obtain by [22, Chapter II, 2.1]

$$\begin{aligned} M_j^{(k,\nu)}w(x) &\leq C2^{-2|j|}\sup_{t>0}(\psi^t * Rw)(x) \\ &\leq C2^{-2|j|}(MRw)(x)\int_{\mathbb{R}^3}\psi(y)dy. \end{aligned}$$

The L^p boundedness of the maximal operator M ([21, Chapter I, 1.3]) implies

$$\|M_j^{(k,\nu)} w\|_p \le C 2^{-2|j|} \|Rw\|_p,$$

as long as $Rw \in L^p(\mathbb{R}^3)$. It remains to show that the sublinear operator R is bounded in $L^p(\mathbb{R}^3)$. Using the cylindrical coordinate $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$, $x_3 = z$, we set

$$w_{(\rho,z)}(\theta) = w(\rho\cos\theta, \rho\sin\theta, z).$$

Then we have

$$Rw(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} |w_{(\rho,z)}(\theta - t)| \frac{dt}{t} \le 2^9 (M w_{(\rho,z)})(\theta).$$

By Lemma 3.3 we find

$$\|M w_{(\rho,z)}\|_{p,I} \le C \|w_{(\rho,z)}\|_{p,I}, \qquad I = (0, 2\pi).$$

Hence,

$$\begin{aligned} \|Rw\|_p^p &\le C \int_{\mathbb{R}} \int_0^\infty \rho \int_0^{2\pi} (M w_{(\rho,z)})(\theta)^p d\theta d\rho dz \\ &\le C \int_{\mathbb{R}} \int_0^\infty \rho \int_0^{2\pi} w_{(\rho,z)}(\theta)^p d\theta d\rho dz = C\|w\|_p^p, \end{aligned}$$

which implies the estimate for $M_j^{(k,\nu)}$. By use of

$$\mathcal{R}w(x) = \sup_{r>0} \int_{2^{-4}r}^{2^4 r} |w(O(t)x)| \frac{dt}{t}$$

instead of (3.29), the same estimate for $\mathcal{M}_j^{(k,\nu)}$ can be proved similarly. □

Proof of Theorem 3.1. In view of (3.20), we use Proposition 3.3 as well as

$$\|H_{k\nu,j}\|_{1,\mathbb{R}^3} \le C 2^{-2|j|} \int_{\mathbb{R}^3} \psi(x) dx,$$

which is implied by (3.23), to see that

$$\begin{aligned} |\langle (ST_j^{(\ell,m)} F)^2, w\rangle| &\le C \sum_{\mu,\nu,k} \|H_{k\nu,j}\|_{1,\mathbb{R}^3} \|M_j^{(k,\nu)} w\|_{q/(q-2),\mathbb{R}^3} \|SF_{\mu\nu}\|_{q,\mathbb{R}^3}^2 \\ &\le C \left(2^{-2|j|}\right)^2 \|w\|_{q/(q-2),\mathbb{R}^3} \sum_{\mu,\nu} \|F_{\mu\nu}\|_{q,\mathbb{R}^3}^2, \end{aligned}$$

for all $w \in L^{q/(q-2)}(\mathbb{R}^3)$. By duality and by (3.17) we arrive at

$$\|T_j^{(\ell,m)}F\|_{q,\mathbb{R}^3} \le C2^{-2|j|}\|F\|_{q,\mathbb{R}^3}, \tag{3.30}$$

with some $C > 0$ independent of $F \in C_0^\infty(\mathbb{R}^3)^9$, $j \in \mathbb{Z}$ and $1 \le \ell, m \le 3$. Hence, as long as $2 < q < \infty$,

$$T = \left(T^{(\ell,m)}\right)_{1\le \ell,m\le 3} \quad \text{with} \quad T^{(\ell,m)} = \sum_{j=-\infty}^{\infty} T_j^{(\ell,m)}$$

is well-defined as a bounded operator on $L^q(\mathbb{R}^3)^9$. For $1 < q < 2$, we use the adjoint operator T^* given by (3.10). The same argument as above implies that T^* is also a bounded operator on $L^{q/(q-1)}(\mathbb{R}^3)^9$; so, T is L^q-bounded for $1 < q < 2$ as well. We have thus proved (3.6) for $1 < q < \infty$.

Let $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$. By [17, Lemma 2.2] there is $F \in L^q(\mathbb{R}^3)^9$ such that

$$\nabla \cdot F = f, \qquad \|F\|_{q,\mathbb{R}^3} \le C\|f\|_{-1,q,\mathbb{R}^3}. \tag{3.31}$$

We take $F_k \in C_0^\infty(\mathbb{R}^3)^9$ so that $\|F_k - F\|_{q,\mathbb{R}^3} \to 0$ as $k \to \infty$. Let u_k be the solution (3.3) with $f = \nabla \cdot F_k$. For each k and $m \in \mathbb{N}$, we take a constant vector $b_k^{(m)} \in \mathbb{R}^3$ satisfying

$$\int_{B_m} (u_k(x) + b_k^{(m)})dx = 0$$

so that

$$\|u_k + b_k^{(m)}\|_{q,B_m} \le C_m\|\nabla u_k\|_{q,B_m} \le C_m\|\nabla u_k\|_{q,\mathbb{R}^3} \le C_m\|F_k\|_{q,\mathbb{R}^3}$$

by the Poincaré inequality and by (3.6). Therefore, there exist $u^{(m)} \in W^{1,q}(B_m)^3$ and $V \in L^q(\mathbb{R}^3)^9$ such that

$$\|u_k + b_k^{(m)} - u^{(m)}\|_{q,B_m} \to 0, \qquad \|\nabla u_k - V\|_{q,\mathbb{R}^3} \to 0 \qquad (k \to \infty)$$

with $\nabla u^{(m)}(x) = V(x)$ (a.a. $x \in B_m$). We first set

$$\widetilde{u} = u^{(1)} \quad \text{on } B_1; \quad b_k = b_k^{(1)}.$$

Consider next the case $m = 2$; since $\nabla u^{(2)}(x) = V(x) = \nabla u^{(1)}(x) = \nabla \widetilde{u}(x)$ for a.a. $x \in B_1 \subset B_2$, the difference $u^{(2)}(x) - \widetilde{u}(x) =: a$ is a constant vector and

$$
\begin{aligned}
|B_1|^{1/q}|b_k^{(2)} - b_k - a| &= \|b_k^{(2)} - b_k - a\|_{q,B_1} \\
&\le \|u_k + b_k - \widetilde{u}\|_{q,B_1} + \|u_k + b_k^{(2)} - u^{(2)}\|_{q,B_2} \to 0
\end{aligned}
\tag{3.32}
$$

as $k \to \infty$. One extends $\widetilde{u}$ by

$$
\widetilde{u} = u^{(2)} - a \quad \text{on } B_2.
$$

Then (3.32) implies

$$
\|u_k + b_k - \widetilde{u}\|_{q,B_2} \le \|u_k + b_k^{(2)} - u^{(2)}\|_{q,B_2} + |B_2|^{1/q}|b_k^{(2)} - b_k - a| \to 0
$$

as $k \to \infty$. We repeat this procedure for $m = 3, 4, \cdots$. By induction there is a function $\widetilde{u} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ so that

$$
\|u_k + b_k - \widetilde{u}\|_{q,B_m} + \|\nabla u_k - \nabla\widetilde{u}\|_{q,\mathbb{R}^3} \to 0 \tag{3.33}
$$

as $k \to \infty$ for all $m \in \mathbb{N}$. In terms of (1.1), it follows from (3.33) together with $Lu_k = \nabla \cdot F_k$ that

$$
Lb_k = \omega \wedge b_k = L(u_k + b_k) - \nabla \cdot F_k \to L\widetilde{u} - \nabla \cdot F \qquad \text{in } \mathcal{D}'(\mathbb{R}^3)^3
$$

as $k \to \infty$. But, then, there is a constant vector $b \in \mathbb{R}^3$ such that

$$
\omega \wedge b_k \to \omega \wedge b = Lb
$$

as $k \to \infty$. Cosequently, we get

$$
L(\widetilde{u} - b) = \nabla \cdot F \qquad \text{in } \mathcal{D}'(\mathbb{R}^3)^3
$$

and $u = \widetilde{u} - b$ is the desired solution. By (3.33) we have $\|\nabla u_k - \nabla u\|_{q,\mathbb{R}^3} \to 0$ and, therefore, the estimate (3.6) holds true for the obtained solution u as well. This together with (3.31) implies the estimate (3.2).

It remains to prove the uniqueness. We use the duality method. Let us consider the adjoint equation

$$
L^*v \equiv -\Delta v + (\omega \wedge x) \cdot \nabla v - \omega \wedge v = \nabla \cdot F \tag{3.34}
$$

with $F \in C_0^\infty(\mathbb{R}^3)^9$. This admits the solution

$$
v(x) = \int_0^\infty O(t)[e^{t\Delta}\nabla \cdot F](O(t)^T x)dt,
$$

where one has only to replace $O(t)$ by $O(t)^T$ in the formula (3.3). By the argument for (3.3) we have $v \in \widehat{W}^{1,r}(\mathbb{R}^3)^3$ for all $r \in (1,\infty)$ with $\|\nabla v\|_{r,\mathbb{R}^3} \leq C\|F\|_{r,\mathbb{R}^3}$. We now let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be a weak solution of $Lu = 0$ in $\widehat{W}^{-1,q}(\mathbb{R}^3)^3$. One can take v as a test function to get

$$\langle Lu, v\rangle = 0.$$

Similarly, one takes u as a test function for (3.34) in $\widehat{W}^{-1,q/(q-1)}(\mathbb{R}^3)^3$ to obtain

$$\langle L^*v, u\rangle = \langle \nabla\cdot F, u\rangle.$$

Therefore,

$$\langle \nabla\cdot F, u\rangle = 0.$$

Since $F \in C_0^\infty(\mathbb{R}^3)^9$ is arbitrary, we obtain $u = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ by Lemma 3.1. Namely, u is a constant vector; but, it should be a constant multiple of ω because $\omega \wedge u = 0$. □

To complete the proof of Theorem 2.1, we need

Lemma 3.4 *Let $v \in \mathcal{S}'(\mathbb{R}^3)$ be the solution of*

$$-\Delta v - (\omega \wedge x)\cdot\nabla v = 0 \qquad \text{in } \mathbb{R}^3.$$

Then supp $\widehat{v} \subset \{0\}$.

Proof. This was shown in [6], but we give the proof for completeness. We first see that

$$|\xi|^2\widehat{v} - (\omega\wedge\xi)\cdot\nabla_\xi\widehat{v} = 0 \qquad \text{in } \mathbb{R}^3_\xi.$$

For any $\varphi \in C_0^\infty(\mathbb{R}^3_\xi \setminus \{0\})$, the adjoint equation

$$|\xi|^2\eta + (\omega\wedge\xi)\cdot\nabla_\xi\eta = \varphi \qquad \text{in } \mathbb{R}^3_\xi$$

is solvable; in fact,

$$\eta(\xi) = \int_0^\infty e^{-|\xi|^2 t}\varphi(O(t)^T\xi)dt \in C_0^\infty(\mathbb{R}^3_\xi \setminus \{0\})$$

is a solution. Hence, we have

$$\langle \widehat{v}, \varphi\rangle = \langle \widehat{v}, |\xi|^2\eta + (\omega\wedge\xi)\cdot\nabla_\xi\eta\rangle = \langle |\xi|^2\widehat{v} - (\omega\wedge\xi)\cdot\nabla_\xi\widehat{v}, \eta\rangle = 0,$$

which completes the proof. □

Proof of Theorem 2.1. Since

$$\nabla \cdot [(\omega \wedge x) \cdot \nabla u - \omega \wedge u] = (\omega \wedge x) \cdot \nabla(\nabla \cdot u) = \nabla \cdot [(\omega \wedge x)\nabla \cdot u],$$

we formally obtain from the problem (2.4)

$$p = -\nabla \cdot (-\Delta)^{-1}[f + \nabla g + (\omega \wedge x)g].$$

Since $(-\Delta)^{-1}$ can be justified as a bounded operator from $\widehat{W}^{-1,q}(\mathbb{R}^3)$ to $\widehat{W}^{1,q}(\mathbb{R}^3)$ ([12], [18]), we get

$$\|p\|_{q,\mathbb{R}^3} \leq C\|f + \nabla g + (\omega \wedge x)g\|_{-1,q,\mathbb{R}^3}, \tag{3.35}$$

which implies

$$\|f - \nabla p\|_{-1,q,\mathbb{R}^3} \leq C\left(\|f\|_{-1,q,\mathbb{R}^3} + \|\nabla g + (\omega \wedge x)g\|_{-1,q,\mathbb{R}^3}\right).$$

Theorem 3.1 thus provides a solution $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$. Since the obtained u and p fulfill

$$(\Delta + (\omega \wedge x) \cdot \nabla)(\nabla \cdot u - g) = 0,$$

Lemma 3.4 yields $\nabla \cdot u = g$ in $L^q(\mathbb{R}^3)$. The estimate (3.2) together with (3.35) implies (2.5). This completes the proof of Theorem 2.1. □

4 Exterior problem

In this section we will prove Theorem 2.2 for the exterior problem (2.2). We combine Theorem 2.1 for the whole space problem with the following lemma on the interior one. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and consider the usual Stokes problem with the inhomogeneous divergence condition

$$-\Delta u + \nabla p = f, \quad \nabla \cdot u = g \quad \text{in } \Omega; \quad u|_{\partial\Omega} = 0. \tag{4.1}$$

Lemma 4.1 (Cattabriga [3], Solonnikov [20], Kozono and Sohr [17]) *Let Ω be as above and $1 < q < \infty$. Suppose that*

$$f \in W^{-1,q}(\Omega)^3, \quad g \in L^q(\Omega), \quad \int_\Omega g(x)dx = 0.$$

Then the problem (4.1) possesses a unique (up to an additive constant for p) weak solution $\{u,p\} \in W_0^{1,q}(\Omega)^3 \times L^q(\Omega)$ *subject to the estimate*

$$\|\nabla u\|_{q,\Omega} + \|p - \overline{p}\|_{q,\Omega} \leq C\left(\|f\|_{-1,q,\Omega} + \|g\|_{q,\Omega}\right), \tag{4.2}$$

where $\overline{p} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega p(x)dx$.

To begin with, we employ a localization procedure to derive the following estimate, which will be refined later, see Proposition 4.2.

Lemma 4.2 *Let* $3/2 < q < \infty$. *Given* $f \in \widehat{W}^{-1,q}(D)^3$, *let*

$$\{u,p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$$

be a weak solution to the problem (2.2). Fix $\rho > \rho_0 > 0$ *so large that* $\mathbb{R}^3 \setminus D \subset B_{\rho_0}$, *and take* $\psi \in C_0^\infty(B_\rho;[0,1])$ *such that* $\psi = 1$ *on* B_{ρ_0}. *Then*

$$\begin{aligned} &\|\nabla u\|_q + \|p\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q} \\ &\leq C\left(\|f\|_{-1,q} + \|u\|_{q,D_\rho} + \|p\|_{-1,q,D_\rho} + \left|\int_{D_\rho}\psi(x)p(x)dx\right|\right), \end{aligned} \tag{4.3}$$

with some $C > 0$.

Proof. By use of the cut-off function ψ fixed above, we decompose the solution $\{u,p\}$ as

$$\begin{cases} u = U + V, & U = (1-\psi)u, \quad V = \psi u, \\ p = \sigma + \tau, & \sigma = (1-\psi)p, \quad \tau = \psi p. \end{cases} \tag{4.4}$$

Then $\{U,\sigma\}$ is a weak solution of

$$-\Delta U - (\omega\wedge x)\cdot\nabla U + \omega\wedge U + \nabla\sigma = Z_1, \quad \nabla\cdot U = -u\cdot\nabla\psi \qquad \text{in } \mathbb{R}^3,$$

where

$$Z_1 = (1-\psi)f + 2\nabla\psi\cdot\nabla u + [\Delta\psi + (\omega\wedge x)\cdot\nabla\psi]u - (\nabla\psi)p.$$

Similarly, $\{V,\tau\}$ is a weak solution of

$$-\Delta V + \nabla\tau = Z_2, \quad \nabla\cdot V = u\cdot\nabla\psi \quad \text{in } D_\rho; \quad V|_{\partial D_\rho} = 0,$$

where

$$Z_2 = \psi[f + (\omega \wedge x) \cdot \nabla u - \omega \wedge u] - 2\nabla\psi \cdot \nabla u - (\Delta\psi)u + (\nabla\psi)p.$$

It therefore follows from Theorem 2.1 and Lemma 4.1, respectively, that

$$\begin{aligned} &\|\nabla U\|_{q,\mathbb{R}^3} + \|\sigma\|_{q,\mathbb{R}^3} \\ &\leq C\|Z_1\|_{-1,q,\mathbb{R}^3} + C\|u \cdot \nabla\psi\|_{q,\mathbb{R}^3} + C\|(\omega \wedge x)(u \cdot \nabla\psi)\|_{-1,q,\mathbb{R}^3}, \end{aligned} \tag{4.5}$$

and that

$$\begin{aligned} &\|\nabla V\|_{q,D_\rho} + \|\tau\|_{q,D_\rho} \\ &\leq C\|Z_2\|_{-1,q,D_\rho} + C\|u \cdot \nabla\psi\|_{q,D_\rho} + \frac{1}{|D_\rho|^{1-1/q}} \left| \int_{D_\rho} \tau(x)dx \right|. \end{aligned} \tag{4.6}$$

Let $\phi \in C_0^\infty(\mathbb{R}^3)^3$. We then have

$$\begin{aligned} |\langle(1-\psi)f, \phi\rangle| &\leq \|f\|_{-1,q}\|\nabla[(1-\psi)\phi]\|_{q/(q-1)} \\ &\leq \|f\|_{-1,q}\left(\|\nabla\phi\|_{q/(q-1)} + C\|\phi\|_{q/(q-1),D_\rho}\right) \\ &\leq C\|f\|_{-1,q}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}. \end{aligned}$$

Here, we have used the condition $q > 3/2$, so that $q/(q-1) < 3$, to apply the Sobolev inequality

$$\|\phi\|_{q/(q-1),D_\rho} \leq C\|\phi\|_{r,D_\rho} \leq C\|\phi\|_{r,\mathbb{R}^3} \leq C\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3},$$

where $1/r = (q-1)/q - 1/3$. Similarly, we obtain

$$\begin{aligned} &|\langle 2\nabla\psi \cdot \nabla u + [\Delta\psi + (\omega \wedge x) \cdot \nabla\psi]u, \phi\rangle| \\ &\leq C\|u\|_{q,D_\rho}\left(\|\nabla\phi\|_{q/(q-1),D_\rho} + \|\phi\|_{q/(q-1),D_\rho}\right) \\ &\leq C\|u\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}, \end{aligned}$$

and

$$\begin{aligned} |\langle(\nabla\psi)p, \phi\rangle| &\leq \|p\|_{-1,q,D_\rho}\|\nabla[(\nabla\psi)\phi]\|_{q/(q-1),D_\rho} \\ &\leq C\|p\|_{-1,q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}, \end{aligned}$$

as well as

$$|\langle(\omega\wedge x)(u\cdot\nabla\psi),\phi\rangle| \le C\|u\|_{q,D_\rho}\|\phi\|_{q/(q-1),D_\rho}$$
$$\le C\|u\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}.$$

In view of (4.5), we collect the estimates above to find

$$\|\nabla U\|_{q,\mathbb{R}^3}+\|\sigma\|_{q,\mathbb{R}^3}\le C\left(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}\right). \tag{4.7}$$

In the same way, we see that

$$|\langle Z_2,\phi\rangle|\le C\left(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}\right)\|\nabla\phi\|_{q/(q-1),D_\rho}$$

for all $\phi\in C_0^\infty(D_\rho)^3$; here, we have used the Poincaré inequality and so the condition $q>3/2$ is not necessary. This combined with (4.6) implies that

$$\begin{aligned}&\|\nabla V\|_{q,D_\rho}+\|\tau\|_{q,D_\rho}\\ &\le C\left(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}+\left|\int_{D_\rho}\psi(x)p(x)dx\right|\right).\end{aligned} \tag{4.8}$$

By (4.5) and (4.6) we obtain

$$\begin{aligned}&\|\nabla u\|_q+\|p\|_q\\ &\le C\left(\|f\|_{-1,q}+\|u\|_{q,D_\rho}+\|p\|_{-1,q,D_\rho}+\left|\int_{D_\rho}\psi(x)p(x)dx\right|\right),\end{aligned}$$

which together with

$$\|(\omega\wedge x)\cdot\nabla u-\omega\wedge u\|_{-1,q}=\|f+\Delta u-\nabla p\|_{-1,q}\le\|f\|_{-1,q}+\|\nabla u\|_q+\|p\|_q$$

yields (4.3). □

We next show the existence and summability of weak solutions to the problem (2.2) and of those to the adjoint one

$$\begin{cases}-\Delta v+(\omega\wedge x)\cdot\nabla v-\omega\wedge v-\nabla\pi=f & \text{in } D,\\ -\nabla\cdot v=0 & \text{in } D,\\ v=0 & \text{on } \partial D.\end{cases} \tag{4.9}$$

for nice force terms f, being in a dense subspace of $\widehat{W}^{-1,q}(D)^3$; see Lemma 3.1.

Lemma 4.3 *Let $F \in C_0^\infty(D)^9$. Then the problem (2.2) with $f = \nabla \cdot F$ has a weak solution $\{u, p\}$ of class*

$$u \in \widehat{W}_0^{1,r}(D)^3, \qquad p \in L^r(D) \qquad \text{for } 3/2 < \forall r < \infty. \tag{4.10}$$

The same assertion for the adjoint problem (4.9) holds true as well.

Proof. We first employ the standard L^2 technique. When $q = 2$, one can take $\varphi = u$ in (2.3) to get

$$\|\nabla u\|_2^2 = \langle \nabla \cdot F, u \rangle$$

since

$$\int_{D_R} [(\omega \wedge x) \cdot \nabla u] \cdot u dx = \frac{1}{2} \int_{\partial D_R} n \cdot (\omega \wedge x)|u|^2 d\sigma = 0$$

for every sufficiently large $R > 0$, where n is the unit exterior normal vector to the boundary ∂D_R. We thus have the a priori estimate

$$\|\nabla u\|_2 \leq \|F\|_2.$$

Then the Galerkin method (traced back to Fujita [7]) provides a distribution solution

$$u \in \widehat{W}_0^{1,2}(D)^3 \subset L^6(D)^3, \qquad p \in L^2_{loc}(\overline{D}),$$

see Galdi [8], [10]. As in (4.4), we use a cut-off technique to split the solution $\{u, p\}$ into flows $\{U, \sigma\}$ for the whole space problem and $\{V, \tau\}$ for the interior one. Along the same line as in the proof of Lemma 4.2, Theorem 2.1 and Lemma 4.1 lead to

$$u \in \widehat{W}_0^{1,r}(D)^3, \qquad p \in L^r(D) \qquad \text{for } 3/2 < \forall r < 6.$$

By the same argument once more, we obtain (4.10) for the problem (2.2). The problem (4.9) is nothing but (2.2) with $\{p, \omega\}$ replaced by $\{-\pi, -\omega\}$, and so the same assertion holds. □

As a corollary, we have the following uniqueness assertion.

Proposition 4.1 *Let $1 < q < 3$. Suppose that $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ is a weak solution to the problem (2.2) with $f = 0$. Then $\{u, p\} = \{0, 0\}$.*

Proof. Consider the adjoint problem (4.9) with $f = \nabla \cdot F$, where $F \in C_0^\infty(D)^9$. By Lemma 4.3 there is a weak solution $\{v, \pi\}$ of class (4.10). Since $q/(q-1) > 3/2$, one can put $\varphi = v$ in (2.3) with $f = 0$:

$$\langle \nabla u, \nabla v \rangle - \langle (\omega \wedge x) \cdot \nabla u - \omega \wedge u, v \rangle = 0.$$

Similarly, one can take u as a test function for $(4.9)_1$ to get

$$\langle \nabla v, \nabla u \rangle + \langle (\omega \wedge x) \cdot \nabla v - \omega \wedge v, u \rangle = \langle \nabla \cdot F, u \rangle.$$

From two equalities above it follows that $\langle \nabla \cdot F, u \rangle = 0$ for all $F \in C_0^\infty(D)^9$. By Lemma 3.1 we get

$$\langle f, u \rangle = 0,$$

for all $f \in \widehat{W}^{-1,q/(q-1)}(D)^3$, which yields $u = 0$ in $\widehat{W}_0^{1,q}(D)^3$. Going back to (2.3) with $f = 0$, we find $p = 0$ in $L^q(D)$. This completes the proof. □

By Lemma 4.2 together with Proposition 4.1 we have the following a priori estimate.

Proposition 4.2 *Let $3/2 < q < 3$. Given $f \in \widehat{W}^{-1,q}(D)^3$, let $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ be a weak solution to the problem (2.2). Then the estimate (2.6) holds.*

Proof. Suppose the contrary. Then there exist sequences $f_k \in \widehat{W}^{-1,q}(D)^3$ and $\{u_k, p_k\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ so that

$$\|\nabla u_k\|_q + \|p_k\|_q + \|(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k\|_{-1,q} = 1,$$

while

$$\|f_k\|_{-1,q} \to 0$$

as $k \to \infty$. Then we have

$$\|u_k\|_{1,q,D_\rho} \le \|\nabla u_k\|_{q,D_\rho} + C\|u_k\|_{q_*,D_\rho} \le C\|\nabla u_k\|_q \le C$$

where $1/q_* = 1/q - 1/3$, as well as $\|p_k\|_{q,D_\rho} \le \|p_k\|_q \le 1$. By the Rellich compactness theorem, there are subsequences, which we denote by u_k and p_k again, so that they strongly converge in $L^q(D_\rho)$ and $W^{-1,q}(D_\rho)$, respectively. From (4.2) it follows that $\{u_k, p_k\}$ and $\{(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k\}$ are the Cauchy sequences, respectively, in $\widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ and in $\widehat{W}^{-1,q}(D)^3$; hence, there exists $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ so that

$$\begin{cases} \|\nabla u_k - \nabla u\|_q + \|p_k - p\|_q \to 0, \\ \|[(\omega \wedge x) \cdot \nabla u_k - \omega \wedge u_k] - [(\omega \wedge x) \cdot \nabla u - \omega \wedge u]\|_{-1,q} \to 0, \end{cases} \tag{4.11}$$

as $k \to \infty$. It easily turns out that the pair $\{u, p\}$ is a weak solution to (2.2) with $f = 0$. Since $q < 3$, Lemma 4.1 tells us that $\{u, p\} = \{0, 0\}$, which contradicts

$$\|\nabla u\|_q + \|p\|_q + \|(\omega \wedge x) \cdot \nabla u - \omega \wedge u\|_{-1,q} = 1.$$

This completes the proof. □

Proof of Theorem 2.2. The uniqueness part follows from Proposition 4.1. Given $f \in \widehat{W}^{-1,q}(D)^3$, we take $F_k \in C_0^\infty(D)^9$ so that $\|\nabla \cdot F_k - f\|_{-1,q} \to 0$ as $k \to \infty$. By Lemma 4.3 there is a solution $\{u_k, p_k\}$ of class (4.10) to the problem (2.2) with the force $\nabla \cdot F_k$. One can take $r = q$ in (4.10) since $q > 3/2$. By Proposition 4.2 one can use (2.6) to show that there exists $\{u, p\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ so that the same convergence properties as in (4.11) hold. We see that the pair $\{u, p\}$ obtained above is a weak solution to (2.2) with the estimate (2.6). We have completed the proof. □

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