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The existence and the decay estimate of the Green functions of higher order elliptic operators with non-decaying complex-valued coefficients

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Abstract

Consider a uniformly elliptic operator of $2m$-th order :

$$Au \equiv \Sigma_{|\alpha|\leq 2m} a_\alpha \partial^\alpha u$$

in $\mathbf{R}^N (N \geq 2)$. Assume that the top order coefficients a_α ($|\alpha| = 2m$) belong to $W^{m,\infty}(\mathbf{R}^N)$ and real-valued while the lower order coefficients are bounded and may be complex-valued. Then, the resolvent $(A-\lambda)^{-1}$ with an arbitrary $\lambda \in \rho(A)$ can be expressed as an integral operator with a kernel function $R_\lambda(x, \xi)$ which decays exponentially as $|x - \xi| \to \infty$ (Theorem 15). In addition, the eigenfunction corresponding to a discrete spectrum decays exponentially as $|x| \to \infty$ (Theorem 16).

1 Basic Assumptions and Notations

We consider the uniformly elliptic operator of $2m$-th order:

$$Au \equiv \sum_{|\alpha|\leq 2m} a_\alpha \partial^\alpha u$$

with $m = 1, 2, \cdots$ in $L^p(\mathbf{R}^N)$ $(1 < p < \infty, N \geq 2$: arbitrary). Here we use the multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ to denote

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_N \\ \partial^\alpha u &= (\partial/\partial x)^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \end{aligned}$$

as well as we use the notation

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N}$$

later for

$$\xi = (\xi_1, \cdots, \xi_N).$$

We make several hypotheses on its coefficients.

(H1) Smoothness and real-valuedness of top order coefficients.

$$a_\alpha \in W^{m,\infty} = W^{m,\infty}(\mathbf{R}^N) \quad (|\alpha| = 2m)$$

and they are real-valued for all $|\alpha| = 2m$.

(H2) Uniform ellipticity.

$$\sum_{|\alpha|=2m} a_\alpha \xi^\alpha \geq \delta |\xi|^{2m}$$

with some constant $\delta > 0$.

(H3) Boundedness and complex-valuedness of the lower order coefficients.

$$a_\alpha \in L^\infty(\mathbf{R}^N) \quad (|\alpha| \leq 2m - 1)$$

and they may even be complex-valued.

In addition, A_p $(1 < p < \infty)$ denotes the operator in $L^p(\mathbf{R}^N)$ determined by

$$A_p u \equiv Au, \quad u \in \mathrm{Dom}(A_p) = W^{2m,p} = W^{2m,p}(\mathbf{R}^N)$$

We use also the following notation for convenience:

$$f(x) \vee g(x) = \max\{f(x), g(x)\}.$$

2 Bessel Potentials

First let us introduce the Bessel potentials, following Schechter[3] and Stein [4]
Definition Given any real $\alpha > 0$, let

$$\begin{aligned} G_\alpha(x) &= \frac{(4\pi)^{-N/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-s-|x|^2/4s} s^{-(N-\alpha)/2-1} ds \\ &= \frac{(4\pi)^{-N/2}}{\Gamma(\alpha/2)} |x|^{-N+\alpha} \int_0^\infty e^{-|x|^2 s - 1/4s} s^{-(N-\alpha)/2-1} ds \\ &= \frac{2^{-(N+\alpha-2)/2}\pi^{-N/2}}{\Gamma(\alpha/2)} |x|^{-(N-\alpha)/2} K_{(N-\alpha)/2}(|x|) \end{aligned}$$

for $x \in \mathbf{R}^N$. Here $K_\nu(r)$ with arbitrary real parameter ν is the modified Bessel function of the second kind (sometimes called MacDonald's function). Note that $K_\nu(r)$ is a positive and strictly decreasing function on $(0, \infty)$. This fact directly follows from some of its integral representation. (see (5) or (7) of §6.22 of Watson[5].)

Lemma 1 *Let $N \geq 2$ be the dimension of the space $\mathbf{R}^N$.*

$$G_\alpha * G_\beta = G_{\alpha+\beta} \qquad (\alpha, \beta > 0 : \textit{real}).$$

Moreover, $G_{2j}(x - \xi)$ is the intregral kernel which represents a homeomorphic map $(-\Delta + 1)^{-j}$ from $L^p(\mathbf{R}^N)$ into $W^{2j,p}(\mathbf{R}^N)$ for every $1 < p < \infty$.

Proof See Schechter [3, Lemma 6.2] or Stein [4,p132] for the proof of this and the other properties of Bessel potentials.

Various estimates of the Bessel potentials follow from those of the modified Bessel functions of the second kind which we collect below.

Lemma 2 *Let ν be an arbitrary real and n be an arbitrary positive integer.*

$$K_\nu(x) \sim \{\sqrt{\pi/2}\}x^{-1/2}e^{-x}$$

holds as $x \to \infty$. Similarly

$$\begin{aligned} K_0(x) &\sim -\log x \\ K_n(x) = K_{-n}(x) &\sim 2^{n-1}(n-1)!x^{-n} \\ K_{n-1/2}(x) = K_{-n+1/2}(x) &\sim \{2^{-n-1/2}\pi^{1/2}(2n+2)!/(n+1)!\}x^{-n+1/2} \end{aligned}$$

as $x \to 0$.

Now we state the estimates of $G_j(x)$ ($j = 1, 2, \cdots$) which will be used in this paper.

Lemma 3

$$G_j(x) \leq \begin{cases} C(|x|^{j-N} \vee |x|^{(j-N-1)/2})e^{-|x|} & (1 \leq j \leq N-1) \\ C\{(-\log|x|) \vee |(|x|+1)^{-1/2}\}e^{-|x|} & (j = N) \\ C(|x|^{(j-N-1)/2} \vee 1)e^{-|x|} & (j \geq N+1) \end{cases}$$

in $\mathbf{R}^N$ with some common $C > 0$ for a finitely many j's .

3 Preliminaries

We state a direct consequence of Tanabe [8] [9] in a way convenient later. Let us begin with the divergence form operator with the same top order terms as A.

Lemma 4 *There exists a divergence form operator*

$$A^0 u \equiv \sum_{|\alpha|=|\beta|=m} \partial^\alpha a_{\alpha\beta} \partial^\beta u$$

with domain $W^{2m,p}$ such that the top order terms

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^{\alpha+\beta} u$$

are the same as those of

$$Au \equiv \sum_{|\gamma| \le 2m} a_\gamma \partial^\gamma u.$$

Moreover the dual $A^{0'}$

$$A^{0'}u \equiv \sum_{|\alpha|=|\beta|=m} \partial^\beta a_{\alpha\beta} \partial^\alpha u$$

is also an operator with domain $W^{2m,p}$ *which satisfies the basic assumption (H1),(H2), and(H3).*

Proof. Consider each of top order terms $a_\gamma \partial^\gamma u \quad (|\gamma| = 2m)$ of A. One can find two multi-indices α andβ satisfying $\alpha + \beta = \gamma$ and $|\alpha| = |\beta| = m$. Thus

$$a_\gamma \partial^\gamma u \equiv a_\gamma \partial^\alpha \partial^\beta u$$

We put

$$a_{\alpha\beta} = a_\gamma = a_{\alpha+\beta}$$

for each $\gamma = \alpha + \beta$. Hence, the operator with divergence form

$$A^0 u \equiv \sum_{|\alpha|=|\beta|=m} \partial^\alpha a_{\alpha\beta} \partial^\beta u$$

has the same top order terms as Au and domain $W^{2m,p}$. Recall the basic assumption $a_\gamma \in W^{m,\infty}$ for $|\gamma| = 2m$. □

Next is a version of Tanabe's result in the form convenient to us.

Theorem 5 *Let* A^0 *be the elliptic operator obtained in Lemma 4 whose top order terms coincide with those of* A. *Then there exists a positive* $\lambda_0 > 0$ *such that* $[\lambda_0, \infty) \subset \rho(A_p^0) = \rho(A_q^{0'})$ *for all* $1 < p < \infty$. *Moreover, all* $(A_p - \lambda)^{-1}$ $(1 < p < \infty)$ *with* $\lambda \ge \lambda_0$ *have a kernel* $K_0(x,\xi)$ *independent of* $1 < p < \infty$ *and* C^{2m-1} *for* $x \ne \xi$ *such that*

$$(A_p^0 - \lambda)^{-1} f(x) = \int_{\mathbf{R}^N} K_0(x,\xi) f(\xi) d\xi,$$

$$(A_q^{0'} - \lambda)^{-1} g(\xi) = \int_{\mathbf{R}^N} K_0(x,\xi) g(x) d\xi$$

for all $f \in L^p$ *and* $g \in L^q$ $(1/p + 1/q = 1)$. *Moreover, for all multi-indices* $\alpha \ge 0$ *with* $|\alpha| \le 2m - 1$,

$$|\partial_x^\alpha K_0(x,\xi)| \le \begin{cases} C|x-\xi|^{2m-N-|\alpha|} e^{-c|\lambda|^{1/2m}|x-\xi|} & \text{if } |\alpha| > 2m - N \\ C(-\log|\lambda|^{1/2m}|x-\xi| \vee 1) e^{-c|\lambda|^{1/2m}|x-\xi|} & \text{if } |\alpha| = 2m - N \\ C|\lambda|^{(|\alpha|+N)/2m-1} e^{-c|\lambda|^{1/2m}|x-\xi|} & \text{if } |\alpha| < 2m - N \end{cases}$$

See Tanabe [8] [9;p210] for the proof. An easier proof can be obtained if one modifies the argument in Miyazaki[3] slightly.

Corollary *With appropriate $\lambda_0 > 0$,*

$$|(\partial/\partial x)^\alpha K(x,\xi)| \le CG_{2m-|\alpha|}(x-\xi) \qquad (0 \le |\alpha| \le 2m-1)$$

for some constant $C > 0$. Note that each $G_{2m-|\alpha|}$ is a Bessel potential.

Proof. Immediate if we consider also Lemma 3.

Lemma 6 *Let the assumptions be the same as in Theorem 5. If $u \in L^p$ $(1 < p < \infty)$ satisfies*

$$A^0 u \equiv \sum_{|\alpha|=|\beta|=m} \partial^\alpha a_{\alpha\beta}(x)\partial^\beta u = f(x) \in L^p$$

weakly, i.e.,

$$\int A^{0\prime}\varphi(x)u(x)dx = \int \sum_{|\alpha|=|\beta|=m} \{\partial^\beta a_{\alpha\beta}(x)\partial^\alpha\varphi(x)\}u(x)dx = \int \varphi(x)f(x)dx$$

for all $\varphi(x) \in C_0^\infty$. Then

$$u \in W^{2m,p}$$

Proof. Let $\lambda_0 > 0$ be as in Theorem 5. Then there exists $U \in W^{2m,p}$ such that

$$(A^0 - \lambda_0)U = -\lambda_0 u + f \in L^p.$$

This turns out to be

$$\int (A^{0\prime} - \lambda_0)\varphi(x)U(x)dx = \int (-\lambda_0 u(x) + f(x))\varphi(x)dx$$

in the weak form. Subtracting this from the equation in the assumption, we have

$$\int (A^{0\prime} - \lambda_0)\varphi(x)\{u(x) - U(x)\}dx = 0.$$

Since $\{(A^{0\prime}-\lambda_0)\varphi; \varphi \in C_0^\infty\}$ is dense in $\mathrm{Ran}(A^{0\prime}-\lambda_0) = L^q$, then $u(x)-U(x) \equiv 0$. Hence

$$u = U \in W^{2m,p}$$

Q.E.D.

Lemma 7 *Let $A^\dagger$ be the operator with the same top order terms as in A and bounded lower order coefficients. If $u \in W^{2m,1}_{loc} \cap W^{2m-1,p}$ $(1 < p < \infty)$ satisfies*

$$A^\dagger u = f \in L^p,$$

then

$$u \in W^{2m,p}$$

Proof. Let A^0 be the operator in Lemma 4 (as well as in Theorem 5 and Lemma 6). Since $(A^0 - A^\dagger)u$ contains only the derivatives of u of order less than $2m$,

$$A^0 u = (A^0 - A^\dagger)u + f \in L^p$$

by assumption. Clearly $u \in L^p \subset W^{2m,1}_{loc} \cap W^{2m-1,p}$ satisfies this equation weakly. Therefore the previous lemma 6 ensures

$$u \in W^{2m,p}$$

Q.E.D.

The kernel function $K_0(x,\xi)$ in Theorem 5, together with its derivatives up to the $2m-1$st order are L^1-valued function continuously dependent on the parameter ξ as the next lemma shows.

Lemma 8 *Let a function $K(x,\xi)$ in $(x,\xi) \in \mathbf{R}^N \times \mathbf{R}^N$ be continuous in $x \neq \xi$ and satisfies*

$$|K(x,\xi)| \leq C|x-\xi|^{1-N} e^{-\epsilon|x-\xi|}$$

for some constants $C > 0$ and $\epsilon > 0$. Then

$$K(\bullet,\xi) \in L^1(\mathbf{R}^N)$$

is a family of L^1 functions dependent continuously (in norm sense) on the parameter ξ.

Proof. Fixing ξ,we regard

$$K(x+\Delta\xi, \xi+\Delta\xi)$$

as a family of functions in x with a new parameter $\Delta\xi$. Thus

$$|K(x+\Delta\xi,\xi+\Delta\xi)| \leq C|x-\xi|^{1-N}e^{-\epsilon|x-\xi|}.$$

Together with the dominated convergence theorem, this implies that $K(x+\Delta\xi,\xi+\Delta\xi)$ a L^1norm-continous family with $\Delta\xi$. (Recall $K(x,\xi)$ is continuous in $x \neq \xi$.) In other words,

$$\|K(\bullet+\Delta\xi,\xi+\Delta\xi) - K(\bullet,\xi)\|_{L^1} \to 0 \quad (\Delta\xi \to 0)$$

that is,

$$\|K(\bullet,\xi+\Delta\xi) - K(\bullet-\Delta\xi,\xi)\|_{L^1} \to 0 \quad (\Delta\xi \to 0).$$

On the other hand,

$$\|K(\bullet-\Delta\xi,\xi) - K(\bullet,\xi)\|_{L^1} \to 0 \quad (\Delta\xi \to 0)$$

holds as is well known. Therefore

$$\|K(\bullet,\xi+\Delta\xi) - K(\bullet,\xi)\|_{L^1} \to 0 \quad (\Delta\xi \to 0).$$

Q.E.D. □

Corollary *For each α with $|\alpha| \le 2m-1$,*

$$(\partial)^\alpha K_0(\bullet, \xi) \in L^1(\mathbf{R}^N)$$

depends continuously on $\xi \in \mathbf{R}^N$.

Now we turn to the operator of our problem.

4 Exisistence and Estimates of the Resolvent Kernel

We seek the solution u of

$$(A-\lambda)u = f \in L^p$$

in the form $u = \int K_0(x,\xi)v(\xi)d\xi$ with $v \in L^p$ where $K_0(x,\xi)$ is the kernel function of $(A^0 - \lambda_0)^{-1}$ with a fixed λ_0 sufficiently large (see Theorem 5). Substituting this into the above equation, we have

$$v + (A - A^0 + \lambda_0 - \lambda)K_0 v = f.$$

Here $A - A^0$ contains only lower order terms. Formally, the last equation can be solved by successive iteration although the resulting infinite series diverges in general. However, we can use the first several terms as parametrix to obtain the true solution as in the below.

Lemma 9 *For an arbitrary $\lambda \in \mathbf{C}$ (possibly, $\lambda \in \sigma(A_p), 1 < p < \infty$), there exist kernel functions $\Gamma_\lambda(x,\xi) = \Gamma_\lambda(x,\xi;\{a_\alpha\}) \in C^{2m-1}$ for $x \ne \xi$ and $Q_\lambda(x,\xi) = Q_\lambda(x,\xi;\{a_\alpha\})$ such that*

$$(A-\lambda)\int_{\mathbf{R}^N} \Gamma_\lambda(x,\xi)f(\xi)d\xi = f(x) - \int_{\mathbf{R}^N} Q_\lambda(x,\xi)f(\xi)d\xi$$

for all $f \in L^p$ with arbitrary $1 < p < \infty$. Here the integral in the left side (resp. the right side) turns out to be a $W^{2m,p}$function (resp. L^pfunction), therefore the equation is regarded in the usual sense. On the other hand, the kernel functions are estimated

$$|\partial_x^\alpha \Gamma_\lambda(x,\xi)| \le \begin{cases} C\{|x-\xi|^{2m-N-|\alpha|} \vee |x-\xi|^{(2m-1)(N+1)/2-|\alpha|/2}\}e^{-|x-\xi|} & \text{if} |\alpha| > 2m-N \\ C\{(-\log|x-\xi|) \vee |x-\xi|^{(2m-1)(N+1)/2-|\alpha|/2}\}e^{-|x-\xi|} & \text{if} |\alpha| = 2m-N \\ C|x-\xi|^{(2m-1)(N+1)/2-|\alpha|/2}e^{-|x-\xi|} & \text{if} |\alpha| < 2m-N \end{cases}$$

for $|\alpha| \le 2m-1$, and

$$|Q_\lambda(x,\xi)| \le C\{1 \vee |x-\xi|^{(2m-1)(N+1)/2}\}e^{-|x-\xi|}.$$

Here the constant $C > 0$ is uniform for any $\xi \in \mathbf{R}^N$, determined only by $\sum_{|\alpha|=2m} |a_\alpha|_{W^{m,\infty}}, \sum_{|\alpha|\le 2m-1} |a_\alpha|_{W^{m,\infty}}, |\lambda|$, m and N. In addition, $Q_\lambda(\bullet,\xi)$ is a family in L^p $(1 < p < \infty$:arbitrary)depending continuously on ξ. Further, $\Gamma_\lambda(x,\xi)$ and $Q_\lambda(x,\xi)$ are polynomial in λ.

Proof Let $K_0(x,\xi)$ be the kernel function of $(A^0-\lambda_0)^{-1}$ as in Theorem 5. It is easy to see that the operator (linear in λ)

$$K \equiv -(A - A^0 + \lambda_0 - \lambda)K_0$$

from L^p ($0 < p < 1$: arbitrary) into itself has the kernel function $K(x,\xi)$ such that

$$|K(x,\xi)| \le C(G_1(x-\xi) + G_{2m}(x-\xi)).$$

Here the constant $C > 0$ can be expressed as

$$C = c(m,N)\Big(\sum_{\|\alpha|=2m} \|a_\alpha\|_{W^{m,\infty}} + \sum_{\|\alpha|\le 2m-1} \|a_\alpha\|_{L^\infty} + |\lambda| + |\lambda_0|\Big)$$

with $c(m,N)$ depending only on the dimension N of $\mathbf{R}^N$ and the order $2m$ of the operator A. Therefore, using the positivity of $G_j(x-\xi)$ for all x,ξ,j, we know that its j-th repeated kernel (polynomial in λ) has the estimate

$$\begin{aligned}
|K^{(j)}(x,\xi)| &\equiv \overbrace{K * K * \cdots * K}^{j}(x,\xi) \\
&\equiv \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\cdots\int_{\mathbf{R}^N} K(x,\xi_1)K(\xi_1,\xi_2)K(\xi_{j-1},\xi)d\xi_1 d\xi_2\cdots d\xi_{j-1} \\
&\le C^j\left(G_1+G_{2m}\right)^{(j)}(x-\xi) = C^j\sum_{k=0}^{j}\binom{j}{k}G_1^{(j-k)} * G_{2m}^k \\
&= C_0\sum_{k=0}^{j} G^{(j-k+2mk)} \le C_1\left(G_j(x-\xi)+G_{2mj}(x-\xi)\right).
\end{aligned}$$

where C_1 is another constant depending on $\sum_{|\alpha|=2m}|a_\alpha|_{W^{m,\infty}}, \sum_{|\alpha|<2m}|a_\alpha|_{L^\infty}, |\lambda|$, m and N. Now using the kernel function $K_0(x,\xi)$ of the operator $(A^0-\lambda_0)^{-1}$, we define

$$\Gamma_\lambda(x,\xi) \equiv K_0(x-\xi) + K_0 * K(x,\xi) + K_0 * K^{(2)}(x,\xi) + \cdots + K_0 * K^{(N)}(x,\xi)$$

and regard ξ as a parameter. It is clear that $\Gamma_\lambda(x,\xi)$ is a polynomial in λ . Using the estimates of $K^{(j)}(x,\xi)$ and $(\partial/\partial x)^\alpha K(x,\xi)$ and $K_0(x,\xi) \in C^{2m-1}$ for $x \ne \xi$, we obtain $\Gamma_\lambda(x,\xi) \in C^{2m-1}$ and the estimates of $(\partial/\partial x)^\alpha\Gamma_\lambda(x,\xi)$.

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \Gamma_\lambda(x,\xi)\right| \le C_2\left(G_{2m-|\alpha|}(x-\xi) + G_{2m(N+1)-|\alpha|}(x-\xi)\right) \quad (|\alpha| \le 2m-1).$$

So one can obtain the estimates of the last, using those of G_α (see Lemma 3).

Recalling $K_0(x,\xi)$ is a kernel of $(A^0-\lambda_0)^{-1}$ in any $f \in L^p$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^N}\Gamma_\lambda(x,\xi)f(\xi)d\xi \\
&= (A^0-\lambda_0)^{-1}f(x) + (A^0-\lambda_0)^{-1}\int_{\mathbf{R}^N}\{K(x,\xi)+K^{(2)}(x,\xi)+\cdots+K^{(N)}(x,\xi)\}f(\xi)d\xi
\end{aligned}$$

Hence the integral on the left side turns out to be a $W^{2m,p}$ function and

$$\begin{aligned}
&(A^0-\lambda_0)\int_{\mathbf{R}^N}\Gamma_\lambda(x,\xi)f(\xi)d\xi\\
=\ & f(x)+\int_{\mathbf{R}^N}\{K(x,\xi)+K^{(2)}(x,\xi)+\cdots+K^{(N)}(x,\xi)\}f(\xi)d\xi\\
&+\int_{\mathbf{R}^N}K^{(N+1)}(x,\xi)f(\xi)d\xi-\int_{\mathbf{R}^N}K^{(N+1)}(x,\xi)f(\xi)d\xi\\
=\ & f(x)+(-A+A^0-\lambda_0+\lambda)\int_{\mathbf{R}^N}\{K_0(x,\xi)+K_0*K(x,\xi)+\cdots+K_0*K^{(N)}(x,\xi)\}f(\xi)d\xi\\
&-\int_{\mathbf{R}^N}K^{(N+1)}(x,\xi)f(\xi)d\xi\\
=\ & f(x)+(-A+A^0-\lambda_0+\lambda)\int_{\mathbf{R}^N}\Gamma_\lambda(x,\xi)f(\xi)d\xi-\int_{\mathbf{R}^N}K^{(N+1)}(x,\xi)f(\xi)d\xi
\end{aligned}$$

Rewriting it, we obtain

$$(A-\lambda)\int_{\mathbf{R}^N}\Gamma_\lambda(x,\xi)f(\xi)d\xi=f(x)-\int_{\mathbf{R}^N}K^{(N+1)}(x,\xi)f(\xi)d\xi$$

Therefore, it suffices to put

$$Q_\lambda(x,\xi)\equiv K^{(N+1)}(x,\xi).$$

The last kernel function is estimated as follows.

$$|K^{(N+1)}(x,\xi)|\le C\left(G_{N+1}(x-\xi)+G_{2m(N+1)}(x-\xi)\right)\le C(1+|x-\xi|^{m(N+1)})e^{-|x-\xi|}$$

Note that the above construction shows $Q_\lambda(x,\xi)$ is a polynomial in λ and its degree is at most $N+1$. Let us finally consider the continuous dependence of $Q_\lambda(\bullet,\xi)=-K^{(N+1)}(\bullet,\xi)\in L^p$ on ξ. It follows from Lemma 8 that $\partial^\alpha K_0(\bullet,\xi)\in L^1$ $(|\alpha|\le 2m-1)$, consequently $\{K(\bullet,\xi)\}\in L^1$ depends continuously on ξ in L^1. On the other hand $K(x,\xi)$ defines a continuous (bounded) integral operator from L^1 to L^1 by virtue of its estimate. Therefore

$$K^{(N+1)}(\bullet,\xi)\in L^1$$

depends continuously on ξ in L^1-norm. This fact together with the boundedness of $K^{(N+1)}(x,\xi)$ ensures its continuous dependence on ξ, even in L^p with arbitrary $1<p<\infty$.

□

Lemma 10 *Let the assumptions and the notation be the same as in the previous lemma 9. Assume further that $\lambda\in\rho(A_p)$ for a given $1<p<\infty$. Then $S_\lambda(\bullet,\xi)=(A_p-\lambda)^{-1}Q_\lambda(\bullet,\xi)$ satisfies*

$$\|S_\lambda(\bullet,\xi)\|_{W^{2m,p}},\ \|S_\lambda(\bullet,\xi)\|_{B^{2m-1}}\le M$$

for some constant $M > 0$ *independent of* ξ, *and it depends continuously on* ξ *with respect to both norms* $\|\bullet\|_{W^{2m,p}}$ *and* $\|\bullet\|_{B^{2m-1}}$. *Moreover, the kernel function*

$$R_\lambda(x,\xi) \equiv \Gamma_\lambda(x,\xi) + S_\lambda(x,\xi)$$

yields a solution

$$u(x) = \int_{\mathbf{R}^N} R_\lambda(x,\xi) f(\xi) d\xi \in W^{2m,p}$$

to

$$(A-\lambda)u = f \tag{1}$$

for an arbitrary $f \in L_0^\infty$ *(*L_0^∞ *is the set of the* L^∞ *function with compact support).*

Proof. From the estimate in the previous lemma,

$$Q_\lambda(\bullet,\xi) \in L^p$$

is continuous in ξ and has an estimate independent of ξ. Hence the assumption $\lambda \in \rho(A_p)$ of the present lemma ensures that

$$S_\lambda(\bullet,\xi) = (A_p - \lambda)^{-1} Q_\lambda(\bullet,\xi) \in W^{2m,p}$$

depends continuously on ξ and satisfies

$$\|S_\lambda(\bullet,\xi)\|_{W^{2m,p}} \le M$$

with $M > 0$ independent of ξ. Therefore, regarded as the limit of a Riemann sum,

$$\int_{\mathbf{R}^N} S_\lambda(x,\xi) f(\xi) d\xi \in W^{2m,p}$$

is easily seen for $f \in C_0^0$. This fact as well as the closedness of the operators A guarantees

$$\begin{aligned}
&(A(x)-\lambda)\int_{\mathbf{R}^N} S_\lambda(x,\xi) f(\xi) d\xi \\
&= \int_{\mathbf{R}^N} (A(x)-\lambda) S_\lambda(x,\xi) f(\xi) d\xi \\
&= -\int_{\mathbf{R}^N} Q_\lambda(x,\xi) f(\xi) d\xi
\end{aligned}$$

for $f \in C_0^0$. Now we prove the same holds also for $f \in L_0^\infty$. Let

$$\mathrm{supp}(f) \subset \Omega$$

with an open bounded set Ω. Then there exists a seququence $f_n \in C_0^0$ $(n = 1, 2, \cdots$ such that

$$\mathrm{supp}(f_n) \subset \Omega, \quad \sup_x |f_n(x)| \le \sup_x |f(x)|, \quad f_n(x) \to f(x) (\text{a.e.}x).$$

The estimate of $(S_\lambda f)(x) \equiv \int_\Omega S_\lambda(x,\xi)f(\xi)d\xi$:

$$\int_{\mathbf{R}^N} |(S_\lambda f)(x)|^p dx \le \int_{x\in\mathbf{R}^N}\int_{\xi\in\Omega} |S_\lambda(x,\xi)|^p dx d\xi \left(\int_{\xi\in\Omega} |f(\xi)|^q d\xi\right)^{p/q}$$

guarantees

$$(S_\lambda f_n)(x) \equiv \int_\Omega S_\lambda(x,\xi)f_n(\xi)d\xi \to \int_\Omega S_\lambda(x,\xi)f(\xi)d\xi \quad (\text{in} L^p).$$

Similarly

$$\int_\Omega Q_\lambda(x,\xi)f_n(\xi)d\xi \to \int_\Omega Q_\lambda(x,\xi)f(\xi)d\xi \quad (\text{in} L^p).$$

Again, by the closedness of A, we obtain

$$(A(x)-\lambda)\int_{\mathbf{R}^N} S_\lambda(x,\xi)f(\xi)d\xi = -\int_{\mathbf{R}^N} Q_\lambda(x,\xi)f(\xi)d\xi$$

for an arbitrary $f \in L_0^\infty$.

Now let

$$R_\lambda(x,\xi) \equiv \Gamma_\lambda(x,\xi) + S_\lambda(x,\xi)$$

Then recalling the property of $\Gamma_\lambda(x,\xi)$ in the previous lemma , we know

$$u(x) = \int_{\mathbf{R}^N} R_\lambda(x,\xi)f(\xi)d\xi \in W^{2m,p}$$

for $f \in L_0^\infty$ and that u is the solution of $(A-\lambda)u = f$ for $f \in L_0^\infty$.

What remains to be proved is the boundedness and the continuity of the kernel function $S_\lambda(x,\xi)$ with respect to the norm $\| \bullet \|_{B^{2m-1}}$. We have only to apply the next lemma. □.

Lemma 11 *Let us pertubate only the lower order coefficients. Suppose that* $S_\lambda(\bullet,\xi) = S_\lambda(\bullet,\xi;\{a_\alpha\}_{|\alpha|\le 2m-1}) \in W^{2m,p}$ *has a uniform estimate*

$$\|S_\lambda(\bullet,\xi)\|_{W^{2m,p}} \le C_1$$

and that $Q_\lambda(\bullet,\xi) = Q_\lambda(\bullet,\xi;a,V)$ *has another type of uniform estimate*

$$|Q_\lambda(x,\xi)| \le C_2 e^{-|x-\xi|/2}.$$

Here $C_1 > 0$ *and* $C_2 > 0$ *are constants uniform for any* $\xi \in \mathbf{R}^N$ *and* λ *in an open set* $U \subset \mathbf{C}$ *determined only by* $\sum_{|\alpha|<2m} |a_\alpha|_{L^\infty}$ *and the set* U. *Suppose also that*

$$(A-\lambda)S_\lambda(\bullet,\xi) = Q_\lambda(\bullet,\xi)$$

holds. Then $S_\lambda(\bullet,\xi) \in B^{2m-1} \cap W^{2m,p}$ *and satisfies*

$$|S_\lambda(\bullet,\xi)|_{B^{2m-1}} \le C(C_1,C_2).$$

Here the constant $C > 0$ *is uniform for any* $\xi \in \mathbf{R}^N$ *determined only by* $\sum_{|\alpha|<2m} |a_\alpha|_{L^\infty}, U$. *Moreover, if* $S_\lambda(\bullet,\xi) \in W^{2m,p}$ *and* $Q_\lambda(\bullet,\xi) \in L^r$ *depend continuously on* ξ *for arbitrary* $1 < r < \infty$, *then so does also*

$$S_\lambda(\bullet,\xi) \in B^{2m-1}.$$

Proof. Rewriting the equation

$$(A^0 - \lambda_0)S_\lambda(\bullet,\xi) = (A^0 - A - \lambda_0 + \lambda)\{S_\lambda(\bullet,\xi)\} + Q_\lambda(\bullet,\xi)$$

with A^0 and λ_0 in Lemma 4 and Theorem 5, we obtain

$$S_\lambda(\bullet,\xi) = (A^0 - \lambda_0)^{-1}(A^0 - A - \lambda_0 + \lambda)\{S_\lambda(\bullet,\xi)\} + (A^0 - \lambda_0)^{-1}Q_\lambda(\bullet,\xi). \quad (2)$$

Note that $Q_\lambda(\bullet,\xi) \in L^r$ and $(A^0 - \lambda_0)^{-1}Q_\lambda(\bullet,\xi) \in W^{2m.p}$ for any $(1 < r < \infty)$.

Now, by the Sobolev embedding theorem, we have

$$S_\lambda(\bullet,\xi) \in W^{2m,r} \subset W^{2m-1,r_1}$$

with $r_1 > p$ such that

$$-N/r_1 = -N/p + 1.$$

This guarantees that the first term on the right side of (2) belongs to W^{2m,r_1}, while the second term belongs to $W^{2m,r}$ with arbitrary $1 < r < \infty$. Therefore

$$S_\lambda(\bullet,\xi) \in W^{2m,r_1} \subset W^{2m-1,r_2}$$

with $r_2 > r_1$ such that

$$-n/r_2 = -N/r_1 + 1.$$

In this way, we obtain

$$r_1 < r_2 < r_3 < \cdots$$

with

$$-N/r_2 = -N/r_1 + 1, -N/r_3 = -N/r_2 + 1, \cdots$$

successively, and eventually reaches r_j such that

$$-N/r_j + 1 > 0.$$

Now this implies

$$S_\lambda(\bullet,\xi) \in W^{2m,r_j} \subset B^{2m-1}.$$

Finally, its continuous dependence on ξ is clear from the above construction. Q.E.D. □

Now we concentrate ourselves to obtain the exponential decay of $S_\lambda(x,\xi)$ in Lemma 10.

Lemma 12 *Let $\lambda_0 \in \rho(A_p)$ and $\tilde{A}_p$ be the perturbation of A_p in lower order term:*

$$\tilde{A}_p u \equiv \sum_{|\alpha|=2m} a_\alpha(x)(\partial/\partial x)^\alpha u + \sum_{|\alpha|\le 2m-1} \tilde{a}_\alpha(x)(\partial/\partial x)^\alpha u$$

considered in the same L^p $(1 < p < \infty)$ with the same domain $\mathrm{Dom}(\tilde{A}_p) = W^{2,p}$. *Then, there exists a constant $\delta > 0$ determined only by $\|(A_p - \lambda)^{-1}\|_{L^p \to W^{2m,p}}$ such that if*

$$|\lambda - \lambda_0| < \delta, \|\tilde{a}_\alpha(\bullet) - a_\alpha(\bullet)\|_{L^\infty} < \delta \qquad (|\alpha| \le 2m-1)$$

then

$$\lambda \in \rho(\tilde{A}_p)$$

and

$$\|(\tilde{A}_p - \lambda)^{-1}\|_{L^p \to W^{2m,p}} \quad \leq 2\|(A_p - \lambda_0)^{-1}\|_{L^p \to W^{2m,p}}.$$

Proof. We consider successive approximation.

$$\begin{aligned} & (\tilde{A}_p - \lambda)(A_p - \lambda_0)^{-1} \\ = & \ \{(A_p - \lambda_0) + (\lambda_0 - \lambda) + \sum_{|\alpha| \leq 2m-1} (\tilde{a}_\alpha(x) - a_\alpha(x))(\partial/\partial x)^\alpha\}(A_p - \lambda_0)^{-1} \\ = & \ I + S \end{aligned}$$

where S is the operator expressed as

$$\{(\lambda_0 - \lambda) + \sum_{|\alpha| \leq 2m-1} (\tilde{a}_\alpha(x) - a_\alpha(x))(\partial/\partial x)^\alpha\}(A_p - \lambda_0)^{-1}.$$

On the other hand,

$$\|u\|_{W^{2m,p}} = \sum_\alpha \|\partial^\alpha u\|_{L^p}.$$

implies

$$\|\partial^\alpha (A_p - \lambda)^{-1}\|_{L^p \to L^p} \leq \|(A_p - \lambda)^{-1}\|_{L^p \to W^{2m,p}}$$

for all $|\alpha| \leq 2m - 1$. Now we determine $\delta > 0$ by

$$\delta + \delta \sum_\alpha \| \left(\frac{\partial}{\partial x}\right)^\alpha u\|_{L^p} = \frac{1}{2}.$$

Thus

$$|\lambda - \lambda_0| < \delta, |\tilde{a}_\alpha(\bullet) - a_\alpha(\bullet)| < \delta \qquad (|\alpha| \leq 2m - 1$$

implies

$$\|S\|_{L^p \to L^p} \leq 1/2.$$

Hence

$$\|(\tilde{A}_p - \lambda)^{-1}\|_{L^p \to W^{2m,p}} = \|(A_p - \lambda_0)^{-1}\|_{L^p \to W^{2m,p}} \|(I + S)^{-1}\| \leq 2\|(A_p - \lambda_0)^{-1}\|_{L^p \to W^{2m,p}}.$$

Q.E.D.

Now we compare the resolvent kernels of A_p and A_p^η which is determined by

$$A^\eta u = e^{\eta \cdot x} A(e^{-\eta \cdot x} u).$$

Note especially that A and A^η have the same top order terms.

Lemma 13 *Let $\lambda \in \rho(A_p)$ $(1 < p < \infty$: arbitrary). Let also a perturbation A_p^η of the operator A_p in L^p be determined by*

$$A^\eta u = e^{\eta\cdot x} A(e^{-\eta\cdot x} u)$$

with a small parameter $\eta \in \mathbf{R}^N$. Let $\Gamma_\lambda(x,\xi)$ be as in Lemma 9. Then $\lambda \in \rho(A_p^\eta)$ and there exists a kernel function $S_\lambda^\eta(x,\xi)$ such that

$$u(x) = \int_{\mathbf{R}^N} \{e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi) + S_\lambda^\eta(x,\xi)\} f(\xi) d\xi \in W^{2m,p}$$

for any $f \in L_0^\infty$ and it represents the solution of

$$(A^\eta - \lambda)u = f.$$

Moreover,

$$\|S_\lambda^\eta(\bullet,\xi)\|_{W^{2m,p}}, \|S_\lambda^\eta(\bullet,\xi)\|_{B^{2m-1}} \leq M$$

holds for all sufficiently small $\eta \in \mathbf{R}^N$ and all $\xi \in \mathbf{R}^N$ with a certain constant $M > 0$.

Proof. First, we prove

$$v(x) = \int_{\mathbf{R}^N} e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi) f(\xi) d\xi \in W^{2m,p}.$$

The estimate of $(\partial/\partial x)^\alpha \Gamma_\lambda(x,\xi)$ $(|\alpha| \leq 2m-1)$ shows

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi)\right| \leq |x-\xi|^{1-N} e^{-|x-\xi|/2}$$

and $v \in W^{2m-1,p}$. On the other hand, $e^{-\eta\cdot\xi} f(\xi) \in L_0^\infty \subset L^p$ ensures

$$v(x) = e^{\eta\cdot x}\int_{\mathbf{R}^N} \Gamma_\lambda(x,\xi) e^{-\eta\cdot\xi} f(\xi) d\xi \in W_{\mathrm{loc}}^{2m,p} \subset W_{\mathrm{loc}}^{2m,1}$$

and

$$\begin{aligned}
(A^\eta - \lambda)v(x) &= e^{\eta\cdot x}(A-\lambda)\int_{\mathbf{R}^N} \Gamma_\lambda(x,\xi) e^{-\eta\cdot\xi} f(\xi) d\xi \\
&= e^{\eta\cdot x}\{e^{-\eta\cdot x} f(x) - \int_{\mathbf{R}^N} Q_\lambda(x,\xi) e^{-\eta\cdot\xi} f(\xi) d\xi\} \\
&= f(x) - \int_{\mathbf{R}^N} e^{\eta\cdot(x-\xi)} Q_\lambda(x,\xi) f(\xi) d\xi).
\end{aligned}$$

The estimate of $|Q_\lambda(x,\xi)| \leq e^{-|x-\xi|/2}$ ensures

$$(A^\eta - \lambda)v(x) \in L^p$$

Hence Lemma 7 guanrantees

$$v \in W^{2m,p}$$

Now we define

$$S_\lambda^\eta(\bullet,\xi) = (A_p^\eta - \lambda)^{-1}\{e^{\eta\cdot(x-\xi)}Q_\lambda(\bullet,\xi)\} \in W^{2m,p}$$

Hence

$$S_\lambda^\eta(\bullet,\xi) \in B^{2m-1}$$

follows. Thus

$$u(x) = \int_{\mathbf{R}^N} \{e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi) + S_\lambda^\eta(x,\xi)\}f(\xi)d\xi \in W^{2m,p}$$

is the solution of

$$(A^\eta - \lambda)u = f \in L_0^\infty.$$

Q.E.D. □

We are now on the position to prove the exponential decay of $S_\lambda(x,\xi)$ in Lemma 10.

Lemma 14 *Let the assumptions and the notations be the same as in Lemmas 9 and 10. Assume further that $\lambda_0 \in \rho(A_p)$ for a given $1 < p < \infty$. Then*

$$S_\lambda(\bullet,\xi) \in W^{2m,r} \cap B^{2m-1} \qquad (1 < r < \infty\text{:arbitrary})$$

for each $\xi \in \mathbf{R}^N$ and

$$|\left(\frac{\partial}{\partial x}\right)^\alpha S_\lambda(x,\xi)| \leq Ce^{-\epsilon|x-\xi|} \qquad (|\alpha| \leq 2m-1)$$

with some constants $C > 0$ and $\epsilon > 0$ uniform in the neighborhood of $\lambda = \lambda_0$. Moreover, as a family in $W^{2m,r}$ with arbitrary $1 < r < \infty$, $S_\lambda(\bullet,\xi)$ also depends continuously on ξ.

Proof. Let

$$(A(x) - \lambda)e^{-\eta\cdot x}u = e^{-\eta\cdot x}f(x) \in L_0^\infty. \tag{3}$$

have a solution $u \in C_0^{2m}$. Here L_0^∞ is the set of bounded function with compact support. Thus Lemma 10 guarantees that

$$e^{-\eta\cdot x}u(x) = \int_{\mathbf{R}^N} R_\lambda(x,\xi)e^{-\eta\cdot\xi}f(\xi)d\xi = \int_{\mathbf{R}^N}\{\Gamma_\lambda(x,\xi) + S_\lambda(x,\xi)\}e^{-\eta\cdot\xi}f(\xi)d\xi.$$

Therefore

$$u(x) = \int_{\mathbf{R}^N} e^{\eta\cdot(x-\xi)}\{\Gamma_\lambda(x,\xi) + S_\lambda f(\xi)\}d\xi.$$

On the other hand, expanding (3), we have

$$(A^\eta(x) - \lambda)u = f(x) \in L_0^\infty.$$

Hence Lemma 13 ensures

$$u(x) = \int_{\mathbf{R}^N} \{e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi) + S_\lambda^\eta(x,\xi)\} f(\xi)d\xi.$$

Combining the two equations, we have

$$\int_{\mathbf{R}^N} e^{\eta\cdot(x-\xi)}\{\Gamma_\lambda(x,\xi)+S_\lambda(x,\xi)\}f(\xi)d\xi = \int_{\mathbf{R}^N} \{e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi)+S_\lambda^\eta(x,\xi)\}f(\xi)d\xi$$

for any $f(x) = (A^\eta - \lambda)u = e^{\eta\cdot x}(A-\lambda)e^{-\eta\cdot x}u$ with $u \in C_0^{2m}$. Since the set of such f is dense in L^p, we have

$$e^{\eta\cdot(x-\xi)}\{\Gamma_\lambda(x,\xi) + S_\lambda(x,\xi)\} = e^{\eta\cdot(x-\xi)}\Gamma_\lambda(x,\xi) + S_\lambda^\eta(x,\xi)$$

$$S_\lambda(x,\xi) = e^{-\eta\cdot(x-\xi)}S_\lambda^\eta(x,\xi).$$

Here Lemma 9 guarantees

$$|\left(\frac{\partial}{\partial x}\right)^\alpha S_\lambda^\eta(x,\xi)| \le M \qquad (|\alpha| \le 2m-1)$$

with some constant $M > 0$ independent of η near $0 \in \mathbf{R}^N$. Therefore

$$|S_\lambda(x,\xi)| \le M \inf_\eta e^{-\eta\cdot(x-\xi)} \le Me^{-\epsilon|x-\xi|}.$$

for some constant $\epsilon > 0$. Hence we have

$$|\left(\frac{\partial}{\partial x}\right)^\alpha S_\lambda(x,\xi)| \le Me^{-\epsilon|x-\xi|} \qquad (|\alpha| \le 2m-1)$$

inductively (replacing the constant $M > 0$ if necessary). On the other hand,

$$(A-\lambda)S_\lambda(\bullet,\xi) = Q(\bullet,\xi) = O(e^{-|\bullet-\xi|/2}) \in L^r$$

for arbitrary $(1 < r < \infty)$. Therefore Lemma 7 ensures

$$S_\lambda(\bullet,\xi) \in W^{2m,r}$$

with any $1 < r < \infty$ and it continuously depends on ξ. (Recall the similar property of $Q_\lambda(x,\xi)$). Q.E.D. □.

Theorem 15 *$\rho(A_p)$ does not depend on $1 < p < \infty$. And the resolvent $(A_p - \lambda)^{-1}$ with $\lambda \in \rho(A_p)$ can be written as an integral operator*

$$((A_p - \lambda)^{-1}f)(x) = \int_{\mathbf{R}^N} R_\lambda(x,\xi)f(\xi)d\xi.$$

with the kernel function $R_\lambda(x,\xi)$ *independent of* $1 < p < \infty$ *which belongs to* C^{2m-1} *in* $x \neq \xi$ *for each fixed* ξ. *It also satisfies*

$$|(\partial/\partial_x)^\alpha R_\lambda(x,\xi)| \leq$$
$$\begin{cases} C\{|x-\xi|^{2m-N-|\alpha|}\}e^{-\epsilon|x-\xi|} & \mathit{if}|\alpha| > 2m-N \\ C\{(-\log|x-\xi|) \vee 1\}e^{-\epsilon|x-\xi|} & \mathit{if}|\alpha| = 2m-N \\ C|x-\xi|^{(2m-1)(N+1)/2-|\alpha|/2}e^{-\epsilon|x-\xi|} & \mathit{if}|\alpha| < 2m-N \end{cases}$$

for $|\alpha| \leq 2m-1$. *Here* $\epsilon > 0$ *and* $C > 0$ *are constants uniform in the neighborhood of each* $\lambda \in \rho(A_p)$.

Proof. Let $p \in (1,\infty)$ and $\lambda \in \rho(A_p)$ be arbitrarily fixed. Choose any other $r \in (1,\infty)$ arbitrarily. We need only to prove $\lambda \in \rho(A_r)$. Recall the continuous dependence of $S_\lambda(\bullet,\xi) \in W^{2m,r}$ on $\xi \in \mathbf{R}^N$ (see Lemma 14)and its property (see Lemma 10):

$$(A-\lambda)S_\lambda(\bullet,\xi) = Q_\lambda(\bullet,\xi).$$

Put

$$R_\lambda(x,\xi) = \Gamma_\lambda(x,\xi) + S_\lambda(x,\xi)$$

whose estimates follow immediately from Lemmas 9 and 14.

Then the same argument as in Lemma 10 guarantees

$$u(x) = \int_{\mathbf{R}^N} R_\lambda(x,\xi)f(\xi)d\xi \in W^{2m,r}, \quad (A-\lambda)u = f(x)$$

holds at least for $f \in L_0^\infty$.

Let us prove generally

$$R_\lambda f(x) = \int_{\mathbf{R}^N} R_\lambda(x,\xi)f(\xi)d\xi \in W^{2m,r}, \quad (A_r-\lambda)R_\lambda f = f$$

for any $f \in L^r$. Note that R_λ maps L^r continuously into itself by the exponential decay of the kernel function $R_\lambda(x,\xi)$. First we choose a sequence $f_n \in L_0^\infty$ with $f_n \to f$ in L^r. Thus $u_n = R_\lambda f_n \in W^{2m,r}$ and

$$\begin{aligned} (A_r-\lambda)u_n = f_n \quad &\to \quad f \quad \text{in } L^r \\ u_n = R_\lambda f_n \quad &\to \quad R_\lambda f \quad \text{in } L^r \end{aligned}$$

The closedness of the operator A_r ensures

$$u = R_\lambda f \in W^{2m,r} = \mathrm{Dom}(A_r), \quad (A_r-\lambda)u = f.$$

Finally, it suffices only to prove

$$R_\lambda(A_r-\lambda)u = u$$

for an arbitrary $u \in W^{2m,r}$. There exists an approximate sequence $u_n \in C_0^{2m} \subset W^{2m,p} \cap W^{2m,r}$ such that $u_n \to u$ in $W^{2m,r}$.

$$R_\lambda(A_r-\lambda)u_n = u_n$$

On the left side, R_λ is a bounded operator from L^r into itself and $(A_r - \lambda)u_n \to (A_r - \lambda)u$ in L^r from the assumption. On the right side, clearly, $u_n \to u$ in L^r. Therefore

$$R_\lambda(A_r - \lambda)u = u \quad (u \in W^{2m,r}).$$

Together with the above obtained

$$(A_r - \lambda)R_\lambda f = f \quad (f \in L^r),$$

we have $R_\lambda = (A_r - \lambda)^{-1}$ and

$$\lambda \in \rho(A_r).$$

Q.E.D. □

We define the discrete spectrum of operators to state the final theorem correctly.

Definition. Let A be an operator in a Banach space X and $\sigma(A)$ be its spectrum. $\lambda_0 \in \sigma(A)$ is called discrete spectrum if it is a pole of the resolvent $(A - \lambda)^{-1}$ as a function in λ, and the generalized eigenspace E corresponding to λ_0 is finite dimensional.

Remark. See Kato [1, p.180] or Yosida [6, p.228] for the Laurent expansion around the general isolated singularity of $(A - \lambda)^{-1}$.

Theorem 16 *Let λ_0 be a discrete spectrum of A_p (independent of $1 < p < \infty$). Then each eigenfunction f (aswell as generalized eigenfunction) corresponding to λ_0 satisfies*

$$|f(x)| \le Ce^{-\epsilon|x|}$$

with certain constants $C > 0$ and $\epsilon > 0$

Proof Consider the Laurent expansion of the operator $(A_p - \lambda)^{-1}$ around $\lambda = \lambda_0$. Its expression with kernel functions is

$$\sum_{k \ge -n} (\lambda - \lambda_0)^k T_k(x, \xi)$$

where

$$T_k(x,\xi) = \frac{-1}{2\pi i} \int_{|\lambda - \lambda_0| = \delta} (\lambda - \lambda_0)^{-k-1} R_\lambda(x,\xi) d\lambda$$

with some small $\delta > 0$.

The spectral projection to the subspace of $W^{2m,p}$ corresponding the isolated spectrum $\{\lambda_0\}$ is expressed by $T_{-1}(x,\xi)$. Recall

$$R_\lambda(x,\xi) = \Gamma_\lambda(x,\xi) + S_\lambda(x,\xi)$$

and $\Gamma_\lambda(x,\xi)$ is a polynomial in λ (See Lemma 5). Thus

$$T_{-1}(x,\xi) = \frac{-1}{2\pi i} \int_{|\lambda - \lambda_0| = \delta} S_\lambda(x,\xi) d\lambda.$$

Meanwhile,

$$|S_\lambda(x,\xi)| \le Ce^{-\epsilon|x-\xi|}$$

holds on $|\lambda - \lambda_0| = \delta$. Therefore

$$|T_{-1}(x,\xi)| \le Ce^{-\epsilon|x-\xi|}.$$

Since $T_{-1}(x,\xi)$ is the kernel function of the projection to the generalized eigenspace E corresponding to λ_0, it reresents a function in E for each ξ. The proof is complete. □

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