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Hamiltonian volume minimizing properties of Lagrangian submanifolds[1]

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1 Introduction

The equator on S^2 has the least length among all its images under area bisecting deformations. This is a well-known theorem by Poincaré and a special case of isoperimetric inequality for closed curves on S^2. This theorem stands in the intersection of symplectic geometry and Riemannian geometry. In fact, we can interpret (S^2, the area form) as a symplectic (Kähler) manifold and the equator as a minimal Lagrangian submanifold. Moreover, area bisecting deformations of the equator are nothing but *Hamiltonian deformations.* Therefore, the above theorem has the feature that some symplectic assumptions give rise to a Riemannian result.

Considering S^2 as $\mathbb{C}P^1$ and the equator as a real form $\mathbb{R}P^1 \subset \mathbb{C}P^1$, it is natural to generalize Poincaré's theorem to the case $\mathbb{R}P^n \subset \mathbb{C}P^n$. In 1990, Y.-G. Oh [5] and B. Kleiner actually obtained the following theorem (see also [2]):

Theorem 1 (Kleiner-Oh). *The standard real form $\mathbb{R}P^n \subset \mathbb{C}P^n$ has the least volume among all its images under Hamiltonian isotopies.*

A minimal Lagrangian submanifold with such a property is said to be *Hamiltonian volume minimizing.*

In this article, we show that the product of equators in $S^2(1) \times S^2(1) (\cong Q_2(\mathbb{C}))$ is also Hamiltonian volume minimizing. More precisely,

Theorem 2 (IOS [4]). *Let $L := S^1(1) \times S^1(1)$ be a totally geodesic Lagrangian torus in $(S^2(1) \times S^2(1), \omega_0 \oplus \omega_0)$, where ω_0 denotes the standard Kähler form of $S^2(1) \cong \mathbb{C}P^1$. Then for any Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(S^2 \times S^2)$, we have*

$$\mathrm{vol}(\phi(L)) \geq \mathrm{vol}(L).$$

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Moreover, if $\mathrm{vol}(\phi(L)) = \mathrm{vol}(L)$ *holds, then there exists an isometry* g *of* $S^2(1) \times S^2(1)$ *such that* $\phi(L) = gL$.

In section 2, we review some standard notions from symplectic geometry.

In section 3, we explain an unified method of proving the Hamiltonian volume minimizing properties of real forms in Hermitian symmetric spaces of compact type and pose a conjecture in terms of integral geometry.

In section 4, we prove the conjecture in section 3 (Conjecture 4) in the case $S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$ and establish its Hamiltonian volume minimizing property and the uniqueness modulo isometric group actions.

2 Lagrangian submanifolds and their Hamiltonian deformations

Let (M, ω) be a $2n$-dimensional closed symplectic manifold with symplectic 2-form ω and L be an n-dimensional closed submanifold of M. Then L is said to be *Lagrangian* if $\omega|_L = 0$. Hamiltonian isotopies of (M, ω) are defined as follows. If a smooth function $F : M \times [0,1] \to \mathbb{R}$ is given, then we can uniquely define the vector field X_t on M for each $t \in [0,1]$ such that

$$\omega(X_t, \cdot) = dF(\cdot, t).$$

Therefore, we have the flow $\{\phi_t\}_{t\in[0,1]}$ of diffeomorphisms on M defined by the differential equation

$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x))$$

with initial condition $\phi_0 = id_M$. The time 1-map ϕ_1 of this flow is called a *Hamiltonian diffeomorphism*. The set of all Hamiltonian diffeomorphisms is denoted by $\mathrm{Ham}(M, \omega)$. We can check that $\mathrm{Ham}(M, \omega)$ is a subgroup of the identity component $\mathrm{Diff}_0(M)$ of the diffeomorphism group of M.

By definition, a Hamiltonian diffeomorphism ϕ of M preserves the symplectic structure (*i.e.*, $\phi^*\omega = \omega$). Therefore, if L is a Lagrangian submanifold of M, then $\phi(L)$ is also Lagrangian.

In the next section, we restrict our attention to Kähler manifolds to introduce the volume functional on the space $\{\phi(L) |\ \phi \in \mathrm{Ham}(M, \omega)\}$.

3 Lagrangian intersection theorem, Poincaré formula and Hamiltonian volume minimizing property

Let (M, ω, J) be a closed connected Kähler manifold. Trivial examples of Hamiltonian volume minimizing Lagrangian submanifolds are *special La-*

grangian submanifolds in Ricci-flat Kähler manifolds. In fact, they are calibrated submanifolds and homologically volume minimizing. But, in general, it is difficult to check whether a minimal Lagrangian submanifold L in M is Hamiltonian volume minimizing or not, if L is not a calibrated submanifold. One method we use here is a combination of Lagrangian intersection theorems in symplectic geometry and Poincaré formulas in integral geometry. This method was first pointed out by Oh and Kleiner.

From now on, we restrict our interest to the case where (M, ω, J) is a Hermitian symmetric space of compact type. It is an important example of Kähler-Einstein manifolds with positive Ricci curvature.

Let τ be a canonical involution on M. Then

$$L := \mathrm{Fix}\ \tau$$

is a totally geodesic Lagrangian submanifold in M (which is called a *real form* of M). It seems worthwhile to verify the Hamiltonian volume minimizing property for such a pair (M, L).

For such a case, a Lagrangian intersection theorem has already established by Oh ([8],[6] and [7]).

Theorem 3 (Oh). *Let (M, ω) be a compact symplectic manifold such that there exists an integrable almost complex structure J for which the triple (M, ω, J) becomes a compact Hermitian symmetric space. Let $L = \mathrm{Fix}\ \tau$ be the fixed point set of an anti-holomorphic involutive isometry τ on M. Assume that the minimal Maslov number of L is greater than or equal to 2. Then for any Hamiltonian diffeomorphism ϕ of M such that L and $\phi(L)$ intersect transversely, the inequality*

$$\sharp(L \cap \phi(L)) \geq \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2) \tag{1}$$

holds.

Here we have to explain the minimal Maslov number of L. For any smooth map of pairs $w : (D^2, \partial D^2) \to (M^{2n}, L^n)$, we have a unique trivialization of the pull-back bundle $w^*TM \cong D^2 \times \mathbb{C}^n$ as a symplectic vector bundle up to homotopy. This defines a map from $S^1 \cong \partial D^2$ to

$$\Lambda(\mathbb{C}^n) := \{L |\ L : \text{Lagrangian plane in } \mathbb{C}^n\}.$$

Using a well-known Maslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$, we can define

$$I_{\mu,L}(w) := \mu(\partial D^2) \in \mathbb{Z}.$$

This is called the *Maslov index* of w. We can show that $I_{\mu,L}$ defines a homomorphism on $\pi_2(M, L)$ and is invariant under Hamiltonian isotopies of M.

The *minimal Maslov number* Σ_L of the Lagrangian submanifold L in M is defined as the positive generator of the subgroup

$$\{I_{\mu,L}(w)|\ w : (D^2, \partial D^2) \to (M, L)\}$$

of $\mathbb{Z}$.

Here, we state our conjecture.

Conjecture 4 (IOS). *Let (M, ω, J) be a Hermitian symmetric space of compact type. Let $L = \mathrm{Fix}\ \tau$ be the fixed point set of a canonical involution τ on $M = G/K$. If any second variation of the volume functional at L on the space $\{\phi(L)|\ \phi \in Ham(M, \omega)\}$ is non-negative, then we have*

$$C\mathrm{vol}(L)\mathrm{vol}(N) \geq \int_G \sharp(L \cap g(N))d\mu(g)$$

where

$$C = \frac{(\sum \mathrm{rank} H_*(L; \mathbb{Z}_2))\mathrm{vol}(G)}{\mathrm{vol}(L)^2}$$

for any Lagrangian submanifold N.

The assumption that any second variation of the volume functional at L on the space $\{\phi(L)|\ \phi \in \mathrm{Ham}(M, \omega)\}$ is non-negative is, of course, a necessary condition for L to be Hamiltonian volume minimizing. A minimal Lagrangian submanifold satisfying such a property is said to be *Hamiltonian stable.* Hamiltonian stabilities of all real forms in compact irreducible Hermitian symmetric spaces are completely determined by Amarzaya-Ohnita ([1]). This is another reason why we investigate real forms in compact Hermitian symmetric spaces.

Proposition 5. *Under the same assumption as Conjecture 4, if Conjecture 4 is true, then the totally geodesic Lagrangian submanifold $L = \mathrm{Fix}\ \tau$ with $\Sigma_L \geq 2$ in $(G/K, \omega, J)$ is Hamiltonian volume minimizing.*

Proof. By Theorem 3 and Conjecture 4, we have

$$\begin{aligned} C\mathrm{vol}(L)\mathrm{vol}(\phi(L)) &\geq \int_G \sharp(L \cap g \circ \phi(L))d\mu(g) \\ &\geq \int_G \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2)d\mu(g) \\ &= \mathrm{vol}(G) \sum_{i=0}^{\dim L} \mathrm{rank} H_i(L, \mathbb{Z}_2) \\ &= C\mathrm{vol}(L)^2. \end{aligned}$$

Hence,

$$\mathrm{vol}(\phi(L)) \geq \mathrm{vol}(L).$$

$\square$

In the next section, we prove the above conjecture in the case $S^1(1) \times S^1(1) \subset S^2(1) \times S^2(1)$.

4 Poincaré formula for Lagrangian surfaces in a product of 2-spheres

Here we review the generalized Poincaré formula in Riemannian homogeneous spaces obtained by Howard [3].

Let U be a finite dimensional real vector space with an inner product, and V and W vector subspaces of dimensions p and q in U, respectively. Take orthonormal bases $v_1, \dots, v_p$ and $w_1, \dots, w_q$ of V and W, and define

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|,$$

which is the angle between V and W.

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric which is also invariant under elements of K. This metric induces a G-invariant Riemannian metric on G/K. We denote by o the origin of G/K. For x and y in G/K and vector subspaces V in $T_x(G/K)$ and W in $T_y(G/K)$, we define $\sigma_K(V, W)$, the angle between V and W, by

$$\sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1}V, dk_o^{-1}(dg_y)_o^{-1}W)d\mu_K(k)$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$.

Theorem 6 (Howard). *Let G/K be a Riemannian homogeneous space and assume that G is unimodular. Let N and L be submanifolds of G/K with* $\dim N + \dim L \geq \dim(G/K)$. *Then*

$$\int_G \mathrm{vol}(N \cap gL)d\mu_G(g) = \int_{N \times L} \sigma_K(T_x^\perp N, T_y^\perp L)d\mu(x, y)$$

holds.

The linear isotropy representation induces an action of K on the Grassmannian manifold $G_p(T_o(G/K))$ consisting of all p dimensional subspaces in the tangent space $T_o(G/K)$ at o in a natural way. Although $\sigma_K(T_x^\perp N, T_y^\perp L)$

is defined as an integral on K, we can consider that it is defined as an integral on an orbit of K-action on the Grassmannian manifold. So $\sigma_K(\cdot,\cdot)$ can be regarded as a function defined on the product of the orbit spaces of such K-actions. In the case where G/K is a real space form, $\sigma_K(T_x^\perp N, T_y^\perp L)$ is constant since K acts transitively on the Grassmannian manifold. This implies that the Poincaré formula is expressed as a constant times of the product of the volumes of N and L. In general, such K-actions are not transitive. However, if we can define an invariant for orbits of this action, which is called an *isotropy invariant*, then using this we can express the Poincaré formula more explicitly.

Next, we define isotropy invariants for surfaces in $S^2 \times S^2$ and give an explicit Poincaré formula for its Lagrangian surfaces.

Let G be the identity component of the isometry group of $S^2 \times S^2$, that is, $G = SO(3) \times SO(3)$. Then the isotropy group K at $o = (p_1, p_2)$ in $S^2 \times S^2$ is isomorphic to $SO(2) \times SO(2)$, and $S^2 \times S^2$ is expressed as a coset space G/K. Assume thet G is equipped with an invariant metric normalized so that G/K becomes isometric to the product of unit spheres. We decompose the tangent space $T_o(G/K)$ as

$$T_o(G/K) = T_{p_1}(S^2) \oplus T_{p_2}(S^2).$$

We take orthonormal bases $\{e_1, e_2\}$ and $\{e_3, e_4\}$ of $T_{p_1}(S^2)$ and $T_{p_2}(S^2)$, respectively, then a complex structure on $T_o(G/K)$ is given by

$$Je_1 = e_2, \ \ Je_2 = -e_1, \ \ Je_3 = e_4, \ \ Je_4 = -e_3.$$

We consider the oriented 2-plane Grassmannian manifold $\tilde{G}_2(T_o(G/K))$. Take an origin $V_o := \text{span}\{e_1, e_2\}$ and express $\tilde{G}_2(T_o(G/K))$ as a coset space

$$\tilde{G}_2(T_o(G/K)) = SO(4)/(SO(2) \times SO(2)) =: G'/K'.$$

Now we study the K-action on $\tilde{G}_2(T_o(G/K))$ and define isotropy invariants. In this case the actions of K and K' on $\tilde{G}_2(T_o(G/K))$ are equivalent by $\mathrm{Ad} : K \to K'$. Therefore it suffices to consider the orbit space of the isotropy action of $\tilde{G}_2(T_o(G/K))$. It is well known that the orbit space of the isotropy action of a symmetric space of compact type can be identified with a fundamental cell of a maximal torus. Hence we can define the isotropy invariant by a coordinate of a maximal torus. We denote by $\mathfrak{g}'$ and $\mathfrak{k}'$ the Lie algebra of G' and K', respectively. Then we have a canonical decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$, where

$$\mathfrak{m}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \ \middle| \ X \in M_2(\mathbb{R}) \right\}.$$

We take a maximal abelian subspace $\mathfrak{a}'$ of $\mathfrak{m}'$ as follows:

$$\mathfrak{a}' = \left\{ \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \ \middle| \ X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \theta_1, \theta_2 \in \mathbb{R} \right\}.$$

Then the set of positive restricted roots of $(\mathfrak{g}', \mathfrak{k}')$ with respect to $\mathfrak{a}'$ is

$$\Delta = \{\theta_1 + \theta_2, \theta_1 - \theta_2\}.$$

So we have a fundamental cell C of $\mathfrak{a}'$:

$$C = \left\{ Y = \begin{pmatrix} O & X \\ -{}^tX & O \end{pmatrix} \middle| X = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \begin{array}{c} 0 \leq \theta_1 + \theta_2 \leq \pi \\ 0 \leq \theta_1 - \theta_2 \leq \pi \end{array} \right\}.$$

Thus the isotropy invariants in this case are given by $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$. It is easy to see that the geometric meaning of $\theta_1 - \theta_2$ is the Kähler angle of 2-dimensional subspace Exp$Y$ of $T_o(G/K)$. On the other hand, there is the other complex structure J' which is defined by

$$J'e_1 = e_2, \ J'e_2 = -e_1, \ J'e_3 = -e_4, \ J'e_4 = e_3$$

on $T_o(G/K)$. We can also check that $\theta_1 + \theta_2$ is the Kähler angle of Exp$Y$ with respect to J'.

Using these isotropy invariants we obtain the following formula from Theorem 6.

Theorem 7 (IOS [4]). *Let N and L be Lagrangian surfaces in $S^2 \times S^2$. We assume that L is a product of curves in S^2. Then we have*

$$\int_G \sharp(L \cap gN) d\mu(g) = 4\mathrm{vol}(L) \int_N \mathrm{length}(\mathrm{Ellip}(\sin^2 \theta_x, \cos^2 \theta_x)) d\mu(x),$$

where $2\theta_x - \pi/2$ is the Kähler angle of $T_x^\perp N$ with respect to J' and Ellip(α, β) *denotes an ellipse defined by*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

Theorem 7 yields the following immediately.

Corollary 8. *Let N and L be surfaces of $S^2 \times S^2$. Suppose that N is Lagrangian and L is a product of curves in S^2. Then the following inequality holds:*

$$\int_{SO(3)\times SO(3)} \sharp(L \cap gN) d\mu(g) \leq 16\mathrm{vol}(N)\mathrm{vol}(L). \tag{2}$$

Moreover, the equality holds if and only if the Lagrangian surface N is also a product of curves in S^2.

Proof of Theorem 2. Let $L := S^1(1) \times S^1(1)$. Since $\Sigma_L = 2$ and

$$\frac{\sum_{i=0}^2 \mathrm{rank} H_i(L, \mathbb{Z}_2)\mathrm{vol}(SO(3) \times SO(3))}{\mathrm{vol}(L)^2} = \frac{4 \cdot (8\pi^2 \cdot 8\pi^2)}{(4\pi^2)^2} = 16,$$

the Lagrangian submanifold L is Hamiltonian volume minimizing by Proposition 5.

Suppose that $\mathrm{vol}(\phi(L)) = \mathrm{vol}(L) = 4\pi^2$. In this case, by the proof of Proposition 5, inequality (2) must satisfy the equality. So $\phi(L)$ must be a product of closed curves l_1 and l_2 in $S^2(1)$. If one of these curves is not area bisecting, we can reduce the volume of $\phi(L) = l_1 \times l_2$ by a Hamiltonian diffeomorphism $\tilde{\phi} \in \mathrm{Ham}(S^2) \times \mathrm{Ham}(S^2) \subset \mathrm{Ham}(S^2 \times S^2)$ in view of the isoperimetric inequality on $S^2(1)$. This contradicts that L is Hamiltonian volume minimizing. Hence, l_1 and l_2 are area bisecting. Consequently, closed curves l_1 and l_2 must be great circles by the isoperimetric inequality. Therefore, the Hamiltonian diffeomorphism ϕ is nothing but an isometry g of $S^2(1) \times S^2(1)$. □

References

[1] A. Amarzaya and Y. Ohnita, *Hamiltonian stability of certain minimal Lagrangian submanifolds in complex projective spaces*, Tohoku Math. J. **55** (2003), 583–610.

[2] A. B. Givental, *The Nonlinear Maslov index*, London Mathematical society Lecture Note Series **151** (1990), 35-43

[3] R. Howard, *The kinematic formula in Riemannian homogeneous spaces*, Mem. Amer. Math. Soc., No.509, **106** (1993).

[4] H. Iriyeh, H. Ono and T. Sakai, *Integral geometry and Hamiltonian volume minimizing property of a totally geodesic Lagrangian torus in $S^2 \times S^2$*, Proc. Japan Acad., **79**, Ser. A (2003), 167–170. arXiv:mathDG/0310432.

[5] Y.-G. Oh, *Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), 501-519.

[6] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, I*, Comm. Pure Appl. Math. **46** (1993), 949-993.

[7] Y.-G. Oh, *Addendum to "Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, I"*, Comm. Pure Appl. Math. **48** (1995), 1299-1302.

[8] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks, III: Arnold-Givental Conjecture*, The Floer Memorial Volume, Birkhauser, Progress in Math. **133** (1995), 555-573.