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Central Limit Theorems for The Random Iterations of 1-dimensional Transformations

Hiroshi ISHITANI
Department of Mathematics,
Mie University
石谷 寛 三重大学教育学部

1 Introduction

Let T be a nonsingular transformation on the unit interval $I=[0,1]$ with the following properties:

(1) There is a countable partition $\{ I_j : j=1,2,\cdots \}$ of I into such subintervals that for each $j=1,2,\ldots$ the restriction T_j of T to I_j is monotonic and can be extended to a C^2-function on the closure I_j.

(2) The collection $\{ J_j := T(I_j);\ j=1,2,\ldots \}$ consists of a finite number of different subintervals.

In the case that there exists a positive integer n_0 for which T satisfies

$$\gamma(T^{n_0}) := \inf|(T^{n_0})'(x)| > 1$$

and T has the unique and weakly mixing invariant measure, J.Rousseau-Egele ([15]) got the central limit theorem, using the so-called "Fourier transform technique" which had been used to obtain limit theorems for Markov processes (cf.[3],[8],[14]). In the more general case, central limit theorems of mixed type for such transformations were given in [5]. That is, under suitable assumptions on the function f and the probability measure ν, the distribution function $\nu\{\sum_{k=0}^{n-1} f(T^k x)/\sqrt{n} < z\}$ is asymptotically a convex combination of normal distribution functions.

On the other hand, it seems more natural to consider that T itself might be slightly but randomly perturbed for each step, if we successively calculate $f(T^k x)$ by a computer. Moreover, when $f(T^k x)$ is a variable in the nature, for example a population at time k of some insect, it is reasonable to think so. In [11],[12] and [13], T. Morita studied the ergodic properties of "random iterations" of transformations and got the random ergodic theorem. The aim of this article is to generalize the central limit theorem of mixed type to random iterations. By virtue of the results in [11] and [13], we can apply the Fourier transform technique to this, and we can obtain the analogous result to that of the case of a single transformation.

Another aim of this article is to generalize central limit theorems of mixed type for a transformation in [5] : the result for random iterations contains the following Theorem as a special case.

Theorem. *Let T satisfy* (1) and (2). *Assume that there exists a positive number n_0 for which we have* $\gamma(T^{n_0}) := \inf|(T^{n_0})'(x)| > 1$, *and that ν is an absolutely continuous probability measure. Then, there exist nonnegative constants $a_j\,(j = 1, 2, \dots, M)$ with $\sum_{j=1}^{M} a_j = 1$ such that if f is a function of bounded variation, we have for some $\sigma_j \geq 0\,(j = 1, 2, \dots, M)$*

$$\lim_{n\to\infty} \nu\{\sum_{k=0}^{n-1}(f(T^k x) - f^*)/\sqrt{n} < y\} = \sum_{j=1}^{M} a_j F(0, \sigma_j; y)$$

at the continuity point of the left hand side, where $F(0, \sigma^2; y)$ stands for the distribution function of $N(0, \sigma^2)$ and $f^* = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(T^k x)/n$. If we assume further that $\sigma_j > 0$ for all j and that $d\nu/dm$ has a version of bounded variation, then we have

$$\sup_y |\nu\{\sum_{k=0}^{n-1}(f(T^{-1}x) - f^*)/\sqrt{n} < y\} - \sum_{j=1}^{M} a_j F(0, \sigma_j; y)| \leq C/\sqrt{n}$$

for some constant $C > 0$.

Central limit theorems for β-transformations, α-continued fraction transformations, Wilkinson's piecewise linear transformations and unimodal linear transformations were given as corollaries to this theorem.

We give our results and an idea of their proofs in §2. Although those are analogous to the results in [5], remark the following. First, we could obtain the improvement of the rate of convergence, by slightly changing the method. Second, it is shown that the number M of possibly different limiting normal distributions is equal to the number of ergodic invariant measures, which was not yet proved in [5]. Last of all, note that there is a remarkable difference between random and deterministic cases. That is, the number of different limiting normal distributions, which appear in the central limit theorem of mixed type for random iterations, is far smaller than in the deterministic cases. Therefore, the ordinary central limit theorem easily holds in the case of random iterations.

In §3, some examples and applications are discussed. First, we give a central limit theorem for random iterations of unimodal linear transformations. Second, the central limit theorem for the random time iteration of dyadic transformation is given as a corollary to the results in §2. Central limit theorem of mixed type for a class of "dynamical system with stochastic perturbations" ([9]) can be also obtained as a corollary to the result in §2.

2 Definitions and Results

We denote by m the Lebesgue measure on the interval $I = [0,1]$ and by $(L^1(m), \|\cdot\|_m)$ the Banach space of Lebesgue integrable functions. A transformation T is said to be m-nonsingular, if $m(A) = 0$ implies $m(T^{-1}A) = 0$. Let us write T^n for the n-th iteration of T.

We shall begin by defining the random iteration of m-nonsingular transformations:

(i) Let Y be a complete separable metric space, $\mathcal{B}(Y)$ be its topological Borel field and p be a probability measure on $(Y, \mathcal{B}(Y))$.

(ii) Define $\Omega := \Pi_{i=1}^{\infty} Y$ and let us write $\mathcal{B}(\Omega)$ for the topological Borel field of Ω. We equip the product measure $P := \Pi_{i=1}^{\infty} p$ on $(\Omega, \mathcal{B}(\Omega))$.

(iii) Let $\{T_y\colon y \in Y\}$ be a family of m-nonsingular transformations on the unit interval I such that the mapping $(x, y) \to T_y x$ is measurable.

In order to study the behavior of the random iterations, we consider the skew product transformation $S : I \times \Omega \to I \times \Omega$ defined by

$$S(x, \omega) := (T_{\omega_1} x, \sigma\omega) \tag{2.1}$$

for $(x, \omega) \in I \times \Omega$, where ω_1 stands for the first coordinate of ω and $\sigma : \Omega \to \Omega$ is the shift transformation to the left. Remark that we have

$$S^n(x, \omega) = (T_{\omega_n} \circ T_{\omega_{n-1}} \circ \ldots \circ T_{\omega_1} x, \sigma^n \omega). \tag{2.2}$$

Therefore, we can consider the random iteration as $\pi_1 S^n(x, \omega)$, writing $\pi_1 : I \times \Omega \to I$ for the projection onto I. Under these settings, T.Morita ([11]) investigated the existence of invariant measures and their mixing properties. His method is also useful for our purpose.

Since T_y are m-nonsingular transformations, S is a nonsingular transformation on $(I \times \Omega, \mathcal{B}(I \times \Omega), m \times P)$. Therefore, we can define the Perron-Frobenius operator $\mathcal{L} : L^1(m \times P) \to L^1(m \times P)$ corresponding to S by

$$\int\int g \cdot \mathcal{L} f \, dmdP = \int\int f(x, \omega) g(S(x, \omega)) dmdP \tag{2.3}$$

for all $g \in L^\infty(m \times P)$, where $L^\infty(m \times P)$ denotes the Banach space of $(m \times P)$-essentially bounded functions. It is well known that the operator $\mathcal{L}$ is linear, positive and satisfies the various convenient properties ([5]). Similarly, we define the Perron-Frobenius operator $\Phi_y : L^1(m) \to L^1(m)$ corresponding to T_y .

Lemma 4.1 in [11] can be rewritten as follows:

Proposition 2.1 *(i) If $(\mathcal{L}f)(x, \omega) = \lambda f(x, \omega)$ for $|\lambda| = 1$, then f does not depend on ω. (ii)For any $f \in L^1(m)$, we have*

$$(\mathcal{L}f)(x, \omega) = \int (\Phi_y f)(x) p(dy) \qquad m \times P - a.e., \tag{2.4}$$

and hence $\mathcal{L}f \in L^1(m)$.

This proposition ensures us to consider $\mathcal{L}$ as an operator on $L^1(m)$, and we can treat our problem similarly to the case of a single transformation, given in [5].

For $f : [0,1] \to C$, we denote the total variation of f by $var(f)$. Let V be the set of functions $f \in L^1(m)$ which have the version $\tilde{f}$ with $var(\tilde{f}) < \infty$. V is a subspace of $L^1(m)$, but not closed. Put

$$\|f\|_V := \|f\|_m + v(f) \tag{2.5}$$

for $f \in V$,where $v(f) := \inf\{var(\tilde{f}) : \tilde{f} \text{ is a version of } f\}$. Then we can easily prove that $(V , \|\cdot\|_V)$ is a Banach space and

$$\|fg\|_V \leq 2\|f\|_V\|g\|_V \tag{2.6}$$

for $f \in V$ and $g \in V$ (cf.[5],[15]).

Definition 1 *We call that the skew-product S satisfies the condition (A), if its Perron-Frobenius operator $\mathcal{L}$ on $L^1(m)$ can be regarded as an operator on V, and if it fulfills the following:*

(A) For the Perron-Frobenius operator $\mathcal{L}$ of S, there exist a positive integer n_0 and real numbers $0 < \alpha < 1$, $0 < \beta < \infty$ such that

$$v(\mathcal{L}^{n_0}f) \leq \alpha v(f) + \beta\|f\|_m$$

for all $f \in V$.

A single m-nonsingular transformation T is also said to satisfy the condition (A), if the same property holds for its Perron-Frobenius operator Φ.

This condition (A) plays an essential role in our discussion. In order to get the concrete and sufficient condition for this, we need the followings.

Definition 2 *By $\mathcal{D}_\infty$ we denote the set of transformations T of $I := [0,1]$ satisfying:*

(1) There is a countable partition $\{ I_j : j = 1, 2, \cdots \}$ of I into such subintervals that for each $j = 1, 2, \ldots$ the restriction T_j of T to I_j is monotonic and can be extended to a C^2- function on the closure $\bar{I}_j$.

(2) The collection $\{ J_j := T(I_j); j = 1, 2, \ldots \}$ consists of a finite number of different subintervals.

(3) T satisfies $\gamma(T) := inf|T'(x)| > 0$.

The following inequality has been given by J. Rousseau-Egele ([15]).

Proposition 2.2 *Suppose that T belongs to $\mathcal{D}_\infty$, and let Φ be the Perron-Frobenius operator of (T,m). Then we have*

$$v(\Phi f) \leq \alpha(T)v(f) + \beta(T)\|f\|_m \tag{2.7}$$

for each $f \in V$, where we write $\alpha(T) := 2(\gamma(T))^{-1}$
and

$$\beta(T) := \sup\{m(J_j)^{-1} : j = 1, 2, ..\} + \sup_{1\leq j}\{(\sup_{x\in J_j} |(T_j^{-1})''(x)|)/(\inf_{x\in J_j} |((T_j)'(x)|)\}.$$

This proposition shows that if for $T \in \mathcal{D}_\infty$ there exists a positive integer n_0 for which T satisfies

$$\gamma(T^{n_0}) := \inf|(T^{n_0})'(x)| > 1,$$

then the condition (A) is satisfied. Note that β-transformations, unimodal linear transformations, α-continued fraction transformations and so on satisfy this condition (cf.[5]). For the random iterations we can show the following sufficient conditions.

Proposition 2.3 *Let the family $\{T_y : y \in Y\}$ be contained in $\mathcal{D}_\infty$. Suppose that the inequalities*

$$\int\int ... \int \alpha(T_{y_n} \circ T_{y_{n-1}} \circ ... \circ T_{y_1})p(dy_n)p(dy_{n-1})...p(dy_1) < 1 \tag{2.8}$$

and

$$\int\int ... \int \beta(T_{y_n} \circ T_{y_{n-1}} \circ ... \circ T_{y_1})p(dy_n)p(dy_{n-1})...p(dy_1) < \infty \tag{2.9}$$

hold for some n. Then, this family satisfies the condition (A).

Under the assumption (A) we can get the following proposition, which is similar to Proposition 1.2 in [5] (see also [11]).

Proposition 2.4 *Suppose that the skew product satisfies the condition (A). Then there exist a positive integer M and nonnegative functions $g_1(x)$, $g_2(x)$,$\cdots$,$g_M(x)$, belonging to V ,such that $\{ g_i > 0 \} \cap \{ g_j > 0 \} = \phi$ ($i \neq j$) , $d\mu_j \times dP := g_j dm \times dP$ ($j = 1, 2, \cdots, M$) are invariant probability measures under S and all other S-invariant $(m \times P)$-absolutely continuous probabilities are convex combinations of $\mu_j \times P's$. Moreover (S , $\mu_j \times P$) ($j = 1, 2, \cdots, M$) are ergodic.*

In the sequel we shall use the following notations. Let $\pi_1 : I \times \Omega \to I$ be the projection onto I. For a function $f(x)$ on [0,1] we denote

$$\mathcal{S}_n(f)(x,\omega) = \sum_{k=0}^{n-1} f(\pi_1 S^k(x,\omega)) = \sum_{k=0}^{n-1} f(T_{\omega_k} \circ T_{\omega_{k-1}} \circ ... \circ T_{\omega_1} x)$$

and $b_j = \mu_j(f) = \int f d\mu_j$, if it has the meaning for each $j = 1, 2, \cdots, M$. Since f and $b_j = \mu_j(f)$ appear at the same time, there will be no confusion. It is known that the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\pi_1 S^k(x,\omega)) = f^*(x) \ \ (m \times P - a.e.) \tag{2.10}$$

exists for all $f \in \cap_{j=1}^{M} L^\infty(\mu_j)$ ([5], [13]). Similarly to Lemma 1.3 in [5], we can get the following

Lemma 2.1 *Under the condition (A) we have that for any $f \in V$ the limit*

$$\lim_{n\to\infty} \int \left(\sum_{k=0}^{n-1} (f(\pi_1 S^k(x,\omega)) - b_j)/\sqrt{n} \right)^2 d\mu_j dP = \sigma_j \tag{2.11}$$

exists for each $j = 1, 2, \cdots, M$.

We define

$$F(b, \sigma^2; y) := (\frac{1}{\sigma\sqrt{2\pi}}) \int_{-\infty}^{y} \exp \frac{-(x-b)^2}{2\sigma^2} dx$$

for $\sigma^2 > 0$ and

$$F(b, 0; y) := \begin{cases} 1 & (b \le y) \\ 0 & (y < b) . \end{cases}$$

Under these notations we give our results.

Theorem 1 *(Central limit theorem of mixed type). Let the condition (A) for the family $\{ T_y \colon y \in Y \}$ be satisfied and ν be an m-absolutely continuous probability measure. Then, there exist nonnegative constants a_i with $\sum_{j=1}^{M} a_j = 1$ such that we have, for all functions $f \in V$,*

$$\lim_{n\to\infty} (\nu \times P)\{ \sum_{k=0}^{n-1} (f(T^k x) - f^*)/\sqrt{n} < y \} = \sum_{j=1}^{M} a_j F(0, \sigma_j; y)$$

at the continuity point of the left hand side. If we assume further that $\sigma_j > 0$ for all j and that $d\nu/dm \in V$, we have

$$\sup_y |(\nu \times P)\{ \sum_{k=0}^{n-1} (f(T^k x) - f^*)/\sqrt{n} < y \} - \sum_{j=1}^{M} a_j F(0, \sigma_j; y)| \le C/\sqrt{n}$$

for some some $C > 0$.

Remark 2.1 *If the parameter set Y consists of a single point, Theorem 1 implies the theorem in §1 , which is the improvement of Theorems 1 and 2 in [5]. Remark that the rate of convergence in Theorem 1 is the best possible and better than those of Theorems 1 and 2 of [5]. Note also that the number of different normal distributions in Theorem 1 is equal to the number of ergodic S-invariant measures.*

As corollaries to Theorem 1 , ordinary central limit theorems for 1-dimensional transformations, which are improvements of Theorems 3 and 4 in [5], are obtained.

Theorem 2 *If a single transformation T satisfies the condition (A) and has a unique m-absolutely continuous invariant probability measure μ , then, for any m – absolutely continuous probability measure ν and for any $f \in V$, there exists $\sigma^2 \geq 0$ such that we have*

$$\lim_{n\to\infty} \nu\{\sum_{k=0}^{n-1}(f(T^k x) - b)\sqrt{n} < y\} = F(0,\sigma^2; y)$$

at any continuity point of F, where $b = \int f d\mu$. In case $\sigma^2 \neq 0$ and $d\nu/dm \in V$, we have

$$\sup_y |\nu\{\sum_{k=0}^{n-1}(f(T^k x) - b)/\sqrt{n} < y\} - F(0,\sigma^2; y)| \leq C/\sqrt{n}$$

holds for some $C > 0$.

Theorem 3 *If T, defined on I, satisfies the condition (A) and μ is an m – absolutely continuous ergodic T – invariant probability measure, then for any $f \in V$ there exists $\sigma^2 \geq 0$ such that*

$$\lim_{n\to\infty} \mu\{\sum_{k=0}^{n-1}(f(T^k x) - b)/\sqrt{n} < y\} = F(0,\sigma^2; y)$$

at any continuity point of F, where $b = \int f d\mu$. In case $\sigma^2 \neq 0$,

$$\sup_y |\mu\{\sum_{k=0}^{n-1}(f(T^k x) - b)/\sqrt{n} < y\} - F(0,\sigma^2; y)| \leq C/\sqrt{n}$$

for some $C > 0$.

Morita's result (cf. [11], Lemma 5.4) ensures us to insist that, in the case of random iteration, the number M of different normal distributions in Theorem 1 becomes far smaller than in the case of a single transformation. Here, we give the following result, which insists that the ordinary central limit theorem for the random iterations is easier to hold than that for a single transformation. His result shows us the following:

Proposition 2.5 *Assume that there exists $a \in Y$ with $p(a) > 0$ such that T_a has an ergodic invariant measure $g(x)dm$. Suppose also that the skew product S has an ergodic invariant measure $f(x)dm$. Then either $\{g(x) > 0\} \subset \{f(x) > 0\}$ or $\{g(x) > 0\} \cap \{f(x) > 0\} = \phi$ holds.*

3 Applications and Examples

In this section we give some examples, using the above statements.

Example 1 *(Unimodal linear transformations) Let us define the so-called unimodal linear transformation by*

$$T_{(a,b)}(x) := \begin{cases} ax + \frac{(a+b-ab)}{b} & (0 \leq x \leq 1 - \frac{1}{b}) \\ -b(x-1) & (1 - \frac{1}{b} < x \leq 1), \end{cases}$$

where $a > 0$, $b > 1$ and $a + b - ab \geq 0$. In [6] and [7], Sh. Ito, S. Tanaka and H. Nakada investigated in detail how the behavior of T depends on parameter values (a, b). The mappings in question belong to $\mathcal{D}$, but do not always have the property (A): there exist the so-called window cases, in which (A) is not satisfied and $T_{(a,b)}$ does not have the m-absolutely continuous invariant probability measure.

Let $Y = \{\ y_i = (a_i, b_i)\colon\ i = 1, 2, ..., N\ \}$ and $p(y_i) =: p_i > 0$ with $\sum_{i=1}^{N} p_i = 1$. Since Y is a finite set and since $T_{y_i} := T_{(a_i,b_i)}$ belongs to $\mathcal{D}$, the fact $T''_{y_i}(x) \equiv 0$ shows that $\beta(T_{y_k} \circ T_{y_{k-1}} \circ ... \circ T_{y_1})$ is uniformly bounded in $(y_k, y_{k-1}, ..., y_1)$ for all fixed k. This implies that (2.9) holds for all $k > 0$. Therefore, if the inequality

$$\int\int ... \int \left(\gamma(T_{y_k} \circ T_{y_{k-1}} \circ ... \circ T_{y_1})\right)^{-1} p(dy_k)p(dy_{k-1})...p(dy_1) < 1 \qquad (3.1)$$

holds for some k, then we can derive from Proposition 2.3 that the property (A) is satisfied in this case. Hence, if (3.1)is fulfilled, we can apply Theorem 1; and we can get the central limit theorem of mixed type.

As is known in [7], $T_{(a,b)}$ has the unique m-absolutely continuous invariant probability, if and only if it has the property $\gamma(T^k_{(a,b)}) > 1$ for some $k > 0$. If we have, besides (3.1), $\gamma(T_{y_i})^k > 1$ for some $y_i \in Y$ and $k > 0$, we can apply Proposition 2.5 to get the ordinary central limit theorem.

More concretely, $\sum_{i=1}^{N} \left(\min\{a_i, b_i\}\right)^{-1} p_i < 1$ means that the inequality (2.8) is valid by putting $k = 1$, because we have $\gamma(T_{y_i})^{-1} = (\min\{a_i, b_i\})^{-1}$. So we can apply Theorem 4 in §1 and get the ordinary central limit theorem.

Example 2 *(Random time iterations) Let us define $T(x) := 2x$ (mod.1). We denote $Y :=$ { 0,1,2,... }, $p(n) =: p_n \geq$ 0 with $\sum_{n=0}^{\infty} p_n = 1$, and $T_y := T^y$. Under this setting, we easily have the following:*

Proposition 3.1 *Suppose that* $p(0) < 1$. *Let* ν *be an* m*-absolutely continuous probability. Then, for any* $f \in V$, *there exist* $\sigma^2 \geq 0$ *and* b *such that we have*

$$\lim_{n\to\infty} (\nu \times P)\{\sum_{k=0}^{n-1}(f(T^{(\omega_1+\omega_2+\ldots+\omega_k)}x) - b)/\sqrt{n} < y\} = F(0, \sigma^2; y)$$

at any continuity point of F. *In case* $\sigma^2 \neq 0$ *and* $d\nu/dm \in V$,

$$\sup_y |(\nu \times P)\{\sum_{k=0}^{n-1}(f(T^{-1}x) - b)/\sqrt{n} < y\} - F(0, \sigma^2; y)| \leq C/\sqrt{n}$$

holds.

Proof. Clearly, we have $\gamma(T^y) = 2^y$ and hence

$$\int \gamma(T^y)^{-1}p(dy) = \sum_{n=0}^{\infty} 2^{-n}p(n) < \sum_{n=1}^{\infty} p(n) = 1.$$

We also have $\beta(T^y) = 1$ for every $y \in Y$. Proposition 2.3 insists that the property (A) holds for this. It is well-known that the Lebesgue measure m itself is the unique invariant probability for T^n $(n > 0)$. Theorem 4 , therefore, shows our results.

Remark 3.1 *We clearly have the same results, putting* $T(x) := nx$ *(mod.1) for any positive integer* $n \geq 2$. *Moreover, if we think about the family of the so-called* $\beta -$ *transformation*$T(x) := \beta x$ *(mod.1,* $\beta > 1$ *) and put* $Y := \{0, 1, ..., N\}$, *then we can get the same results by changing the proof.*

Example 3 *(Dynamical systems with stochastic perturbations) Let* T *be a transformation belonging to* $\mathcal{D}_\infty$ *with, and* $\{\xi_n : n = 1, 2, ...\}$ *be a sequence of independent and identically distributed random variables. Assume that* $\mathrm{ess.}\sup|\xi_1|$ *is small enough to have* $T(x)+\xi_1 \in [0, 1]$ *(a.e.). Define* $Y = R$, $p(A) := Prob\{\xi_1(\omega) \in A\}$ *and* $T_y(x) := T(x)+y$. Ω *and* S *are defined as before. Then, regarding* $\xi_n = \omega_n$, *we have*

$$\pi_1 S(x, \omega) = T(x) + \xi_1,$$
$$\pi_1 S^2(x, \omega) = T(T(x) + \xi_1) + \xi_2,$$
$$\cdots\cdots\cdot$$

and

$$\pi_1 S^n(x, \omega) = T(T(T(\cdots(T(x) + \xi_1) + \xi_2) + ...) + \xi_n.$$

That is, $\pi_1 S^n(x, \omega) = x_n$, *if* $\{x_n : n = 0, 1, ...\}$ *is defined by*

$$x_n = T(x_{n-1}) + \xi_n, \ x_0 = x.$$

Therefore, we can regard this type of dynamical system with stochastic perturbations (cf. [8]) as a special case of our random iterations.

Clearly, $\gamma(T_y) = \gamma(T)$ *and* $\beta(T_y) = \beta(T)$. *Assuming* $\gamma(T) > 2$, *we can easily see that the central limit theorem of mixed type holds.*

Example 4 *Let us define*

$$T_1(x) := \begin{cases} (1/2)T(2x) & (0 \leq x < (1/2)) \\ (1/2)T(2(x-(1/2))) + (1/2) & ((1/2) \geq x < 1), \end{cases}$$

and

$$T_2(x) := \begin{cases} (1/3)T(3x) & (0 \leq x < (1/3)) \\ (2/3)T((3/2)(x-(1/3))) + (1/3) & ((1/3) \geq x < 1), \end{cases}$$

where $T(x) := 2x \pmod 1$. *Put* $Y := \{1,2\}$, $p_1 := P(\{1\}) > 0$ *and* $p_2 := P(\{2\}) > 0$ *with* $p_1 + p_2 = 1$. *Then it is clear that* T_1 *has two absolutely continuous ergodic probabilities ,whose supports are* $[0,1/2)$ *and* $[1/2,1]$. T_2 *also has two ergodic components,* $[1,1/3)$ *and* $[1/3,1]$. *Since* $\gamma(T_1) = \gamma(T_2) = 2$ $\beta(T_1) = 2$ *and* $\beta(T_2) = 3$ *clearly hold, the skew product* S *satisfies the condition* (A). *Moreover, Proposition 2.5 implies that the skew product* S *has a unique ergodic measure. Hence we have an ordinary central llimit theorem for this random iteration, though for* T_1 *and* T_2 *we have 2 limitting normal distributions.*

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