数理解析研究所講究録 1404 巻 2004 年 64-67

Low discrepancy sequences generated by dynamical system

Makoto Mori
Department of Mathematics, Nihon University
森 真 日本大学文理学部 数学科

1 Intorduction

A sequence $x_1, x_2, \ldots \in [0,1]$ is called uniformly distributed if

$$\lim_{n\to\infty} \frac{\#\{x_i \in J: i \leq n\}}{n} = |J|,$$

where J is an interval and $|J|$ is the Lebesgue measure. The discrepancy of this sequence is defined by

$$D_n = \sup_J \left| \frac{\#\{x_i \in J: i \leq n\}}{n} - |J| \right|.$$

It is well known that the order of the discrepancy D_n is greater than or equals to $\frac{\log n}{n}$. Thus, when the discrepancy is of order $\log n/n$, this sequence is called of low discrepancy.

The quasi-monte Carlo method is the approximation of the integration $\int_0^1 f(x)\,dx$ by

$$\frac{f(x_1) + \cdots + f(x_n)}{n}$$

using quasi random numbers $\{x_i\}$. It is well known that the error term is less than or equals to $D_n \times V(f)$ when f is of bounded variation, where $V(f)$ is the total variation of f. Hence, when we use a low discrepancy sequence, the approximation is best possible.

One of the most famous low discrepancy sequence is the van der Corput sequence using binary expansion. We will study this from the view point of dynamical system.

2 Notations

Let F be a piecewise linear transformation from $[0,1]$ into itself, and its slope satisfy $|F'(x)| \equiv \beta > 1$. Associated with this transformation, there exists a finite partition $\{\langle a\rangle\}_{a\in\mathcal{A}}$ of $[0,1]$, and on each $\langle a\rangle$ F is continuous and monotone. We call this $\mathcal{A}$ an alphabet. A finite sequence $w = a_1\cdots a_n$ of $\mathcal{A}$ a word, and define

$$\langle w\rangle = \cap_{i=1}^{n} F^{-i+1}(\langle a_i\rangle),$$
$$|w| = n.$$

We call this an expression of a dynamical system to a symbolic dynamics.

3 Construction of van der Corput sequence

For a point $x \in [0,1]$ and word w, we define wx by

$$wx \in \langle w\rangle, \quad F^{|w|}(wx) = x.$$

Note that wx does not always exist.

Now we fix a point x, and define an order in wx. For the first, we consider the natural order in $\mathcal{A}$. Define

- if $|w| < |w'|$, then $wx < w'x$.
- if $w = a_1\cdots a_n$, $w' = b_1\cdots b_n$ and $a_n\cdots a_1 < b_n\cdots b_1$ in the lexicographical order, then $wx < w'x$.

We call $\{wx\}$ a van der Corput sequence generated by the dynamical system. Actually, when $F(x) = 2x \pmod 1$ and $x = \frac{1}{2}$, we get the original van der Corput sequence.

4 Perron-Frobenius operator

Associated with the dynamical system, we define the Perron-Frobenius operator by

$$Pf(x) = \sum_{y:F(y)=x} f(y)|F'(y)|^{-1}.$$

It is well known that the ergodic properties of the dynamical system are determined by the spectra of this operator. For example,

1. The number of ergodic components of a dynamical system equals the dimension of the eigenspace associated with eigenvalue 1 of P. In our case, the dimension of the eigenspace equals 1.
2. The eigenfunction of the eigenvalue 1 of P gives the density function of the invariant probability measure.
3. P is originally a transformation on L^1, and all the points in the unit circle are eigenvalues with infinite multiplicity.

4. When we restrict P a transformation of the functions with bounded variation, then all the point in the circle with radius $1/\beta$ are eigenvalues with infinite multiplicity. We call $1/\beta$ the essential spectrum radius.

5. The second greatest eigenvalue of P in modulus gives the decay rate of convergence:

$$\int f(x)g(F^n(x))\,d\mu - \int f\,d\mu \times \int g\,d\mu,$$

where f is of bounded variation, g is a bounded measureable function and μ is the invariant porbability measure.

Theorem 1 *The eigenvalues of P in $|z| > 1/\beta$ coincide with the singularities of the dynamical zeta function*

$$\zeta(z) = \exp[\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{p=F^n(p)} |F^{n\prime}(p)|^{-1}].$$

For an interval J, we get

$$\begin{aligned} P^n 1_J(x) &= \sum_{y:F^n(y)=x} 1_J(y)|F^{n\prime}(x)|^{-1} \\ &= \sum_{|w|=n} 1_J(wx)\beta^{-n}. \end{aligned}$$

This equals the number of hits to J of the subsequence of our van der Corput sequence corresponding to the words with length n. On the contrary, when we express the density function of μ by ρ,

and the second greatest eigenvalue of P by η, as a very rough expression, we get

$$P^n 1_J(x) = |J|\rho(x) + O(\eta^n).$$

Since 1_J is of course of bounded variation, we know that η is greater than or equals to $1/\beta$. Thus we get

$$\#\{wx \in J\colon |w| = n\} = |J|\beta^n\rho(x) + O(1).$$

¿From this, we can estimate the discrepancy of our van der Corput sequence.

Definition 1 *We call an endpoint of $\langle a\rangle$ a Markov endpoint if the image of this point coincides with some endpoint of $\langle a\rangle$ $(a \in \mathcal{A})$.*

Theorem 2 *Let us denote by k the number of Markov endpoints of F. If the dynamical zeta function of F has no singularities in $1/\beta < |z| \leq 1$ except 1, we get that the discrepancy D_N of our van der Corput sequence equals of order*

$$\frac{(\log N)^{k+1}}{N}.$$

Expecially, if F is Markov, our van der Corput sequence is of low discrepancy.

5 Extension to higher dimensions

In the higher dimensional cases, we can conclude that if the essential spectrum radius of the Perron-Frobenius operator coincides with its reciprocal of the Jacobian, then we can get the similar results as one dimensional case. However, the estimate of the essential spectrum radius is crucial in general. Until now, we suceeded to costruct low discrepancy sequences for 2 and 3 dimensional cases using special transformations. We used binary expansions, so we believe these sequences

will be better than the Halton sequences, but we have not suceeded to construct higher dimensional cases with dimensions greater than 3 nor general theory for higher dimensional cases.

References

[1] Y.Ichikawa and M.Mori, Discrepancy of van der Corput sequences generated by piecewise linear transformations, preparing

[2] M.Mori, Fredholm determinant for piecewise linear transformations, *Osaka J. Math.*, vol.**27**, 81–116(1990)

[3] M.Mori, Low discrepancy sequences generated by piecewise linear maps, *Monte Carlo Methods and Applications*, vol.**4**, No.2, 141–162(1998)

[4] M.Mori, Discrepancy of sequences generated by piecewise monotone maps, *Monte Carlo Methods and Applications*, vol.**5**, No.1, 55–68(1999)

[5] M.Mori, Fredholm determinant for higher dimensional piecewise linear transformations, *Japanese J. Math.*, vol.**25**, No.2, 317–342(1999)

[6] M.Mori, Construction of two dimensional low discrepancy sequences, *Monte Carlo methods and Applications*, vol.**8**, No.2, 159-170(2002).

[7] S.Ninomiya, Constructing a new class of low-discrepancy sequences by using the β-adic transformation, *Math. Comput. Simul.*, vol.**47**, 405–420(1998)

[8] S.Ninomiya, On the discrepancy of the β-adic van der Corput sequence, *J. Math. Sci. Univ. Tokyo*, vol.**5**, 345–366(1998)