数理解析研究所講究録 1405 巻 2004 年 1-13

On non-topological multivortex condensates in the generalized self-dual Chern-Simons theory

Takashi Suzuki[1] (鈴木貴) and Futoshi Takahashi[2] (高橋太)

Abstract. In this note, we sketch a method of construction of non-topological multivortex (periodic) condensates in the generalized self-dual Chern-Simons gauge theory. We propose a simple proof of the desired property of the linearized operator needed in the construction in use of classical abstract analysis.

1. Introduction. In this note, we are concerned with the existence problem of periodic solutions to a quasilinear elliptic PDE on a periodic cell domain in $\mathbf{R}^2$, which arises from the generalized version of the self-dual Chern-Simons-Higgs gauge theory.

Let Ω be a fundamental cell domain in $\mathbf{R}^2$

$$\Omega = \{x = (x_1, x_2) \in \mathbf{R}^2 | -\frac{a}{2} \le x_1 \le \frac{a}{2}, -\frac{b}{2} \le x_2 \le \frac{b}{2}\}$$

generated by the linearly independent vectors $e_1 = (a, 0)$ and $e_2 = (0, b)$.

Let $p_1, \cdots, p_s \in \Omega \setminus \partial\Omega$ be s distinct vortex points with integer multiplicity $m_1, \cdots, m_s \in \mathbf{N}$ such that $\sum_{j=1}^{s} m_j = N$.

After the reduction first initiated by Taubes [10], the existence of stationary vortex solutions, called the *vortex condensates*, satisfying a suitable gauge-invariant periodicity, that is, 't Hooft boundary condition, is deduced from finding a solution $u = u_\kappa$ to

$$\begin{cases} -\Delta(u - e^u) = \dfrac{4}{\kappa^2} e^u (1 - e^u)^2 - 4\pi \displaystyle\sum_{j=1}^{s} m_j \delta_{p_j} \quad \text{in } \Omega, \\ u : \text{doubly periodic on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\kappa > 0$ is the Chern-Simons coupling parameter.

[1]Graduate School of Engineering Science, Osaka University
[2]Mathematical Institute, Tohoku University

Compare this to the more simpler problem:

$$\begin{cases} -\Delta u = \dfrac{4}{\kappa^2} e^u(1-e^u) - 4\pi \sum\limits_{j=1}^{s} m_j \delta_{p_j} & \text{in } \Omega, \\ u : \text{doubly periodic on } \partial\Omega. \end{cases} \tag{1.2}$$

(1.2) is a semilinear elliptic equation which arises in usual self-dual Chern-Simons-Higgs vortex theory [8] [9], and is studied in [1], [5], [6], [11], [12], [14] and so on. On the other hand, (1.1) is considered in [4] on the whole space $\mathbf{R}^2$ in the relation of generalized self-dual Chern-Simons-Higgs theory [2].

Following [12], we define the notion of topological and non-topological condensates as follows:

- ("topological" N-vortex condensates): $e^{u_\kappa} \to 1$ uniformly on compact sets of $\Omega \setminus \{p_1, \cdots, p_s\}$, as $\kappa \to 0$.
- ("non-topological" N-vortex condensates): $e^{u_\kappa} \to 0$ uniformly on compact sets of $\Omega \setminus \{p_1, \cdots, p_s\}$, as $\kappa \to 0$.

In pure Chern-Simons case (1.2), the existence of topological N-vortex condensates was completely solved affirmatively in [14].

Recently, non-topological N-vortex condensates with any given vortex points in Ω was obtained by Nolasco in [12] (see also [13]), which extends the former work of [14], [11], [6], [5].

But for the generalized Chern-Simons case (1.1), almost nothing is known on the existence of multivortex condensates by now.

Now, we recall the construction by Nolasco in pure Chern-Simons case. Sketch of Nolasco's argument to the equation (1.2) is as follows:

(1) Rewriting (1.2) by periodicity and scaling, we reduce the problem to find a solution $\widehat{u}$ to

$$\begin{cases} -\Delta \widehat{u} = \dfrac{1}{\varepsilon^2} e^{\widehat{u}}(1-e^{\widehat{u}}) - 4\pi \sum\limits_{n\in\mathbf{Z}^2} \sum\limits_{j=1}^{s} m_j \delta_{\widehat{p}_j^n} & \text{in } \mathbf{R}^2, \\ \widehat{u}(x + \frac{1}{\delta} e_i) = \widehat{u}(x) \ (i=1,2), \ \forall x \in \mathbf{R}^2, \end{cases}$$

where $\delta << \varepsilon$ so that $\kappa = 2\varepsilon\delta$, and $\widehat{p}_j^n := \frac{1}{\delta}(p_j + n_1 e_1 + n_2 e_2), n = (n_1, n_2) \in \mathbf{Z}^2, j = 1, \cdots, s$ is a scaled periodic lattice of vortex points.

(2) We make change of unknown functions from $\widehat{u}$ to z:

$$\widehat{u}(x) := \sum_{n\in\mathbf{Z}^2}\sum_{j=1}^{s}\widehat{\varphi}_j^n(x)\widehat{u}_j^n(x) + \varepsilon^2 z(z).$$

Here $\widehat{u}_j^n(x)$ is a radial solution to the equation

$$-\Delta\widehat{u}_j^n = \frac{1}{\varepsilon^2}e^{\widehat{u}_j^n}(1-e^{\widehat{u}_j^n}) - 4\pi m_j\delta_{\widehat{p}_j^n} \quad \text{in } \mathbf{R}^2$$

obtained by Chae-Imanuvilov [3]. $\widehat{u}_j^n$ is constructed as a perturbation of the radial function $\log\widehat{\rho}_j^n$, where

$$\widehat{\rho}_j^n(x) = \frac{8(m_j+1)^2|x-\widehat{p}_j^n|^{2m_j}}{(1+|x-\widehat{p}_j^n|^{2m_j+2})^2}$$

is a radial solution of the singular Liouville equation

$$-\Delta\log\widehat{\rho}_j^n = \widehat{\rho}_j^n - 4\pi m_j\delta_{\widehat{p}_j^n} \quad \text{in } \mathbf{R}^2.$$

Further, $\{\widehat{\varphi}_j^n\}$ is a partition of unity subordinating to a locally finite open covering of $\mathbf{R}^2$, which is defined according to the locations of $\{\widehat{p}_j^n\}$. Thus we define a suitable approximate solution by "gluing" together single vortex radial solutions in use of $\{\widehat{\varphi}_j^n\}$.

(3) Finally we solve the equation satisfied by z using Banach fixed point theorem argument.

In the last step, we need the invertibility of the linearized operator around the approximate solution as above in appropriate weighted spaces, and the bulk of the paper [12] consists of detailed studies of the linearized operator.

In construction of non-topological condensates for (1.1), we can follow the argument of Nolasco and we may "glue" together single radial vortex solutions recently obtained by Chae-Imanuvilov [4] to define an approximate solution. In the existence proof, we will need the invertibility of the linearized operator around the Chae-Imanuvilov solution, as in pure Chern-Simons case.

In this note, we present a simple proof of the invertibility of the linearized operator around the single vortex radial solutions by using perturbation theory of Fredholm operator and the stability of their indices

[7]. The same procedure can shorten and clarify some part of Nolasco's argument also in pure Chern-Simons case, see [13].

2. Preliminaries.

As in [12], we extend the equation (1.1) to all of $x \in \mathbf{R}^2$ by periodicity:

$$\begin{cases} -\Delta(\tilde{u} - e^{\tilde{u}}) = \dfrac{4}{\kappa^2} e^{\tilde{u}}(1 - e^{\tilde{u}})^2 - 4\pi \sum\limits_{n \in \mathbf{Z}^2} \sum\limits_{j=1}^{s} m_j \delta_{p_j^n} \quad \text{in } \mathbf{R}^2, \\ \tilde{u}(x + e_i) = \tilde{u}(x) \ (i = 1, 2) \quad \forall x \in \mathbf{R}^2. \end{cases} \tag{2.1}$$

Here, for $x \in \Omega$ and $n = (n_1, n_2) \in \mathbf{Z}^2$, $\tilde{u}(x) := u(x + n_1 e_1 + n_2 e_2)$ and the set of points $p_j^n := p_j + n_1 e_1 + n_2 e_2$, $j = 1, \cdots, s$ defines a periodic lattice of vortex points.

We introduce a scaling parameter δ and an approximation parameter ε, $\delta << \varepsilon$ so that $\kappa = 2\varepsilon\delta$ and set $\hat{u}(x) := \tilde{u}(\delta x)$, $\widehat{p_j^n} := \frac{1}{\delta} p_j^n$, $j = 1, \cdots, s$. Then the scaled function $\hat{u}$ satisfies

$$\begin{cases} -\Delta(\hat{u} - e^{\hat{u}}) = \dfrac{1}{\varepsilon^2} e^{\hat{u}}(1 - e^{\hat{u}})^2 - 4\pi \sum\limits_{n \in \mathbf{Z}^2} \sum\limits_{j=1}^{s} m_j \delta_{\widehat{p_j^n}} \quad \text{in } \mathbf{R}^2, \\ \hat{u}(x + \hat{e}_i) = \hat{u}(x) \ (i = 1, 2), \ \forall x \in \mathbf{R}^2, \end{cases} \tag{2.2}$$

where $\hat{e}_i := \frac{1}{\delta} e_i$. Note that the scaling $a^2 \delta(x) = \delta(\frac{x}{a})$, $a > 0, x \in \mathbf{R}^2$.

We want to solve (2.2) by the implicit function theorem argument. For this purpose, first we will define a suitable approximate solution, by "gluing" radially symmetric vortex solutions constructed recently by Chae-Imanuvilov [4].

The construction in [4] for the radially symmetric case is as follows.

Consider the following equation of N-vortex condensates at the origin,

$$\begin{cases} -\Delta(u - e^u) = \dfrac{1}{\varepsilon^2} e^u (1 - e^u)^2 - 4\pi N \delta_0 \quad \text{in } \mathbf{R}^2, \\ u(x) \to -\infty \quad \text{as } |x| \to \infty. \end{cases} \tag{2.3}$$

where the case $N = 0$ is allowed.

The key idea of Chae-Imanuvilov is to construct a solution to (2.3) as a perturbation of the radial function $\log \rho_N$, where

$$\rho_N(|x|) = \frac{8(N+1)^2 |x|^{2N}}{(1 + |x|^{2N+2})^2}. \tag{2.4}$$

Note that $\log \rho_N$ is a radial solution of the singular Liouville equation

$$-\Delta \log \rho_N = \rho_N - 4\pi N \delta_0 \quad \text{in } \mathbf{R}^2. \tag{2.5}$$

Further, we introduce an auxiliary function w_N which is a radial solution of

$$-\Delta w = \rho_N(x) w - \rho_N^2(x) - \frac{|\nabla \rho_N|^2}{\rho_N} \quad \text{in } \mathbf{R}^2. \tag{2.6}$$

For the solvability of (2.6) in the weighted space X below, see [4] Lemma 2.1.

Then we make a change of variable in the equation (2.3) as follows:

$$u(|x|) = \log(\varepsilon^2 \rho_N(|x|)) + \varepsilon^2 w_N(|x|) + \varepsilon^2 v(|x|). \tag{2.7}$$

If u is a radial solution of (2.3), new unknown function v must satisfy

$$\begin{aligned}
\Delta v &+ \frac{1}{\varepsilon^2}\rho_N e^{\varepsilon^2(v+w_N)} - \rho_N^2 e^{2\varepsilon^2(v+w_N)} - \frac{1}{\varepsilon^2}\rho_N + \Delta w_N \\
&- \Big[\varepsilon^2 \rho_N e^{\varepsilon^2(v+w_N)}(\Delta v + \Delta w_N) + 2\varepsilon^2 \nabla \rho_N \cdot \nabla(v+w_N) e^{\varepsilon^2(v+w_N)} \\
&+ \varepsilon^4 \rho_N |\nabla(v+w_N)|^2 e^{\varepsilon^2(v+w_N)} + \frac{|\nabla \rho_N|^2}{\rho_N} e^{\varepsilon^2(v+w_N)} - \rho_N^2 e^{\varepsilon^2(v+w_N)} \\
&+ \rho_N^2 e^{2\varepsilon^2(v+w_N)} - \varepsilon^2 \rho_N^3 e^{3\varepsilon^2(v+w_N)}\Big] = 0
\end{aligned} \tag{2.8}$$

in $\mathbf{R}^2$.

Following [3], we set the Hilbert spaces

$$\begin{aligned}
X &= \{u \in W^{2,2}_{loc}(\mathbf{R}^2) : \|u\|_X^2 = (u,u)_X < \infty\}, \\
Y &= \{u \in L^2(\mathbf{R}^2) : \|u\|_Y^2 = (u,u)_Y < \infty\},
\end{aligned} \tag{2.9}$$

where the inner products $(,)_X$ and $(,)_Y$ are defined as

$$\begin{aligned}
(u,v)_Y &= \int_{\mathbf{R}^2} (1+|x|^{2+\alpha}) uv\,dx, \\
(u,v)_X &= (\Delta u, \Delta v)_Y + \int_{\mathbf{R}^2} \frac{uv}{(1+|x|^{2+\alpha})} dx
\end{aligned} \tag{2.10}$$

with $\alpha \in (0, \frac{1}{2})$, and let X^r, Y^r be the space of radially symmetric functions in X, Y respectively.

The following facts are proved in [3] [4]:

Lemma 2.1. *We have*

$$\begin{aligned} |v(x)| &\le \|v\|_X(\log^+|x|+1), \quad \forall x\in\mathbf{R}^2,\ \forall v\in X. && (2.11)\\ |w_N(|x|)| &\le C(\log^+|x|+1), \quad \forall x\in\mathbf{R}^2, \\ w_N(|x|) &= -^{\exists}\tilde{C}\log^+|x|+o(\log|x|)\ (\tilde{C}>0) \quad \textit{as } |x|\to\infty. && (2.12)\end{aligned}$$

Now we define $P_N(v,\varepsilon)$ as the left hand side of (2.8). By (2.6), we can rearrange

$$\begin{aligned} P_N(v,\varepsilon) &= \Delta v+\frac{1}{\varepsilon^2}\rho_N e^{\varepsilon^2(v+w_N)}-\rho_N^2 e^{2\varepsilon^2(v+w_N)}-\frac{1}{\varepsilon^2}\rho_N \\ &- \rho_N w_N+\rho_N^2 \\ &- \Big[\varepsilon^2\rho_N e^{\varepsilon^2(v+w_N)}\Delta v \\ &+ \varepsilon^2\rho_N e^{\varepsilon^2(v+w_N)}\{-\rho_N w_N+\rho_N^2+\frac{|\nabla\rho_N|^2}{\rho_N}\} \\ &+ 2\varepsilon^2\nabla\rho_N\cdot\nabla(v+w_N)e^{\varepsilon^2(v+w_N)}+\varepsilon^4\rho_N|\nabla(v+w_N)|^2e^{\varepsilon^2(v+w_N)} \\ &+ \frac{|\nabla\rho_N|^2}{\rho_N}e^{\varepsilon^2(v+w_N)}-\frac{|\nabla\rho_N|^2}{\rho_N} \\ &- \rho_N^2e^{\varepsilon^2(v+w_N)}+\rho_N^2e^{2\varepsilon^2(v+w_N)}-\varepsilon^2\rho_N^3e^{3\varepsilon^2(v+w_N)}\Big] \qquad (2.13)\end{aligned}$$

for $v\in X^r$ and $\varepsilon>0$ small.

By lemma 2.1, we can check that $P_N(\cdot,\cdot)$ is well-defined and a smooth map from a suitable bounded set $\subset X^r\times\mathbf{R}$ to Y^r. Further if we set $P_N(0,0):=\lim_{\varepsilon\to+0}P_N(0,\varepsilon)$, we can check that $P_N(0,0)=0$ by the fact $e^{\varepsilon^2 w_N}=1+\varepsilon^2 w_N+o(\varepsilon^2)$ and (2.6).

Our goal in this section is to find a continuous mapping $\varepsilon\to v_\varepsilon$ such that $(v_\varepsilon,\varepsilon)$ is in the neighborhood of $(0,0)$ and solves the equation $P_N(v_\varepsilon,\varepsilon)=0$. For this purpose, we set the bounded linear operator

$$L_N^r=D_vP_N(0,0)=\Delta+\rho_N: X^r\to Y^r.$$

Chae-Imanuvilov proved that L_N^r is onto and $\mathrm{Ker}(L_N^r)$ is $span\{\phi_N\}$, where $\phi_N(|x|)=\frac{1-|x|^{2N+2}}{1+|x|^{2N+2}}$. Thus, if we set

$$H_N^r=\{u\in X^r:(u,\phi_N)_X=0\}=X^r/\mathrm{Ker}L_N^r,$$

we can apply the standard implicit function theorem in the neighborhood of the origin in H_N^r, therefore we obtain at least one solution $v_{\varepsilon,N}^*\in H_N^r$ satisfying $P_N(v_{\varepsilon,N}^*,\varepsilon)=0$. By continuity in ε and (2.12), we have

$$|v_{\varepsilon,N}^*(|x|)|\le C(\varepsilon)(\log^+|x|+1) \qquad (2.14)$$

with $C(\varepsilon) = \|v^*_{\varepsilon,N}\|_X \to 0$ as $\varepsilon \to 0$.

Returning back to (2.7),

$$u^*_{\varepsilon,N}(|x|) = \log(\varepsilon^2 \rho_N(|x|)) + \varepsilon^2 w_N(|x|) + \varepsilon^2 v^*_{\varepsilon,N}(|x|). \tag{2.15}$$

is a radial solution of (2.3), which is close to the solution of the Liouville equation.

Furthermore, we have the decay estimate

$$e^{u^*_{\varepsilon,N}} = O(\frac{1}{|x|^{2N+4+\beta(\varepsilon)}}) \quad \text{as } |x| \to \infty \tag{2.16}$$

where $\beta(\varepsilon) > 0, \beta(\varepsilon) \to 0$ as $\varepsilon \to 0$ ([4],(3.6)), so $u^*_{\varepsilon,N}$ is in fact the "non-topological" solution to (2.3).

3. Perturbation theorem of the Fredholm operator.

In this section, we will study the linear property of the radial vortex solutions constructed above. Consider the bounded linear operator

$$A_{\varepsilon,N} := D_v P_N(v^*_{\varepsilon,N}, \varepsilon) : X \to Y$$

for $\varepsilon > 0$ sufficiently small, which is explicitly calculated as

$$\begin{aligned} A_{\varepsilon,N} &= \Delta + \rho_N e^{\varepsilon^2(w_N + v^*_{\varepsilon,N})} - 2\varepsilon^2 \rho_N^2 e^{2\varepsilon^2(w_N+v^*_{\varepsilon,N})} \\ &- \Big[\varepsilon^2 \rho_N e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \Delta \\ &+ \varepsilon^4 \rho_N e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \Delta w_N \\ &+ \varepsilon^4 \rho_N e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \Delta v^*_{\varepsilon,N} \\ &+ 2\varepsilon^4 e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \nabla\rho_N \cdot \nabla(v^*_{\varepsilon,N} + w_N) \\ &+ 2\varepsilon^2 e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \nabla\rho_N \cdot \nabla \\ &+ \varepsilon^6 \rho_N e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} |\nabla(v^*_{\varepsilon,N} + w_N)|^2 \\ &+ 2\varepsilon^4 \rho_N e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \nabla(v^*_{\varepsilon,N} + w_N) \cdot \nabla \\ &- \varepsilon^2 (\rho_N^2 - \frac{|\nabla\rho_N|^2}{\rho_N}) e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} \\ &+ 2\varepsilon^2 \rho_N^2 e^{2\varepsilon^2(w_N+v^*_{\varepsilon,N})} \\ &- 3\,\varepsilon^4 \rho_N^3 e^{3\varepsilon^2(w_N+v^*_{\varepsilon,N})}\Big]. \end{aligned} \tag{3.1}$$

Since $e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} = 1 + \varepsilon^2 w_N + o(\varepsilon^2)$ by (2.12) and (2.14), we have

$$A_{\varepsilon,N} = L_N + \varepsilon^2 B_N + o(\varepsilon^2) \tag{3.2}$$

where L_N and B_N is a bounded linear operator from X to Y defined as

$$L_N := \Delta + \rho_N, \tag{3.3}$$

$$B_N := \rho_N w_N - 2\rho_N^2 - \left(2\nabla\rho_N \cdot \nabla + \rho_N^2 + \frac{|\nabla\rho_N|^2}{\rho_N} + \rho_N\Delta\right). \tag{3.4}$$

We collect important facts about the operators L_N and B_N.

Lemma 3.1.([3]Lemma 2.4, Proposition 2.2, [4]Lemma 2.5) *We have*

(1) $KerL_N = span\{\phi_N, \phi_N^+, \phi_N^-\}$ where

$$\begin{aligned}\phi_N(x) &= \frac{1-|x|^{2N+2}}{1+|x|^{2N+2}},\\ \phi_N^+(x) &= \frac{|x|^{N+1}\cos(N+1)\theta}{1+|x|^{2N+2}},\\ \phi_N^-(x) &= \frac{|x|^{N+1}\sin(N+1)\theta}{1+|x|^{2N+2}}.\end{aligned} \tag{3.5}$$

(2) $ImL_N = \{f \in Y : \int_{\mathbf{R}^2} f\phi_{\pm,N}dx = 0\}$.

(3) $I_N^\pm := (B_N\phi_N^\pm, \phi_N^\pm)_{L^2(\mathbf{R}^2)} < 0 \quad (N \in \mathbf{N}), \quad I_0^\pm = 0$.

On the other hand, it is easy to see

$$(B_N\phi_N^+, \phi_N^-)_{L^2(\mathbf{R}^2)} = (B_N\phi_N, \phi_N^\pm)_{L^2(\mathbf{R}^2)} = 0$$

for $N \in \mathbf{N} \cup \{0\}$.

Now, we set the Hilbert space

$$H_N := \{u \in X : (u, \phi_N)_X = 0\}. \tag{3.6}$$

Note that $\phi_N^\pm \in H_N$ and we have the orthogonal decomposition $X = H_N \oplus H_N^\perp$, where $H_N^\perp = span\{\phi_N\}$.

As in [12], we can prove the following lemma, which says that the operator $A_{\varepsilon,N}|_{H_N} : H_N \to Y$ is injective for $N \neq 0$.

Lemma 3.2. *Assume $N \geq 1$. Then there exists $\varepsilon_0 > 0$ and a constant $C > 0$ (independent of ε) such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $v \in H_N$ we have*

$$\|A_{\varepsilon,N}v\|_Y \geq C\varepsilon^2\|v\|_X. \tag{3.7}$$

For completeness, we show the Proof:

Arguing by contradiction, we suppose ${}^{\exists}\varepsilon_n \to 0$ $(n \to \infty)$,${}^{\exists} v_n \in H_N$ with $\|v_n\|_X = 1$ such that $\frac{1}{\varepsilon_n^2}\|A_{\varepsilon,N}v_n\|_Y \to 0$ as $n \to \infty$. Since X is Hilbert and H_N is weakly closed, there exists a subsequence of v_n (without changing the notation) and $\bar{v} \in H_N$ such that $v_n \rightharpoonup \bar{v}$ weakly in X. Then, it is easy to check that $A_{\varepsilon_n,N}v_n \rightharpoonup L_N\bar{v}$ weakly in Y. So by our assumption that $\|A_{\varepsilon_n,N}v_n\| = o(\varepsilon_n^2)$, we have $L_N\bar{v} = 0$, hence $\bar{v} \in \mathrm{Ker}L_N \cap H_N$.

By Lemma 3.1, we can write $\bar{v} = C_+\phi_N^+ + C_-\phi_N^-$ for some constants C_+, C_-.

We claim that $v_n \to \bar{v}$ strongly in X, so $\bar{v} \not\equiv 0$ and $C_+^2 + C_-^2 \neq 0$.

Indeed, by (3.2) and our assumption, we have $\|L_N(v_n - \bar{v})\|_Y = o(1)$ which leads to

$$\|\Delta(v_n - \bar{v})\|_Y \leq \|\rho_N(v_n - \bar{v})\|_Y + o(1)$$

as $n \to \infty$.

On the other hand, by Rellich-Kondrachev theorem and the growth estimate (2.12) for functions in X, we see the linear bounded operator $K_1 := \rho_N : X \to Y$ is a compact operator. So, above inequality leads to $\Delta v_n \to \Delta\bar{v}$ strongly in Y. Similarly, $K_2 := \frac{1}{(1+|x|^{2+\alpha})^{1/2})} : X \to L^2(\mathbf{R}^2)$ is also compact, so we conclude $v_n \to \bar{v}$ strongly in X and $\|\bar{v}\|_X = 1$.

From this, we reach a contradiction when $N \neq 0$ as follows: Set $\bar{w} = \frac{\bar{v}}{1+|x|^{2+\alpha}} \in Y$. By our assumption,

$$\frac{1}{\varepsilon_n^2}|(A_{\varepsilon_n,N}v_n, \bar{w})_Y| \leq \frac{1}{\varepsilon_n^2}\|A_{\varepsilon_n,N}v_n\|_Y\|\bar{w}\|_Y \to 0.$$

On the other hand, by (3.2) we have

$$\frac{1}{\varepsilon_n^2}(A_{\varepsilon_n,N}v_n, \bar{w})_Y = \frac{1}{\varepsilon_n^2}(L_N v_n, \bar{w})_Y + (B_N v_n, \bar{w})_Y + o(1).$$

Now, for any $n \in \mathbf{N}$, we know by Lemma 3.1 (2) that

$$\frac{1}{\varepsilon_n^2}(L_N v_n, \bar{w})_Y = \frac{1}{\varepsilon_n^2}(L_N v_n, C_+\phi_N^+ + C_-\phi_N^-)_{L^2(\mathbf{R}^2)} \equiv 0,$$

so we obtain $(B_N v_n, \bar{w})_Y = o(1)$.

Letting $n \to \infty$ and noting $v_n \to \bar{v}$ strongly in X, we have

$$(B_N\bar{v}, \bar{w})_Y = (B_N\bar{v}, \bar{v})_{L^2(\mathbf{R}^2)} = 0.$$

However, it holds that

$$\begin{aligned}
& (B_N\bar{v}, \bar{v})_{L^2(\mathbf{R}^2)} \\
= \; & C_+^2(B_N\phi_N^+, \phi_N^+)_{L^2} + C_-^2(B_N\phi_N^-, \phi_N^-)_{L^2} + 2C_+C_-(B_N\phi_N^+, \phi_N^-)_{L^2} \\
= \; & C_+^2 I_N^+ + C_-^2 I_N^- < 0
\end{aligned}$$

by Lemma 3.1 (3) when $N \neq 0$. This contradiction proves the Lemma. □

In pure Chern-Simons case, Nolasco studied the corresponding operator $A^0_{\varepsilon,N}$ which is similar to $A_{\varepsilon,N}$, and proved that $\mathrm{Im}(A^0_{\varepsilon,N}|_{H_N})$ is closed in Y ([12]Lemma 4.3) and $A^0_{\varepsilon,N}|_{H_N}$ is surjective if $\varepsilon > 0$ sufficiently small ([12]Lemma 4.4). As a consequence of these lemmas, she obtained the invertibility of $A^0_{\varepsilon,N}|_{H_N}$ and the upper bound of the operator norm of $(A^0_{\varepsilon,N}|_{H_N})^{-1}$, at least when $N \geq 1$.

In this note, we present a short proof of the facts above for the operator $A_{\varepsilon,N}$, by using the perturbation theory of Fredholm operator by Gohberg and Krein.

We say that a bounded linear operator $T : E \to F$ (E, F Banach spaces, $Dom(A) = E$) is *Fredholm* operator if $\mathrm{Ker}(T)$ is finite dimensional in E and $\mathrm{Im}(T)$ is closed and has finite codimension in F. In this case we define the *index* of T as $\mathrm{Index}(T) := \dim \mathrm{Ker}(T) - \mathrm{codimIm}(T)$.

Theorem 3.3.(Gohberg-Krein [7]) *Let E, F are Banach spaces. Assume the bounded linear operator $T : E \to F$ is a Fredholm operator. Then there exists $\gamma > 0$ such that for any bounded linear operator $B : E \to F$ with $\|B\| < \gamma$, the operator $T + B$ is also Fredholm, and the followings hold:*

(1) $\dim \mathit{Ker}(T + B) \leq \dim \mathit{Ker}(T)$,

(2) $\mathit{codimIm}(T + B) \leq \mathit{codimIm}(T)$,

(3) $\mathit{Index}(T + B) = \mathit{Index}(T)$.

By (3.2), $B_{\varepsilon.N} := A_{\varepsilon,N} - L_N$ is a bounded linear operator from X to Y and $\|B_{\varepsilon,N}\|$ (operator norm) $\to 0$ as $\varepsilon \to 0$.

Now, on the operator $L_N|_{H_N} : H_N \to Y$, we have

- $\mathrm{Ker}(L_N|_{H_N}) = \mathrm{Ker}(L_N) \cap H_N = span\{\phi_N^+, \phi_N^-\}$ by Lemma 3.1 (1). Therefore we have

$$\dim \mathrm{Ker}(L_N|_{H_N}) = 2.$$

- We have

$$L_N(X) = L_N(H_N \oplus H_N^{\perp}) = L_N(H_N \oplus span\{\phi_N\}) = L_N(H_N) \oplus \{0\},$$

and hence it holds that

$$\mathrm{Im}(L_N|_{H_N}) = \mathrm{Im}(L_N) = \{f \in Y : (f, \bar{\phi}_{\pm,N})_{L^2(\mathbf{R}^2)} = 0\}$$

by Lemma 3.1 (2). In particular, $\mathrm{Im}(L_N|_{H_N})$ is closed in Y and

$$\mathrm{codim}\mathrm{Im}(L_N|_{H_N}) = 2.$$

From above we know $L_N|_{H_N} : H_N \to Y$ is a Fredholm operator with Fredholm index $= 0$.

Applying Theorem 3.3 for $E = H_N, F = Y, T = L_N|_{H_N}, B = B_{\varepsilon,N}|_{H_N}$, we conclude that

$$A_{\varepsilon,N}|_{H_N} = L_N|_{H_N} + B_{\varepsilon,N}|_{H_N}$$

is also a Fredholm operator with $\mathrm{Index}(A_{\varepsilon,N}|_{H_N}) = 0$. Lemma 3.2 guarantees that $A_{\varepsilon,N}|_{H_N}$ is injective when $N \geq 1$ and $\varepsilon > 0$ small, we get that $A_{\varepsilon,N}|_{H_N}$ is also surjective when $N \geq 1$ and $\varepsilon > 0$ small.

Now, we conclude the following.

Theorem 3.4. *Assume $N \geq 1$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the operator $A_{\varepsilon,N}|_{H_N} : H_N \to Y$ is invertible. Moreover, there exists a constant $C > 0$ (independent of ε) such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $u \in Y$ we have*

$$\|(A_{\varepsilon,N}|_{H_N})^{-1}u\|_X \leq \frac{C}{\varepsilon^2}\|u\|_Y.$$

Using these estimates, we can follow the method of Nolasco, which succeeded in constructing multivortex condensates in pure Chern-Simons gauge theory. Detailed "gluing" process and the construction of non-topological multivortex condensates in the generalized Chern-Simons theory will be appeared elsewhere.

参考文献

[1] L. Caffarelli, and Y. Yang. *Vortex condensation in the Chern-Simons-Higgs model: an existence theorem,* Comm. Math. Phys. **168** (1995), 321-366.

[2] J. Burzlaff, A. Chakrabarti, and D. H. Tchrakian. *Generalized self-dual Chern-Simons vortices,* Phys. Lett. B. **293** (1992), 127-131.

[3] D. Chae, and O. Imanuvilov. *The existence of non-topological multivortex solutions in the relativistic selfdual Chern-Simons theory,* Comm. Math. Phys. **215** (2000), 119-142.

[4] D. Chae, and O. Imanuvilov. *Non-topological solutions in the generalized self-dual Chern-Simons-Higgs theory,* Calc. Var. **16** (2003), 47-61.

[5] W. Ding, J. Jost, J. Li, X. Peng and G. Wang. *Self duality equations for Ginzburg-Landau and Seiberg-Witten type functional with* 6^{th} *order potential,* Comm. Math. Phys. **217** (2001), 383-407.

[6] W. Ding, J. Jost, J. Li, and G. Wang. *An analysis of the two-vortex case in the Chern-Simons-Higgs model,* Calc. Var. and PDE **7** (1998) 87-97.

[7] I. C. Gohberg, and M.G. Krein. *The basic propositions on defect numbers, root numbers and indices of linear operators,* Uspehi Mat. Nauk. **12** (1957) 43-118. English translate: Amer. Math. Soc. Translations, Ser.2, **13** (1960) 185-264.

[8] J. Hong, Y. Kim, and P. Y. Pac. *Multivortex solutions of the abelian Chern-Simons-Higgs theory,* Phys. Rev. Lett. **64** (1990) 2230-2233.

[9] R. Jackiw, and E. J. Weinberg. *Self-dual Chern-Simons vortices,* Phys. Rev. Lett. **64** (1990) 2234-2237.

[10] A. Jaffe, and C. Taubes. *Vortices and Monopoles,* Birkhäuser, Boston (1980)

[11] M. Nolasco, and G. Tarantello. *Double vortex condensates in the Chern-Simons-Higgs theory,* Calc. Var. and PDE **9** (1999) 31-94.

[12] M. Nolasco. *Non-topological N-vortex condensates for the self-dual Chern-Simons theory,* Comm. Pure and Appl. Math. **56** (2003) 1752-1780.

[13] T. Suzuki, and F. Takahashi. *Existence of the nontopological condensate in self-dual Chern-Simons gauge theory,* Adv. Diff. Eq. **9** no. 5-6 (2004) 539-562.

[14] G. Tarantello. *Multiple condensate solutions for the Chern-Simons-Higgs theory,* J. Math. Phys. **37** (1996) 3769-3796.