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Upper bound of the best constant of the Trudinger-Moser inequality and its application to the Gagliardo-Nirenberg inequality

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We consider the best constant of the Trudinger-Moser inequality in $\mathbb{R}^n$. Let Ω be an arbitrary domain in $\mathbb{R}^n$. It is well known that the Sobolev space $H_0^{n/p,p}(\Omega)$, $1<p<\infty$, is continuously embedded into $L^q(\Omega)$ for all q with $p \leqq q<\infty$. However, we cannot take $q=\infty$ in such an embedding. For bounded domains Ω, Trudinger [18] treated the case $p=n(\geqq 2)$, i.e., $H_0^{1,n}(\Omega)$ and proved that there are two constants α and C such that

$$\|\exp(\alpha|u|^{n'})\|_{L^1(\Omega)} \leqq C|\Omega| \tag{0.1}$$

holds for all $u \in H_0^{1,n}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} \leqq 1$. Here and hereafter p' represents the Hölder conjugate exponent of p, i.e., $p'=p/(p-1)$. Moser [9] gave the optimal constant for α in (0.1), which shows that one cannot take α greater than $1/(n^{n-2}\omega_n^{n-1})$, where ω_n is the volume of the unit n-ball, that is, $\omega_n:=|B_1|=2\pi^{n/2}/(n\Gamma(n/2))$ (Γ : the gamma function). Adams [2] generalized Moser's result to the case $H_0^{m,n/m}(\Omega)$ for positive integers $m<n$ and obtained the sharp constant corresponding to (0.1).

When $\Omega=\mathbb{R}^n$, Ogawa [10] and Ogawa-Ozawa [11] treated the Hilbert space $H^{n/2,2}(\mathbb{R}^n)$ and then Ozawa [14] gave the following general embedding theorem in the Sobolev space $H^{n/p,p}(\mathbb{R}^n)$ of the fractional derivatives which states that

$$\|\Phi_p(\alpha|u|^{p'})\|_{L^1(\mathbb{R}^n)} \leqq C\|u\|_{L^p(\mathbb{R}^n)}^p \tag{0.2}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leqq 1$, where

$$\Phi_p(\xi) = \exp(\xi) - \sum_{j=0}^{j_p-1} \frac{\xi^j}{j!} = \sum_{j=j_p}^{\infty} \frac{\xi^j}{j!}, \quad j_p := \min\{j \in \mathbb{N} \mid j \geqq p-1\}.$$

The advantage of (0.2) gives the scale invariant form. Concerning the sharp constant for α in (0.2), Adachi-Tanaka [1] proved a similar result to Moser's in $H^{1,n}(\mathbb{R}^n)$.

Our purpose is to generalize Adachi-Tanaka's result to the space $H^{n/p,p}(\mathbb{R}^n)$ of the fractional derivatives. We show an upper bound of the constant α in (0.2). Indeed, the following theorem holds :

Theorem 0.1. *Let $2 \leqq p < \infty$. Then, for every $\alpha \in (A_p, \infty)$, there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{n/p,p}(\mathbb{R}^n) \setminus \{0\}$ with $\|(-\Delta)^{n/(2p)}u_k\|_{L^p(\mathbb{R}^n)} \leqq 1$ such that*

$$\frac{\|\Phi_p(\alpha|u_k|^{p'})\|_{L^1(\mathbb{R}^n)}}{\|u_k\|^p_{L^p(\mathbb{R}^n)}} \to \infty \ \text{as} \ k \to \infty,$$

where A_p is defined by

$$A_p := \frac{1}{\omega_n}\left[\frac{\pi^{n/2}2^{n/p}\Gamma(n/(2p))}{\Gamma(n/(2p'))}\right]^{p'}. \tag{0.3}$$

Remark . *Let α_p be the best constant of* (0.2) *, i.e.,*

$$\alpha_p := \sup\{\alpha > 0 \mid \text{The inequality (0.2) holds with some constant } C.\}.$$

Then Theorem 0.1 *implies that $\alpha_p \leqq A_p$ for $2 \leqq p < \infty$.*

Next, if we give a similar type estimate to (0.2) by taking another normalization such as $\|(I - \Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leqq 1$, then we can cover all $1 < p < \infty$. Moreover, when $p = 2$, it turns out that our constant A_2 of (0.3) is optimal. To state our second result, let us recall the rearrangement f^* of the measurable funcition f on $\mathbb{R}^n$. For detail, see Section 2 (Stein-Weiss [16]). We denote by f^{**} the average function of f^*, i.e.,

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau \quad \text{for } t > 0.$$

Our theorem now reads :

Theorem 0.2. *Let* $1 < p < \infty$ *and* A_p *be as in* (0.3).
(i) *For every* $\alpha \in (A_p, \infty)$, *there exists a sequence* $\{u_k\}_{k=1}^{\infty} \subset H^{n/p,p}(\mathbb{R}^n)$ *with* $\|(I-\Delta)^{n/(2p)}u_k\|_{L^p(\mathbb{R}^n)} \leqq 1$ *such that*

$$\|\Phi_p(\alpha|u_k|^{p'})\|_{L^1(\mathbb{R}^n)} \to \infty \text{ as } k \to \infty.$$

(ii) *We define* A_p^* *by*

$$A_p^* = A_p / B_p^{1/(p-1)},$$

where

$$B_p := (p-1)^p \sup \left\{ \int_0^\infty (f^{**}(t) - f^*(t))^p dt \mid \|f\|_{L^p(\mathbb{R}^n)} \leqq 1 \right\}.$$

Then for every $\alpha \in (0, A_p^*)$, *there exists a positive constant* C *depending only on* p *and* α *such that*

$$\|\Phi_p(\alpha|u|^{p'})\|_{L^1(\mathbb{R}^n)} \leqq C \tag{0.4}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ *with* $\|(I-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leqq 1$.

Remark . *Later, we shall show that*

$$1 \leqq B_p \leqq p^p - (p-1)^p \quad \text{for } 1 < p < \infty.$$

In particular, for $2 \leqq p < \infty$, *there holds*

$$B_p = (p-1)^{p-1}. \tag{0.5}$$

In any case, we obtain $A_p^* \leqq A_p$ *for* $1 < p < \infty$.

Since it follows from (0.5) that $B_2 = 1$, we see that $A_2 = A_2^* = (2\pi)^n/\omega_n$ is the best constant of (0.4). Hence, the following corollary holds :

Corollary 0.1. (i) *For every* $\alpha \in ((2\pi)^n/\omega_n, \infty)$, *there exists a sequence* $\{u_k\}_{k=1}^{\infty} \subset H^{n/2,2}(\mathbb{R}^n)$ *with* $\|(I-\Delta)^{n/4}u_k\|_{L^2(\mathbb{R}^n)} \leqq 1$ *such that*

$$\|\Phi_2(\alpha|u_k|^2)\|_{L^1(\mathbb{R}^n)} \to \infty \text{ as } k \to \infty.$$

(ii) *For every* $\alpha \in (0, (2\pi)^n/\omega_n)$, *there exists a positive constant* C *depending only on* α *such that*

$$\|\Phi_2(\alpha|u|^2)\|_{L^1(\mathbb{R}^n)} \leqq C \tag{0.6}$$

holds for all $u \in H^{n/2,2}(\mathbb{R}^n)$ *with* $\|(I-\Delta)^{n/4}u\|_{L^2(\mathbb{R}^n)} \leqq 1$.

It seems to be an interesting question whether or not (0.6) does hold for $\alpha = (2\pi)^n/\omega_n$.

Next, we consider the Gagliardo-Nirenberg interpolation inequality which is closely related to the Trudinger-Moser inequality. Ozawa [14] proved that for $1 < p < \infty$ there is a constant M depending only on p such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq M q^{1/p'} \|u\|_{L^p(\mathbb{R}^n)}^{p/q} \|(-\Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \tag{0.7}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all $q \in [p, \infty)$. Ozawa [13],[14] also showed the fact that (0.2) and (0.7) are equivalent and he gave the relation between α in (0.2) and M in (0.7). Combining his formula with our result, we obtain an estimate of M from below. Indeed, there holds the following theorem :

Theorem 0.3. *Let $2 \leqq p < \infty$. We define M_p and m_p as follows.*

$$M_p := \inf\{M > 0 \mid \text{The inequality (0.7) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [p, \infty).\},$$
$$m_p := \inf\{M > 0 \mid \text{There exists } q_0 \in [p, \infty) \text{ such that the inequality (0.7) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [q_0, \infty).\}.$$

Then there holds

$$M_p \geqq m_p \geqq \frac{1}{(p' e A_p)^{1/p'}}.$$

Since Ozawa [13],[14] gave the relation between the constants α in (0.2) and M in (0.7), we obtain a lower bound of the best constant for the Sobolev inequality in the critical exponent :

Theorem 0.4. *Let $1 < p < \infty$.*
(i) *For every $M > (p' e A_p^*)^{-1/p'}$, there exists $q_0 \in [p, \infty)$ depending only on p and M such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq M q^{1/p'} \|(I - \Delta)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \tag{0.8}$$

holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ and for all $q \in [q_0, \infty)$.
(ii) *We define $\overline{M}_p$ and $\overline{m}_p$ as follows.*

$$\overline{M}_p := \inf\{M > 0 \mid \text{The inequality (0.8) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [p, \infty).\},$$
$$\overline{m}_p := \inf\{M > 0 \mid \text{There exists } q_0 \in [p, \infty) \text{ such that the inequality (0.8) holds for all } u \in H^{n/p,p}(\mathbb{R}^n) \text{ and for all } q \in [q_0, \infty).\}.$$

Then there holds

$$\overline{M}_p \geqq \overline{m}_p \geqq \frac{1}{(p'eA_p)^{1/p'}}.$$

Since we have obtained $A_2 = A_2^*$ for $p = 2$, we see that

$$\frac{1}{\sqrt{2eA_2}} = \frac{1}{\sqrt{2eA_2^*}} = \sqrt{\frac{\omega_n}{2^{n+1}e\pi^n}}.$$

Hence, the above theorem gives the best constant for (0.8). Indeed, we have the following corollary :

Corollary 0.2. (i) *For every* $M > \sqrt{\omega_n/(2^{n+1}e\pi^n)}$, *there exists* $q_0 \in [2, \infty)$ *such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \leqq Mq^{1/2}\|(I - \Delta)^{n/4}u\|_{L^2(\mathbb{R}^n)}$$

holds for all $u \in H^{n/2,2}(\mathbb{R}^n)$ *and for all* $q \in [q_0, \infty)$.

(ii) *For every* $0 < M < \sqrt{\omega_n/(2^{n+1}e\pi^n)}$ *and* $q \in [2, \infty)$, *there exist* $q_0 \in [q, \infty)$ *and* $u_0 \in H^{n/2,2}(\mathbb{R}^n)$ *such that*

$$\|u_0\|_{L^{q_0}(\mathbb{R}^n)} > Mq_0^{1/2}\|(I - \Delta)^{n/4}u_0\|_{L^2(\mathbb{R}^n)}$$

holds.

To prove our theorems, by means of the Riesz and the Bessel potentials, we first reduce the Trudinger-Moser inequality to some equivalent form of the fractional integral. The technique of symmetric decreasing rearrangement plays an important role for the estimate of fractional integrals in $\mathbb{R}^n$. To this end, we make use of O'Neil's result [12] on the rearrangement of the convolution of functions. Such a procedure is similar to Adams [2]. First, we shall show that for every $\alpha \in (0, A_p^*)$, there exists a positive constant C depending only on p and α such that (0.4) holds for all $u \in H^{n/p,p}(\mathbb{R}^n)$ with $\|(I-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)} \leqq 1$. On the other hand, we shall show that the constant α holding (0.2) and (0.4) in $\mathbb{R}^n$ can be also available for the corresponding inequality in bounded domains. Since Adams [2] gave the sharp constant α in the corresponding inequality to (0.1), we obtain an upper bound A_p as in (0.3). For general p, we have $A_p^* \leqq A_p$. In particular, for $p = 2$, there holds $A_2^* = A_2$, which provides us the best constant of (0.4).

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