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# The Generalized Fermat-Steiner Problem with Free Ends

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## 1 Introduction

Let $G = (e_1, e_2, \ldots, e_n)$ be a conneted graph such that the degree of its vertices are all 3 except for the end points. In other words, $G$ is a network with triple junctions. For a given region $\Omega \subset \mathbb{R}^2$, a set of line segment $\Gamma_G$ is called *admissible* for $G$ if $\Gamma_G$ is isomorphic to $G$ and all the end points of $\Gamma_G$ are on $\partial\Omega$.

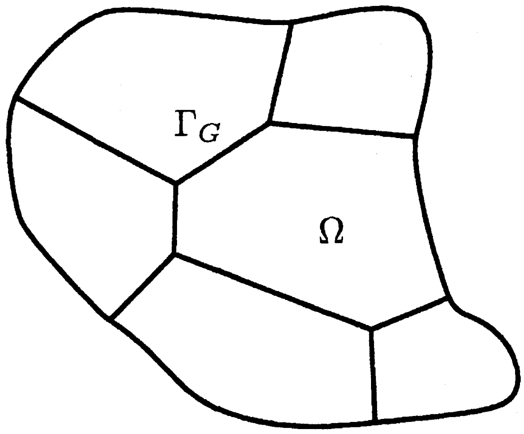


**Figure 1:** An example of $\Gamma_G$.

We assign a positive number $\sigma_i$ to each edge $e_i$, which represents "surface energy." Denote by $\gamma_i$ $(i = 1, 2, \ldots, n)$ component segments of $\Gamma_i$ which correspond to $e_i$. In this study we are concerned with the following problem:

**Problem P.** Find an admissible $\Gamma_G$ for $G$ that minimizes

$$E[\Gamma_G] = \sum_{i=1}^{n} \sigma_i |\gamma_i|, \tag{1}$$

where $|\gamma_i|$ denote the lengths of $\gamma_i$.

This problem arises in grain boundary motions of anealing pure metal. Critical points of $E[\Gamma_G]$ represent stationary states of a curvature-driven motion, which models the grain boundary motions. A curvature-driven motion with a triple junction has been introduced by Mullins [6]. Later, the motion was derived formally by Bronsard and Reitich [1] as the singular limit of a vector-valued Allen-Cahn equation. Bronsard and Reitich [1] also showed short-time existence of the motion. Let $\Gamma_i(t)$ $(i = 1, 2, 3)$ represent curves at time $t > 0$ contained in a two-dimensional bounded region $\Omega$ with smooth boundary $\partial\Omega$. Suppose $\Gamma_i(t)$ $(i = 1, 2, 3)$ meet at one point $m(t)$. The evolving interface that we consider is subject to the following laws:

(M1) The normal velocity of the interface is given by its curvature.

(M2) At the triple junction $m(t)$, the contact angle $\theta_k$ between $\Gamma_i(t)$ and $\Gamma_j(t)$ is given by Young's law, where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$. That is, for positive constants $\sigma_1$, $\sigma_2$, $\sigma_3$,
$$\frac{\sin\theta_1}{\sigma_1} = \frac{\sin\theta_2}{\sigma_2} = \frac{\sin\theta_3}{\sigma_3},$$
where $0 < \theta_k < \pi$ and $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

(M3) At the other end of each curve, $\Gamma_i(t)$ touches $\partial\Omega$ at the right angle.

The interfaces have Energy $E(t)$, which decreases as time goes:

$$E(t) = \sigma_1|\Gamma_1(t)| + \sigma_2|\Gamma_2(t)| + \sigma_3|\Gamma_3(t)|,$$

where $|\Gamma_i(t)|$ $(i = 1, 2, 3)$ mean the lengths of curves $\Gamma_i(t)$. Stationary interfaces of the motion can be viewed as critical points of the energy. In this connection Sternberg and Ziemer [7] have proved the existence of local minimizers of the energy in clover-like regions. Here we remark that stationary interfaces consist of straight line segments.

On the other hand, Ikota and Yanagida [4] have studied stabilities of stationary interfaces of the motion (M1)–(M3) by linearizing corresponding equations around the stationary interfaces. They linearized the equations formally and analyzed the resulting elliptic operator rigorously. Later they have extended their results to stationary interfaces of binary-tree type with more than one triple junctions[5]. The results are stated as follows.

**Theorem 1.1.** *Let $\Gamma = \{\gamma_i\}$ be a stationary interface that is homeomorphic to a binary tree. Denote by $L_i$ the length of $\gamma_i$. Define a characteristic index $D$ by*

$$D = \sum_{\gamma_i \in \Gamma} \sigma_i L_i \times \prod_{\gamma_i \in B} h_i + \sum_{\gamma_i \in B} \left\{ \sigma_i \prod_{\gamma_j \in B \setminus \{\gamma_i\}} h_j \right\},$$

*where $h_i$ denotes the curvature of $\partial\Omega$ at the point of contact with $\gamma_i \in B$. (Note that $h_i$ is taken to be nonpositive if $\Omega$ is convex.)*

(i) *The unstable dimension $N_{\mathrm{U}}$ is given by*

$$N_{\mathrm{U}} = \begin{cases} m-1 & \text{for} \quad (-1)^m D < 0, \\ m & \text{for} \quad (-1)^m D > 0, \end{cases}$$

*where $m = \#\{h_i < 0\}$.*

(ii) *The stationary interface is degenerate (i.e., there exists a zero eigenvalue) if and only if $D = 0$.*

We remark that the index $D$ is independent of the topology of $\Gamma$.

Although Ikota and Yanagida have established a stability criterion assuming the the existence of stationary interfaces, it has not been known whether given regions have stationary interfaces in general. The existence problem can be regarded as a variation of the Fermat-Steiner problem[2].

In [4] and [5], stabilities of stationary states have been studied on the assumption of the existence of stationary states.

In the present study we show that stationary states do exisit for convex $\Omega$. Our problem can be regarded as a variant of the Fermat-Steiner problem, though the treatments are quite different.

The Fermar-Steiner problem is described as follows: for a given triangle $\Delta ABC$, find a point $P$ that minimizes the sum of lengths

$$|PA| + |PB| + |PC|.$$

This problem was proposed by Fermat to Torricelli. Afterwards Steiner considered the same problem and gave a systematic solution. In [2] Gueron and Tessler solved the weighted Fermat-Steiner problem. They also gave an interesting historical survey of the problem.

Now we are in a position to state our result.

**Theorem 1.2.** *Suppose $\Omega$ is convex. Let $n$ be a positive integer and $G$ a binary tree with $n$ triple junctions. Then there exists at least one critical interface of $E$ which is admissible for $G$.*

## 2 Outline of Proof

Before proceeding with Problem P, we consider the two phase separation problem with no triple junctions as an illustration. Let $\Omega$ be a convex domain in $\mathbb{R}^2$. Suppose two points $P_1$ and $P_2$ are on the boundary $\partial\Omega$. We seek a critical interface of $E(P_1P_2) = |P_1P_2|$, the length of a line segment $P_1P_2$.

A simple calculation shows that $P_1P_2$ is critical if and only if $P_1P_2$ intersects with $\partial\Omega$ at the right angle. Thus all we have to do is to find $P_1P_2$ such that $P_1P_2$ are orthogonal to $\partial\Omega$ at both $P_1$ and $P_2$.

We parameterize $\partial\Omega$ by an arc length parameter $s$: $s \mapsto P(s) = (x(s), y(s)) \in \partial\Omega$. By $\tau(s)$ we denote the tangential vector to $\partial\Omega$ at $P(s)$, that is $\tau(s) = (\partial/\partial s)(x(s), y(s))$. For any point $P_1 = P_1(s_1) \in \partial\Omega$ , we can choose $s = s_2$ so that $\tau(s_2)$ is parallel to $\tau(s_1)$.

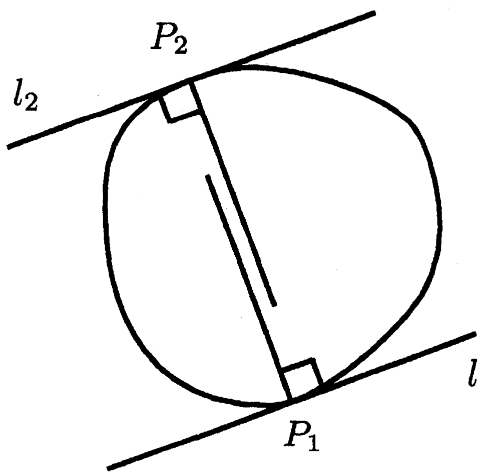


**Figure 2:** Lines $l_1$ and $l_2$ are rotated along $\partial\Omega$.

Then we move $s_1$ and observe variations of the distance $d(l_1, l_2)$, where $l_i$ are tangential lines to $\partial\Omega$ at $P(s_i)$ $(i = 1, 2)$. We can easily see that the distance $d(l_1, l_2)$ is critical if and only if $P_1P_2$ intersects with $\partial\Omega$ orthogonally. Since $d(l_1, l_2)$ has a maximum (and a minimum), the energy $E(P_1P_2)$ has a critical interface.

Now we turn our attention to Problem P. We consider the case where $G$ has a single triple junction; for more triple junctions we can show the existence of critical interfaces by induction. We can easily verify that $\Gamma_G = (\gamma_1, \gamma_2, \gamma_3)$ is critical if and only if the following two conditions are satisfied:

1. $\angle(\gamma_i, \gamma_j) = \theta_k \quad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2).$
2. $\gamma_i \perp \partial\Omega \quad (i = 1, 2, 3).$

Let $\nu(s_i)$ be the unit normal to $\partial\Omega$ at $P(s_i)$ pointing inside $\Omega$. Likewise in the analysis of the two phase problem, we can choose $s_2$ and $s_3$ for $s_1$ such that

$$\angle(\nu(s_i), \nu(s_j)) = \theta_k, \qquad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2).$$

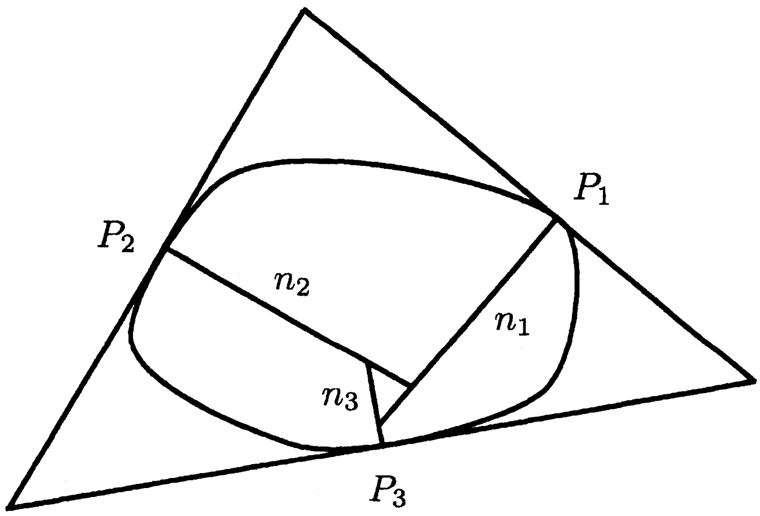


**Figure 3:** Lines $n_1$, $n_2$, $n_3$ are rotated.

Let $\mathcal{T}$ be the triangle composed of $l_1$, $l_2$, $l_3$, where $l_i$ are again the tangential lines at $P_i = P(s_i)$, and $T(s)$ the area of $\mathcal{T}$. Denote by $n_i$ the normal line to $\partial\Omega$ at $P_i$. Then we can prove that $n_1$, $n_2$, $n_3$ meet at one point if and only if $\partial T/\partial s = 0$. This indicates that the $n_i$ $(i = 1, 2, 3)$ make a critical $\Gamma_G$.

## 3 Concluding Remarks

If $\Omega$ is not convex, the approach we took in the previous section does not work in general. We illustrate it in the two phase problem.

Let $a$, $b$ be positive constants. We introduce two graphs in $\mathbb{R}^2$:

$$y = g_1(x) = (x - a)^3,$$
$$y = g_2(x) = x^3 + b.$$

Suppose $\partial\Omega$ is represented by $g_1(x)$ and $g_2(x)$ locally. We parameterize the two parts as $(\xi, (\xi - a)^3)$ and $(-\xi, -\xi^3 + b)$ respectively. Here $\xi$ runs over some interval $(-\delta, \delta)$. Then the distance between $l_1$ and $l_2$ are given by

$$d(l_1, l_2) = (4\xi^3 + 3a\xi^2 + b)/\sqrt{9\xi^4 + 1}.$$

Straightforward calculation shows that

$$\frac{\partial}{\partial \xi} d(l_1, l_2)\bigg|_{\xi=0} = 0.$$

However the two normal lines at $\xi = 0$ do not coincide.

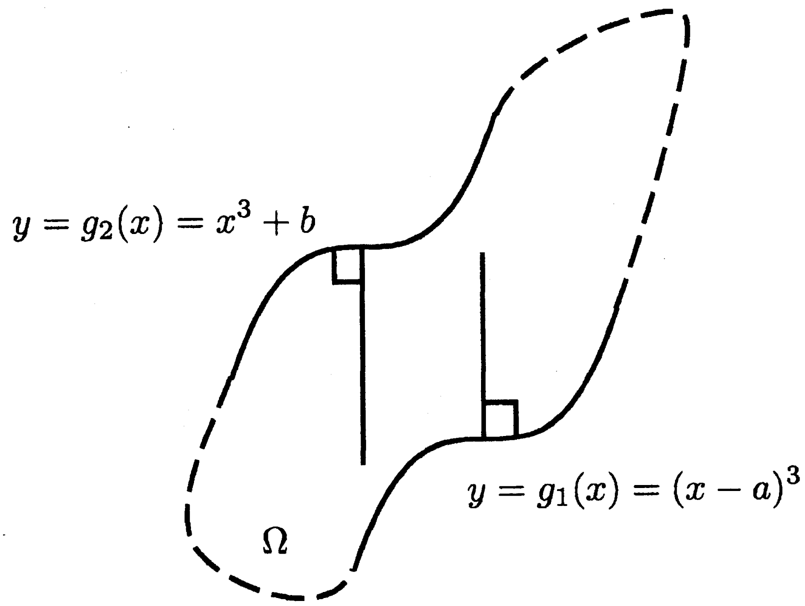


**Figure 4:** The critical lines of the distance $d(l_1, l_2)$ do not coincide.

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