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Decay estimates for the solutions of the system of crystal acoustics for cubic crystals.

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1. The main aim of this report is to study decay estimates for global solutions of the system of linear crystal elasticity in three space dimensions for cubic crystals in the nearly isotropic case. In order to state the problem, we recall that the system of linear elasticity is of form

$$\frac{\partial^2 u_i}{\partial t^2} = \sum_{j,p,q=1}^{3} c_{ijpq} \frac{\partial^2 u_p}{\partial x_j \partial x_q}, i = 1,2,3\ . \tag{1}$$

For crystals, the c_{ijpq} will be real constants (called "stiffness" constants) and depend on the crystal under consideration. At a first glance the system thus apparently depends on the 81 ($= 3^4$) constants c_{ijpq}, but since the c_{ijpq} have to satisfy the conditions

$$c_{ijpq} = c_{jipq} = c_{ijqp} = c_{pqij}, \forall i, \forall j, \forall p, \forall q,$$

the number of "essential" constants is at most 21. Additional restrictions will come from the fact that the system has to be hyperbolic (the condition is that the form

$$\sigma = (\sigma_{ij}) \to \sum_{i,j,p,q=1}^{3} c_{ijpq}\sigma_{ij}\sigma_{pq},$$

must be positive definite on symmetric tensors, i.e., when $\sigma_{ij} = \sigma_{ji}$, $\forall i, \forall j$) and the number of essential constants will decrease further when the crystal under consideration has additional symmetries. Thus, when the crystal is cubic, as we will suppose in this report, then the number of essential constants is 3. (Cf. [13].)

With the system (1) we now associate the initial conditions:

$$u_i(0,x) = f_i(x),\ \frac{\partial u_i(0,x)}{\partial t} = g_i(x),\ x \in\ R^3, i = 1,2,3. \tag{2}$$

Here we shall assume that the functions f_i, g_i, $i = 1, 2, 3$, are C^∞- functions on R^3 and have compact support. Since this problem is hyperbolic (and with constant coefficients) it clearly admits global solutions on R^4. It is moreover standard to observe that the functions $x \to u_i(t,x)$ are compactly supported in x for any fixed t, so it will in particular make sense to consider partial Fourier transforms in x, with t a parameter. We also recall that the characteristic variety of the system in (1) is by definition $\{(\tau,\xi); \tau \in R, \xi \in R^3, P(\tau,\xi) = \det A(\tau,\xi) = 0\}$, where A is the matrix

$$A(\tau,\xi) = \begin{pmatrix} \tau^2 - \sum_{p,q} c_{1p1q}\xi_p\xi_q & -\sum_{p,q} c_{1p2q}\xi_p\xi_q & -\sum_{p,q} c_{1p3q}\xi_p\xi_q \\ -\sum_{p,q} c_{2p1q}\xi_p\xi_q & \tau^2 - \sum_{p,q} c_{2p2q}\xi_p\xi_q & -\sum_{p,q} c_{2p3q}\xi_p\xi_q \\ -\sum_{p,q} c_{3p1q}\xi_p\xi_q & -\sum_{p,q} c_{3p2q}\xi_p\xi_q & \tau^2 - \sum_{p,q} c_{3p3q}\xi_p\xi_q \end{pmatrix}.$$

("det A" is the determinant of A and thus P is the characteristic polynomial of the system.) The polynomial P is immediately seen to be homogeneous and of degree six. "Hyperbolicity" then implies that for every $\xi \in R^3$ the equation $P(\tau,\xi) = 0$ has 6 real roots τ if multiplicities are counted. We also note right away that while in the general theory of partial differential equations tradition has it to work directly with the characteristic surface, in elasticity theory it seems more natural to work in terms of the so-called "slowness surface", which, by definition, is the surface $S = \{\xi \in R^3; P(1,\xi) = 0\}$. It is useful to write the slowness surface in what is called "Kelvin's" form. We do so for the particular case of cubic crystals, when Kelvin's form is (cf [4]):

$$\frac{b\xi_1^2}{1 - c|\xi|^2 + (b-a)\xi_1^2} + \frac{b\xi_2^2}{1 - c|\xi|^2 + (b-a)\xi_2^2} + \frac{b\xi_3^2}{1 - c|\xi|^2 + (b-a)\xi_3^2} = 1, \tag{3}$$

which we shall also write as $G(\xi) = 0$, where

$$G(\xi) = \sum_{i=1}^{3} b\xi_i^2(1-c|\xi|^2+(b-a)\xi_{i+1}^2)(1-c|\xi|^2+(b-a)\xi_{i+2}^2) - \prod_{j=1}^{3}(1-c|\xi|^2+(b-a)\xi_j^2). \tag{4}$$

Here the a, b, c are real constants which can be calculated in terms of the 3 "essential" stiffness constants of a cubic crystal. (Cf. [4] and [11].) The fact that (4) is the slowness surface of a cubic crystal gives some restrictions on the a, b, c. We mention here the following ones calculated in [11]: $c > 0$, $3c - b + a > 0$, $a \neq 0$, $a + c > 0$. As in [4] we shall also assume that $b > 0$. The quantity $d = b - a$ is a measure of the "anisotropy" of the crystal and in a number of arguments we shall have to assume that d is small. We shall say then that we are in the "nearly isotropic" case. When d vanishes, the equation $G(\xi) = 0$ reduces to

$$(1 - c|\xi|^2)^2(1 - (c+b)|\xi|^2) = 0. \tag{5}$$

The slowness surface is thus the union of two spheres, one of which being double. This is a manifestation of the fact that in the isotropic case every solution of the system (1) is a sum of a "transversal wave" and a "longitudinal wave" and that the components of these waves satisfy the classical scalar wave equation (with wave speed which depends on the type of the wave). In the case of effectively anisotropic cubic crystals, the structure of the slowness surface is more complicated and the surface will always have singular points. We shall review the known results about the geometry of the slowness surface for cubic crystals later on. Our interest in the slowness surface comes from the fact that solutions of the system of crystal elasticity can be expressed in terms of Fourier-integrals living on the slowness surface.

Our main estimate for solutions of the Cauchy problem (1), (2) is

Theorem 1. *Assume that (1) is the system of crystal elasticity for some given cubic crystal. Also assume that we are in the nearly isotropic case. Then there is a constant $c' > 0$ and an integer k such that*

$$|u(t,x)| \le c'(1+|t|)^{-1/2} \sum_{j=1}^{3} \sum_{|\alpha|\le k} [|D_x^\alpha f_j|_{L^1(R^3)} + |D_x^\alpha g_j|_{L^1(R^3)}], \forall (t,x) \in R^4, \quad (6)$$

for any solution of the Cauchy problem (1), (2), for which the f_j and g_j are $C^\infty(R^3)$ and have compact supports.

(If $\varphi : R^3 \to C$ is given, we denote by $|\varphi|_{L^1(R^3)}$ its L^1-norm, i.e., $|\varphi|_{L^1}(R^3) = \int_{R^3} |\varphi(\xi)|\, d\xi$.)

Remark 2. *After this authors talk at the conference, I was told by T.Sonobe that aparently the estimate in theorem 1 can be improved, at least in the case $b = 2a$. (Cf. [20].) However, while it still seemed (at the time when the discussion took place) difficult to predict what the "optimal" estimate could be, it seems clear that with respect to the case of the wave equation there will be a "loss" of decay.*

2. The first equation for which decay properties as in theorem 1 have been studied in a systematic way has been the wave equation: cf. e.g. Segal [15], Strauss [22], von Wahl [24], Klainerman [5], Racke [14], Sideris [16]. In the 3-dimensional case, the results obtained in these papers give then a decay of type $c|t|^{-1}$ when $t \to \infty$, whereas a decay of order $ct^{-1/2}$ is typical for the wave equation in two dimensions. Similar results have been obtained for a number of related hyperbolic equations, such as the Klein-Gordon equation, or, sometimes, for more general classes of constant coefficient hyperbolic operators or systems. cf. von Wahl [24], Costa [3], Sideris [17], Sideris-Tu [18], Sugimoto [23]. All papers mentioned so far have in common that they refer to the case of operators with characteristics of constant multiplicity. These results also imply that if the crystal under consideration is

isotropic, then the conclusion in (6) can essentially be improved to $|u(t,x)| \leq c|t|^{-1}$. With respect to the isotropic case, in theorem 1 we therefore have a "loss" of decay of one dimension. It is interesting to note then that the same loss of decay does also appear for the system of crystal optics (cf. Liess [10]). Actually, there are a number of analogies which relate crystal elasticity to crystal optics, to the point that in both cases the loss of decay is related to the fact that the corresponding slowness surfaces have singular points and imbedded curves along which the total curvature vanishes. However, the structure of the singular points in the case of crystal elasticity is more complicated and much less seems to be known on the geometry of the slowness surface for crystal elasticity when compared with the situation in crystal optics.

Let us mention here also briefly that decay estimates for solutions of the system of crystal elasticity in a somewhat different setting have been considered before in a number of papers: cf. e.g. Buchwald [1] (and papers cited therein) and Stodt [21]. (The results of Stodt are described also in Racke [14].) None of these papers however addresses the difficulties related to singular points on the slowness surface.

3. The overall strategy to prove a result like theorem 1 is well-understood nowadays and is in particular similar with the one used in the related case of crystal optics in [10]. Starting point is that the solutions of the Cauchy problem (1), (2) admit rather explicit representations in terms of Fourier integrals involving the Fourier transforms of the Cauchy data $\hat{f}_j$, $\hat{g}_j$, $j = 1,2,3$, of f_j and g_j: cf. Duff, [4]. It is then also no surprize that rather than arriving at an estimate of the form (6), we shall obtain at first the estimate

$$|u(t,x)| \leq c''(1+|t|)^{-1/2} \sum_{k=1}^{3} \{ \sup_{\xi \in R^3} \sum_{|\sigma| \leq d', |\beta| \leq d''} |\xi^{\sigma}| [|\partial_{\xi}^{\beta} \hat{f}_k(\xi)| + |\partial_{\xi}^{\beta} \hat{g}_k(\xi)|] \}, \forall (t,x) \in R^4, \tag{7}$$

for some constants c'', d', d''. The fact that this estimate implies the stronger estimate (6) can be proved as in the corresponding passage on page 65 in [10]. Also cf. Klainerman [8]. An important point is that due to homogeneities in the representation formulas, we can reduce the estimates (7) to estimates of Fourier transforms of densities (with parameters) which live on the slowness surface. It is then possible to localize these estimates on the slowness surface and the contribution of a suitably localized portion near some point P on the slowness surface will depend on the geometric properties of the surface near P.

The main steps in the argument are as follows:

a) at first we have to study curvature properties and the behaviour near singular points of the slowness surface for cubic crystals, with special emphasis on the nearly isotropic case;

b) the next step is to study the representation formulas for the solutions of the Cauchy problem (1), (2);

c) finally, one can reduce the estimate (7) to an estimate of some parametric integrals living on the slowness surface. We should also mention that once this reduction is done, the continuation of the argument depends on wether we are in the regular part of the slowness surface or wether we are near a singular point. Furthermore, the situation will depend on wether the singular point is "conical" or "uniplanar". To treat the uniplanar case, one can at a crucial step apply a result on estimates of Fourier integrals defined on surfaces near uniplanar singularities which has been discussed in [12]. The case of conical singularities is to some extend parallel to the uniplanar case, so in the end one is left with a (rather elementary) discussion of the contribution of the regular part of the slowness surface.

4. We now review some definitions from classical differential geometry and recall the main result in [12]. To fix the terminology, we consider at first a surface $W \subset R^n$ defined in some open neighborhood U of some fixed point $\xi^0 \in W$. (In our case we shall have $n = 3$.) To simplify notations, we shall assume that $\xi^0 = 0$. We assume further that W is given by the equation $\varphi(\xi) = 0$, for some C^∞ function φ defined on U. We shall assume that ξ^0 is a "node", in the sense that $\operatorname{grad}\varphi(\xi^0) = 0$, but $\operatorname{grad}\varphi(\xi) \neq 0$ for $\xi \neq \xi^0$. We also assume that coordinates ξ can be found such that $(\partial/\partial\xi_1)^2\varphi(0) \neq 0$ and apply the Malgrange preparation theorem to write φ locally near 0 in the form $\psi(\xi)[\xi_n^2 + a(\xi')\xi_n + b(\xi')]$, where $\xi' = (\xi_1, \ldots, \xi_{n-1})$ and $\psi(\xi) \neq 0$ near 0. For convenience we assume that $\operatorname{grad} a(0) = 0$ and denote by $\Delta(\xi') = a^2(\xi') - 4b(\xi')$ the local discriminant of φ. Finally, we denote by $J_k\varphi = \sum_{|\alpha|=k}[(\partial/\partial\xi)^\alpha\varphi(0)]\xi^\alpha/\alpha!$. This is a geometric invariant when $J_{k'}\varphi(0) = 0$ for $k' < k$.

Definition 3. *(cf. Sommerville [19].) ξ^0 is said to be a conical singular point of W if $J_2(\xi)\varphi \sim |\xi|^2$ near 0. It is called a unode, or uniplanar singularity, if $J_{k'}\varphi(0) = 0$ for $k' \leq 3$, but $J_4\varphi(0) \neq 0$.*

We now denote by

$$W^+ = \{\xi; \xi_n = (1/2)[-a(\xi') \pm \sqrt{\Delta(\xi')}]\}$$

the two sheets of W in a neighborhood of 0, and by

$$\Gamma'^{\pm} = \{\xi' \in R^{n-1}; [-J_2a(\xi') \pm \sqrt{J_4\Delta(\xi')}] = 1\}. \tag{8}$$

The main geometric assumption considered in [12] for unodes is the following:

$$\text{the } \Gamma'^{\pm} \text{ are smooth and of nowhere vanishing total curvature.} \tag{9}$$

The following result is proved in [12]:

Theorem 4. *Let 0 be an unode of W for which (9) holds. If $f : W \to R$ is sufficiently regular near zero, it follows that there are constants ε, c and d such that*

$$|\int_W \exp[-i\langle x,\xi\rangle] f(\xi)\, dw(\xi)| \leq c(1+|x|)^{(1-n)/2} \sup_{\xi\in W|\xi|\leq\varepsilon, |\alpha|\leq d} |\partial_\xi^\alpha f(\xi)|,$$
$$\text{provided } f(\xi) = 0 \text{ for }, \xi \in W, |\xi| \geq \varepsilon. \tag{10}$$

(Here we denote by dw the surface element on W.)

We want to apply this result to estimate integrals which live on some portion W of the slowness surface S of some given cubic crystal. In particular, we shall have $n = 3$ and and S (respectively W) is two dimensional. We shall denote by $\Gamma'^{\pm}$ the curves associated in (8) with W near some fixed uniplanar singular point of S as above. In order to describe these curves explicitly, we must of course at first calculate a decomposition of a local defining equation of S in the form $\psi(\xi)[\xi_3^2 + a(\xi')\xi_3 + b(\xi')]$, assuming that we have labelled coordinates appropriately. It can be shown that (in suitable coordinates) the curves $\Gamma^{\pm}$ have the form

$$\Gamma^{\pm} = \{(\xi_1,\xi_2) \in R^2; \alpha(\xi_1^2+\xi_2^2) \pm \beta\sqrt{\xi_1^4 + 2\gamma\xi_1^2\xi_2^2 + \xi_2^4} = 1\} \tag{11}$$

for some constants α, β, γ. We shall assume here that $\beta > 0$, $\gamma + 1 > 0$, $\alpha - \beta > 0$, $2\alpha^2 > \beta^2(\gamma+1)$. (These conditions have been introduced in [12]; they are precisely the ones needed if we are to have a non-degenerate uniplanar singularity with two sheets which touch near the singular point only in the singular point itself.) The following result is obtained in [12]:

Theorem 5. *Assume that $\beta > 0$ $\gamma+1 > 0$, $\alpha - \beta > 0$, $2\alpha^2 > \beta^2(\gamma+1)$. Then the curve $\Gamma^+ = \{(\xi_1,\xi_2) \in R^2; \alpha(\xi_1^2+\xi_2^2) + \beta\sqrt{\xi_1^4 + 2\gamma\xi_1^2\xi_2^2 + \xi_2^4} = 1\}$ has no inflection point. Furthermore, $\Gamma^- = \{(\xi_1,\xi_2) \in R^2; \alpha(\xi_1^2+\xi_2^2) - \beta\sqrt{\xi_1^4 + 2\gamma\xi_1^2\xi_2^2 + \xi_2^4} = 1\}$ will have no inflection point, if and only if $(\alpha - \beta\gamma)(-\alpha\sqrt{2(1+\gamma)} + \beta(3-\gamma)) < 0$.*

We do not recall here the exact regularity conditions required in [12] for the function f in theorem 4. Actually, these conditions are somewhat more general than of type "$f \in C^k$ for some sufficiently large k", in that what we want is that after passing to some special kind of polar coordinates (r,ω) in a neighborhood of the singularity, the function f be C^k in (r,ω) up to $r = 0$, for some sufficiently large k. (The right hand side in the inequality in (10) will have then of course to refer to derivatives in these polar coordinates.) A situation similar to this one will appear also for conical singularities.

The proof of theorem 1 is based on a preliminary study of some geometric properties of the slowness surface. A number of these properties can be understood easier if the defining equation of the surface is put in Kelvin's form. When $\xi_1 = \xi_2 = 0$, equation (3) has the positive solutions $\xi_3 = 1/\sqrt{c}$, $\xi_3 = 1/\sqrt{b+c-d}$, the first being double. When $d = 0$ we are in the isotropic case, so $\sqrt{c}$, (being a double root) must be the velocity of the transversal waves, whereas $\sqrt{b+c}$ will correspond to the velocity of the longitudinal waves. Here we recall that in the isotropic case the velocity of the longitudinal waves is known to be strictly bigger than that of the transversal waves (cf. [9], section 22, where it is stated that $\sqrt{c+b} \geq \sqrt{4/3}\sqrt{c}$). "$d$" measures the degree of anisotropy of the crystal, and we are mainly interested in the "nearly isotropic" case, i.e., we shall assume whenever needed, that d is sufficiently small. This shows that we may assume that $b > 0$, although this is not strictly speaking necessary in arguments which refer directly to surfaces defined as in (3), and is not assumed in [4].

5.We shall denote the slowness surface henceforth by S and shall denote by S^1, S^2, S^3, the "outer", "middle" and "inner" sheet of S. More precisely, when we denote by ω generic points on the unit sphere, then we will have 3 values $\rho > 0$ such that the point $\rho\omega$ belongs to S. We denote these values by $\rho_1(\omega), \rho_2(\omega)$ and $\rho_3(\omega)$ respectively, where the numbering is made in such a way that $0 < \rho_3(\omega) \leq \rho_2(\omega) \leq \rho_1(\omega)$. In view of the assumption that $b > 0$, we will have for small d that $\rho_3(\omega) < \rho_2(\omega)$ for all ω. It is then immediate to see that S^3 is smooth and it is also standard to observe that it must be convex. (Cf. [2], [4], and also [11].) When $d = 0$, S^2 coincides with S^1 and all three surfaces are spheres. (Also see the discussion above.) As in the case of crystal optics for biaxial crystals, both surfaces S^1 and S^2 become effectively singular at a finite number of points as soon as $d \neq 0$. Indeed, it is clear that S^1 and S^2 can be singular only when they touch and it is classical (cf. e.g., [4], [11]), that this happens precisely on the six points they have on the coordinate axes and on the eight points on $S^1 \cap S^2$, where $|\xi_1| = |\xi_2| = |\xi_3|$. For later use we mention that

$$\begin{gathered}\text{when } \xi \in S^3, |\xi_1| = |\xi_2| = |\xi_3|, \text{ then } |\xi_1| = 1/\sqrt{3c+b-d}, \\ \text{whereas } \xi \in S^2, |\xi_1| = |\xi_2| = |\xi_3|, \text{ gives } |\xi_1| = 1/\sqrt{3c-d}.\end{gathered} \tag{12}$$

The distance of the latter points to the origin is of course $1/\sqrt{c-d/3}$. The type of singularity at double points is also known: it is uniplanar on the points lying on the coordinate axes and it is conical on the points for which $|\xi_1| = |\xi_2| = |\xi_3|$: cf. again [4] and [11].

Definition 6. *We call a vector ξ^0 "singular" if the half-ray through ξ^0 intersects S in a singular point, and "regular" if this is not the case. (ξ^0 and $-\xi^0$ are of course simultaneously singular or regular.) Likewise, we shall speak about a uniplanarly (respectively conically) singular direction if the intersection of the halfray through ξ^0 with S is a uniplanar (respectively, conical) singular point on S.*

6. In principle, the behaviour of S near some singular point can be understood from a study of the discriminant of the defining polynomial. In the present context we are also interested in detailed information concerning the discriminant since condition (9) is stated in terms of it. Calculations are particularily simple for the case of uniplanar singularities and in fact it is not difficult to calculate the discriminant of the polynomial equation defining S in this case explicitly. To describe how this is done, we shall work in a neighborhood of the point $\xi^0 \in S$, where $\xi_1^0 = \xi_2^0 = 0$, $\xi_3^0 = 1/\sqrt{c}$. We shall denote the variable ξ_3 by σ. As far as the choice of linear coordinates is concerned, we note that $\xi_3 = 1/\sqrt{c}$ is tangent to S at ξ^0, so it is natural to parametrize S near ξ^0 by $\xi' = (\xi_1, \xi_2)$ and assume that the three sheets of S are given in a neighborhood of the $(0,0,1)$ axis by the graph of some functions $\xi_3 = \rho_i(\xi')$. There is no singularity in the isotropic case, so we shall assume $a \neq b$ and denote by $\rho_i(\xi')$, $i = 1,2,3$, continuous functions so that $\rho_i(\xi') \neq \rho_j(\xi')$ if $i \neq j$ and $\xi \neq \xi^0$ is close to ξ^0. Moreover, we assume $G(\xi', \rho_i(\xi')) \equiv 0$ and $\rho_1(0) = \rho_2(0) = 1/\sqrt{c}$, $\rho_3(0) = 1/\sqrt{a+c}$. Also recall that we assumed $a \neq 0$, so there will be no triple roots. (See the assumptions on the constants a, b, c made above.)

We want to obtain information on the local discriminant $\Delta(\xi_1, \xi_2) = (\rho_1(\xi') - \rho_2(\xi'))^2$. Since ξ^0 is an unode, we will now have $\sum_{|\beta|\leq 3} |\partial_{\xi'}^{\beta} \Delta(0)| = 0$ and we have to study $J_4\Delta$. To do this, we shall use the fact that G is an expression in ρ^2. In fact, when written out explicitly, G is of form

$$A_0\rho^6 + A_1(\xi')\rho^4 + A_2(\xi')\rho^2 + A_3(\xi'), \tag{13}$$

for some explicitly calculable coefficients A_j, which are polynomials of degree $2j$ in ξ. We set here $\rho^2 = \sigma$ and denote by $Q(\xi', \sigma)$ the polynomial $A_0\sigma^3 + A_1(\xi')\sigma^2 + A_2(\xi')\sigma + A_3(\xi')$. We denote by $\sigma_i(\xi')$ the functions $\sigma_i(\xi') = \rho_i(\xi')^2$ (so that $Q(\xi', \sigma_i(\xi')) \equiv 0$) and by D the discriminant of Q in the variable σ. It is then immediate that there is a constant $\tilde{\gamma} \neq 0$ so that at ξ^0

$$J_4\Delta = \tilde{\gamma} J_4 D. \tag{14}$$

Of course $\tilde{\gamma} = [(\rho_1 + \rho_2)(0)(\rho_1^2 - \rho_3^2)(0)(\rho_2^2 - \rho_3^2)(0)]^2$. The expression of D can also be calculated explicitly. In fact, it is standard (cf. e.g. [25]) that, calculated in terms of the coefficients A_i, the discriminant D is

$$D = A_1^2A_2^2 - 4A_0A_2^3 - 4A_1^3A_3 - 27A_0^2A_3^2 + 18A_0A_1A_2A_3.$$

Here all the coefficients depend explicitly on ξ_1^2, ξ_2^2 and not directly on ξ_1, ξ_2. We conclude that $J_4\Delta$ is a polynomial in ξ_1^2, ξ_2^2. Of course we can calculate J_4D explicitly. What we get is if we set $d = b - a$:

$$J_4D = c^2d^2[4b^4 - 12b^3d + 13b^2d^2 - 6bd^3 + d^4](\xi_1^4 + \xi_2^4) -$$
$$c^2d^2[4b^4 + 16b^3d - 22b^2d^2 + 12bd^3 - d^4]\xi_1^2\xi_2^2.$$

(All these calculations were valid also for $b = a$. Note incidentally that for this case $J_4 D$ vanishes identically, as it should, since $\rho_1 \equiv \rho_2$ then.) It is perhaps also worth noting that the main term for $d \to 0$ is $4b^4c^2d^2(\xi_1^4 + \xi_2^4 - \xi_1^2\xi_2^2)$, which has order of magnitude $b^4c^2d^2(|\xi_1|^4 + |\xi_2|^4)$.

When we want to apply theorem 4, we need in addition to information on $J_4\Delta$ also related information on "$J_2 a$". (For notations cf. (9).) Actually, what we want is to understand the structure of the quantities $J_2 a \pm \sqrt{J_4\Delta}$, in terms of which condition (9) is formulated. In particular, we need to show that the curves $\Gamma'^{\pm}$ introduced in (8) for our S are quartics of the particular form described in (11), and to understand how one can calculate the coefficients of these quartics in terms of the constants a, b, c (notations are as in (3)) of the cubic crystal at hand. With calculations similar to the ones which led to an understanding of $J_4\Delta$ one can in fact show that $J_2 a \pm \sqrt{J_4\Delta}$ has the form

$$J_2 a \pm \sqrt{J_4\Delta} = -c(\xi_1^2 + \xi_2^2) + dQ_1(\xi_1, \xi_2, d) \pm |d|\sqrt{Q_2(\xi_1, \xi_2, d)}\}, \qquad (15)$$

where Q_1 is a polynomial of order two, and Q_2 a polynomial of order four in (ξ_1, ξ_2), with coefficients which are analytic in d. The fact that condition (9) must hold for small d is then immediate.

While these very explicit calculations are interesting in a neighborhood of the axes, we are also interested in estimates of this discriminant when $d \to 0$ also away from the axes. We mention the following result:

Proposition 7. *Let T be some open cone in R^3 which contains the six coordinate axes. Denote by Ω the set*

$$\Omega = \{\xi \in R^3; \xi \notin T\}. \qquad (16)$$

and by $\tau_1(\xi) \leq \tau_2(\xi) \leq \tau_3(\xi)$, the positive roots of the equation

$$\sum_i \frac{b\xi_i^2}{\tau^2 - c|\xi|^2 + d\xi_i^2} = 1. \qquad (17)$$

We can find c', which does not depend on d, once b, c, T have been fixed, so that

$$\min(|\tau_1(\xi) - \tau_2(\xi)|, |\tau_2(\xi) - \tau_3(\xi)|) \geq c'd \sum_{i,j} |\xi_i^2 - \xi_j^2|/|\xi|^2 \ \forall \xi \in \Omega. \qquad (18)$$

(The most interesting part of the statement is about the behavior when $d \to 0$ and near the rays $|\xi_1| = |\xi_2| = |\xi_3|$.)

7. We can use these results to study the behaviour of the surfaces S^i for $d \to 0$ from a quantitative point of view. The following results are in fact easy to establish for

small d: if we are in a neighborhood of the uniplanar singularity $\xi^0 = (0, 0, 1/\sqrt{c})$, then $\rho_1(\xi') - \rho_2(\xi') \sim d|\xi'|^2$ for $\xi' \to 0$ (notations are here as in subsection 6 above), if ω^0, is a conically singular direction, then $\rho_1(\omega) - \rho_2(\omega) \sim d|\omega - \omega^0|$ and if ω^0 is a regular direction then $\rho_1(\omega) - \rho_2(\omega) \sim d$ in a neighborhood of ω^0 (we have returned to the notations in subsection 5 above).

8. The curvature properties of the wave surface of the system of crystal optics (at smooth points of the surface) are well-established: cf. e.g., [2]. We have not found information of similar quality for the case of the system of elasticity for crystals in the literature. Cf. anyway [13] for some numerical evidence. In this paper we are interested mainly in the case of cubic crystals in the nearly isotropic case. The principal results which we have obtained are the following:

a) when $d \neq 0$, is small, the total curvature will always vanish on entire curves in the smooth part of $S^1 \cup S^2$. It does not vanish however in the nearly isotropic case on S^3,

b) the mean curvature will vanish nowhere in the smooth part of S, at least if we are close to the isotropic case,

The statement in a) is not effectively needed in the proof of theorem 1, but it shows why we need b), and it also shows that it is not possible to obtain decay estimates for the solutions of the system of crystal-elasticity by applying the method of stationary phase in a mechanical way. On the other hand, assuming that we are in the nearly isotropic case and using b), we can still obtain decay results using the stationary phase method in part of the variables, if we treat part of the variables as additional parameters.

8. The proof of a) will be based on the following statement which is perhaps of independent interest:

c) in the nearly isotropic case, the total curvature is negative on S^1 near conical points and positive near uniplanar points.

That c) is true is a consequence of a simple remark on surfaces which have defining equations of the form considered in the following proposition.

Proposition 8. *Let $Q_1 = Q_1(x, y, d)$, $Q_2 = Q_2(x, y, d)$ be positive definite quadratic forms in the variables (x, y) with coefficients which depend in a C^∞ way on d for small d and assume that there are constants $c_1 > 0$, $c_2 > 0$ such that*

$$Q_1(x, y, d) \geq c_1(x^2 + y^2), Q_2(x, y, d) \geq c_2(x^2 + y^2).$$

Also assume that f_1, f_2 are C^∞-functions of (x, y, d), $|d| < 1$, such that $|f_i(x, y, d)| \leq c_3(|x|^3+|y|^3)$, $|\nabla f_i(x, y, d)| \leq c_3(|x|^2+|y|^2)$, $|Hf_i(x, y, d)| \leq c_3(|x|+|y|)$, and denote *by $\tilde{S}$ the surface*

$$\tilde{S} = \{(x, y, z); z = -Q_1(x, y, d) + f_1(x, y, d) + |d|\sqrt{Q_2(x, y, d) + f_2(x, y, d)}\}.$$

(Thus $\tilde{S}$ depends on d.) Then there is c_4, which depends only on c_1, c_2, c_3, d, such that the total curvature $K(P)$ at any point $P \in \tilde{S}$ is strictly negative when $|P| < c_4$.

Remark 9. *It is quite trivial to show that on S^2 there are points of positive total curvature when we make the additional assumption that $d > 0$. In fact, it is obvious that then the distance from the origin will increase near conical points to values bigger than $1/\sqrt{c - d/3}$ (which is the distance from the conically singular points to the origin). Since the distance from the uniplanarly singular points to the origin is $1/\sqrt{c}$, we conclude that the points $\tilde{P} \in S^2$ farthest away from the origin must lie in the smooth part of S^2. At such a point $\tilde{P}$ the total curvature must be positive.*

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