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# APPROXIMATION FOR RECONSTRUCTION OF HOMOGENEOUS OBJECTS IN THE PLANE FROM THEIR TWO PROJECTIONS

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ABSTRACT. In this article, we discuss problems in the reconstruction of the plane sets from their two projections. We first study the reconstruction from the orthogonal projections. We prove that if the orthogonal projections of a plane set $F$ are close to those of some set $G$ which is uniquely determined from their orthogonal projections, then the set $F$ itself is close to the uniquely determined set $G$. In this case, an algorithm for constructing approximate solutions is given. We also mention the reconstruction from two projections which are not necessarily orthogonal.

## 1. INTRODUCTION

We discuss reconstruction of measurable plane sets from their two projections. Let $F \subset \mathbb{R}^2$ be a measurable plane set such that $\lambda_2(F) < \infty$, where $\lambda_i$ is the Lebesgue measure on $\mathbb{R}^i$. Let $f(x, y)$ be the characteristic function of $F$. Define the *horizontal* and the *vertical projections* of $F$ (or equivalently of $f$) as

$$PF(y) := \int_{-\infty}^{\infty} f(x, y)dx \tag{1}$$

and

$$QF(x) := \int_{-\infty}^{\infty} f(x, y)dy, \tag{2}$$

respectively. *The reconstruction problem* of measurable plane sets from their orthogonal projections is as follows:

**Problem 1.1.** *Given two non-negative, integrable functions $f_1$ and $f_2$, find a measurable plane set $F$ such that $PF = f_1$ and $QF = f_2$ almost everywhere.*

This problem was first studied by G.G. Lorentz [5] in 1949. He proved that the answers to Problem 1.1 are classified into three cases; the pair $(f_1, f_2)$ determines a unique set, sets non-uniquely or no set, respectively. In 1988, A. Kuba and A. Volčič [3] gave a reconstruction formula for the uniquely determined sets. They also studied the structure of non-uniquely determined sets (cf. [4]). In 1998, L. Huang and T. Takiguchi [1] proved that the class of uniquely determined bounded sets are stable, applying which they also gave an algorithm to construct approximate sets for the uniquely determined ones from their orthogonal projections possibly containing noise and error.

In practical applications, however, it hardly happens that the set to be reconstructed is a priori known to be uniquely determined. If the projections contain no error, it is possible to judge whether the pair determines a set uniquely or not. In practice, however, it is impossible to obtain the projections without noise and error. Therefore the best we can hope is to construct an approximate set to the original one from a pair of orthogonal projections with noise and error. In this article, we discuss this problem. More concretely, we study the following problems.

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**Problem 1.2.**

(i): *Give a sufficient condition for a non-uniquely determined set $F$, by which we can approximately reconstruct it from its projections possibly containing noise.*

(ii): *Give an algorithm to approximate such non-unique sets.*

Since we may assume the existence of solutions in practical applications, what we have to study is the approximation of non-unique solutions. If Problem 1.2 is solved, then we can approximately reconstruct the sets satisfying the sufficient condition mentioned in Problem 1.2, (i) from their orthogonal projections possibly containing noise and error.

In the second section, we review some known results on Problem 1.1 which are closely related to our main purpose. In Section 3, we give a characterization of non-uniquely determined sets having the same projections. Applying the characterization, we give an answer to Problem 1.2, which is one of our main purpose in this article. In the final section, we make concluding remarks and mention some problems left to be solved for further development, among which we mention the reconstruction from two projections which are necessarily not orthogonal. Throughout this article, unless mentioned otherwise, all discussions are made up to sets of measure zero.

## 2. Known results on Problem 1.1

In this section, we review known results on Problem 1.1 which are closely related to our theme in this article. In 1949, G.G. Lorentz proved that the answers to the problem are classified into three cases. He gave a characterization of each case in terms of projection functions. For introduction of his result, let us prepare some definitions.

**Definition 2.1.** For non-negative, integrable functions $f_1(y)$ and $f_2(x)$, define the projection of $f_1$ by

$$f_{12}(x) := \lambda_1(\{y|\ f_1(y) \geq x\}) \tag{3}$$

for $x \geq 0$ and the projection of $f_2$ by

$$f_{21}(y) := \lambda_1(\{x|\ f_2(x) \geq y\}) \tag{4}$$

for $y \geq 0$. Similarly, the functions $f_{121}$ and $f_{212}$ are defined respectively as

$$f_{121}(y) := \lambda_1(\{x|f_{12}(x) \geq y\}), \tag{5}$$

$$f_{212}(x) := \lambda_1(\{y|f_{21}(y) \geq x\}). \tag{6}$$

The functions $f_{12}$, $f_{21}$, $f_{121}$ and $f_{212}$ thus defined are clearly non-negative, non-increasing and integrable.

**Definition 2.2.** (cf. [6]) A measurable function $f_r(x)$ is called a *rearrangement* of $f(x)$ if

$$\lambda_1(\{x|f(x) \geq y\}) = \lambda_1(\{x|f_r(x) \geq y\}) \tag{7}$$

for all $y$.

The functions $f_{121}$ and $f_{212}$ are rearrangements of $f_1$ and $f_2$, respectively.

**Definition 2.3.** (cf. [6]) For a measurable function $f$, let

$$\begin{aligned} f^*(y) &:= \lambda_1(\{x|f(x) \geq y\}) \\ f^{**}(x) &:= \text{ ess.sup}\{y|f_x(y) \geq x\}. \end{aligned} \tag{8}$$

We call $f^{**}(x)$ the *decreasing rearrangement* of $f(x)$ .

We can then show that $f_{121}$ and $f_{212}$ are the decreasing rearrangement of $f_1$ and $f_2$, respectively. G.G. Lorentz's result is as follows:

**Theorem 2.1.** *Let $f_1(y)$ and $f_2(x)$ be non-negative, integrable functions such that*

$$\int_{-\infty}^{\infty} f_1(y)dy = \int_{-\infty}^{\infty} f_2(x)dx. \tag{9}$$

(i) *The unique case.*
*There exists a unique set in $\mathbb{R}^2$ which has $(f_1, f_2)$ as projections if and only if*

$$\int_0^c f_{12}(x)dx = \int_0^c f_{212}(x)dx, \quad \text{for any } c > 0. \tag{10}$$

(ii) *The non-unique case.*
*There exist plural sets having $(f_1, f_2)$ as projections if and only if*

$$\int_0^c f_{12}(x)dx \geq \int_0^c f_{212}(x)dx, \quad \text{for any } c > 0, \tag{11}$$

*and there is a $c > 0$ for which the strict inequality holds.*

(iii) *The inconsistent case.*
*There exists no set having $(f_1, f_2)$ as projections if and only if*

$$\int_0^c f_{12}(x)dx < \int_0^c f_{212}(x)dx, \quad \text{for some } c > 0. \tag{12}$$

For the unique case, A. Kuba and A. Volčič [3] gave a reconstruction formula in 1988.

**Theorem 2.2.** *If a measurable set $F$ is uniquely determined by a pair of its projections $(f_1, f_2)$, then*

$$F = \{(x, y) | f_1(x) \geq f_{12}(f_1(y))\} \tag{13}$$

*up to a set of measure zero.*

In the same paper [3], they also gave a characterization of non-uniquely determined sets. For a measurable set $P \subset \mathbb{R}^2$, define *the horizontal* and *the vertical translations* of $P$ by

$$\begin{aligned} P_{(s,0)} &:= \{(x, y) | (x - s, y) \in P\}, \\ P_{(0,t)} &:= \{(x, y) | (x, y - t) \in P\}. \end{aligned} \tag{14}$$

A plane set $F$ is called to have $(P, P^{12}; P^1, P^2)$ as *a switching component* if there exist four sets $P, P^1, P^2, P^{12}$ (of positive measure) and two real numbers $s, t \neq 0$ such that

$$P^1 = P_{(s,0)}, \quad P^2 = P_{(0,t)}, \quad P^{12} = P_{(s,t)} \tag{15}$$

and that

$$(P \cup P^{12}) \subseteq F, \quad (P^1 \cup P^2) \cap F = \emptyset. \tag{16}$$

Let a set $F$ have $(f_1, f_2)$ as projections and $(P, P^{12}; P^1, P^2)$ as a switching component. By *to switch the switching components* $(P, P^{12}; P^1, P^2)$ in $F$, we mean the procedure of making another set $\tilde{F} := (F \cup P^1 \cup P^2) \setminus (P \cup P^{12})$ which has the same projections as $F$.

**Theorem 2.3.** (cf. [3]) *A measurable plane set having a finite measure is non-uniquely determined by its projections if and only if it has a switching component.*

In 1998, L. Huang and the author proved the stability in the class of uniquely determined bounded sets.

**Theorem 2.4.** (cf. [1]) *Let $f_1$, $f_2$, $g_1$ and $g_2$ be non-negative, essentially bounded integrable functions. Assume that the pairs of projections, $(f_1, f_2)$ and $(g_1, g_2)$, uniquely determine measurable plane sets $F$ and $G$ (with characteristic functions $f(x,y)$ and $g(x,y)$), respectively. Then we have*

$$||f-g||_{L^1(\mathbb{R}^2)} \le C\cdot \max\{||f_1-g_1||_{L^\infty(\mathbb{R})}, ||f_2-g_2||_{L^\infty(\mathbb{R})}\}, \tag{17}$$

*where*

$$\begin{aligned} C = \max\{&9(||f_1||_{L^\infty(\mathbb{R})}+||g_1||_{L^\infty(\mathbb{R})})+3(||f_2||_{L^\infty(\mathbb{R})}+||g_2||_{L^\infty(\mathbb{R})}),\\ &9(||f_2||_{L^\infty(\mathbb{R})}+||g_2||_{L^\infty(\mathbb{R})})+3(||f_1||_{L^\infty(\mathbb{R})}+||g_1||_{L^\infty(\mathbb{R})}),\\ &6(||f_1||_{L^\infty(\mathbb{R})}+||g_1||_{L^\infty(\mathbb{R})}+||f_2||_{L^\infty(\mathbb{R})}+||g_2||_{L^\infty(\mathbb{R})})\}. \end{aligned} \tag{18}$$

In the same paper, they also gave an algorithm to construct an approximate solution to Problem 1.1 from their orthogonal projections, in the case where the set to be reconstructed is a priori known to be uniquely determined. Let us review their algorithm. Their idea will be applied to construct an approximate solution for non-unique sets in the fourth section, which is one of the main purposes in this article.

First, consider the case where the set $F$ to be reconstructed is a direct sum of finite rectangles whose sides are parallel to the $x$- and $y$- axes. Let $f$ have $(f_1, f_2)$ as projections and

$$\begin{aligned} F_x &:= \{(x,y) \mid 0<y<f_2(x)\},\\ F_y &:= \{(x,y) \mid 0<x<f_1(y)\},\\ F_{yxy} &:= \{(x,y) \mid 0<y<f_{212}(x)\}. \end{aligned} \tag{19}$$

The sets $F_x$, $F_y$ and $F_{yxy}$ are direct sums of finite rectangles. Let

$$F_{yxy} = F^1_{yxy} \oplus \cdots \oplus F^N_{yxy}, \tag{20}$$

where

$$F^i_{yxy} = [x^i_{yxy}, x^i_{yxy}+h^i) \times [y^i_{yxy}, y^i_{yxy}+k^i), \quad i=1,2,\cdots,N, \tag{21}$$

where $k^1 \ge k^2 \ge \cdots \ge k^N$ and $y^i_{yxy}=0$ for any $i=1,2,\cdots,N$. In the simple case where for $F^i_{yxy}$, there exist rectangles

$$\begin{aligned} F^i_x &= [x^i_x, x^i_x+h^i) \times [y^i_{yxy}, y^i_{yxy}+k^i) \subset F_x\\ F^i_y &= [x^i_{yxy}, x^i_{yxy}+h^i) \times [y^i_y, y^i_y+k^i) \subset F_y \end{aligned} \tag{22}$$

such that $F^i_{yxy}$, $F^i_x$ and $F^i_y$ are congruent. In this case the counterpart in $F$ is reconstructed as

$$F^i = [x^i_x, x^i_x+h^i) \times [y^i_y, y^i_y+k^i). \tag{23}$$

In the general case where there may exist several rectangles

$$F^{i_1}_x, \cdots, F^{i_m}_x \quad \text{in the strip} \quad y^i_{yxy} \le y < y^i_{yxy}+k^i \tag{24}$$

and

$$F^{i_1}_y, \cdots, F^{i_n}_y \quad \text{in the strip} \quad x^i_{yxy} \le x < x^i_{yxy}+h^i, \tag{25}$$

where

$$
(26) \qquad \begin{aligned} F_x^{i_\alpha} &= [x_x^{i_\alpha}, x_x^{i_\alpha} + h^{i_\alpha}) \times [y_{yxy}^i, y_{yxy}^i + k^i), \\ F_y^{i_\beta} &= [x_{yxy}^i, x_{yxy}^i + h^i) \times [y_y^{i_\beta}, y_y^{i_\beta} + k^{i_\beta}), \end{aligned}
$$

such that $F_x^{i_\alpha} \cap F_x^{i_\beta} = F_y^{i_\alpha} \cap F_y^{i_\beta} = \emptyset$ for $\alpha \neq \beta$ and

$$
(27) \qquad h^{i_1} + \cdots + h^{i_m} = h, \quad k^{i_1} + \cdots + k^{i_n} = k.
$$

In this case, the counterpart of $F_x^{i_\alpha}$ and $F_y^{i_\beta}$ in $F$ is reconstructed as

$$
(28) \qquad F_x^{i_{\alpha,\beta}} = [x_x^{i_\alpha}, x_x^{i_\alpha} + h^{i_\alpha}) \times [y_y^{i_\beta}, y_y^{i_\beta} + k^{i_\beta})
$$

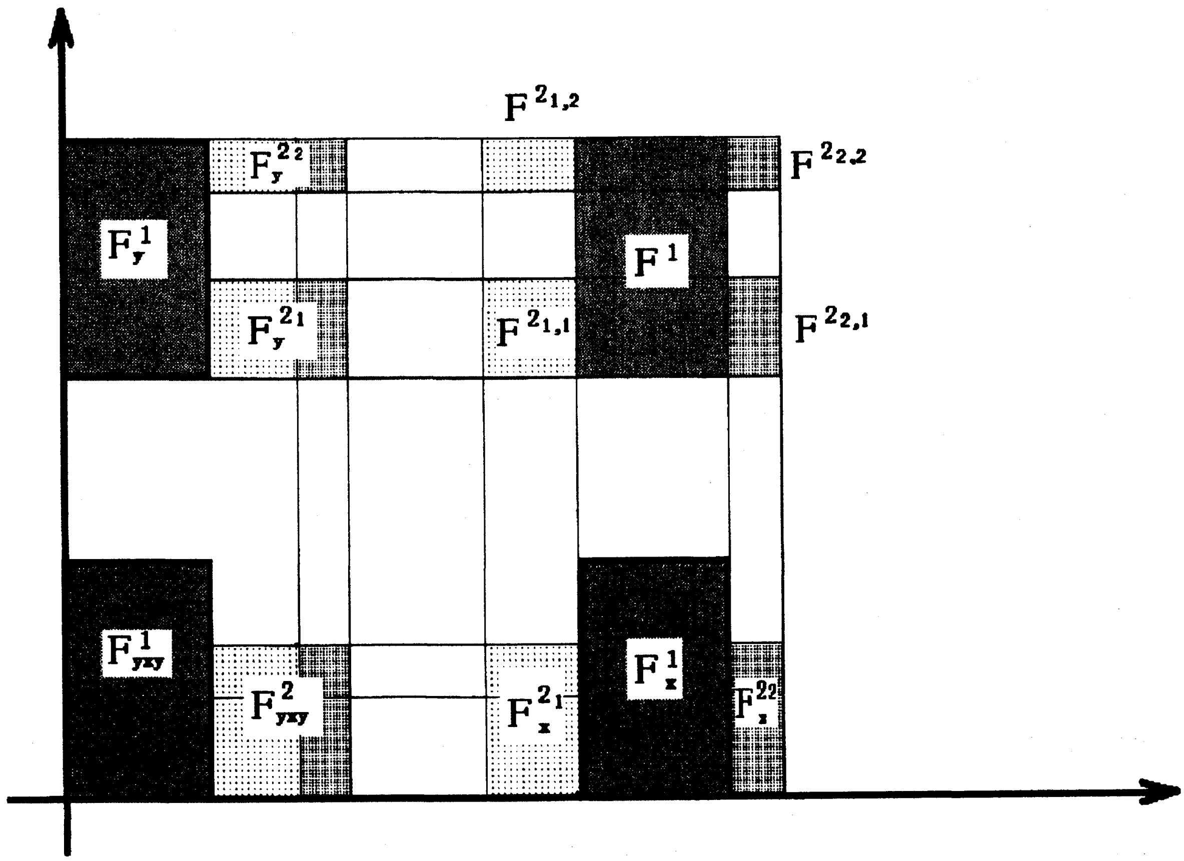
and the counterpart of $F_{yxy}^i$ in $F$ is reconstructed as

$$
(29) \qquad F^i = \sum_{\alpha,\beta} F_x^{i_{\alpha,\beta}}.
$$

Thus

$$
(30) \qquad F = \sum_{i=1}^{N} F^i
$$

is the set to be reconstructed.




In the general case where $F$ is not a direct sum of rectangles, we have only to approximate $f_1$, $f_2$ and $f_{212}$ by suitable step functions. For more detail, confer [1].

Note that this algorithm is able to be applied to a pair of orthogonal projections with noise and error if the pairs are a priori known to determine a set uniquely.

## 3. An answer to Problem 1.2

In this section, we first give a characterization of non-uniquely determined sets which have the same projections. Applying this characterization, we will give an answer to Problem 1.2.

**Proposition 3.1.** *Assume that a pair of projections $(f_1, f_2)$ determines sets non-uniquely. Take a set $F$ whose projections are $(f_1, f_2)$. Then any set having $(f_1, f_2)$ as projections is obtained by switching the switching components in $F$ countable times.*

For the proof of this proposition, confer [7]. Let us prove that a non-uniquely determined set is approximately reconstructed if their projections are *close to* those of a uniquely determined set. This is one of our main purposes in this article.

**Theorem 3.1.** *Assume that the non-unique pair $(f_1, f_2)$ of projections, $f_1, f_2 \in L^1 \cap L^\infty(\mathbb{R})$, satisfies*

$$\|f_{12} - f_{212}\|_{L^1(\mathbb{R})} < \varepsilon. \tag{31}$$

*Then there exists a uniquely determined set $G$ such that*

$$\lambda_2(G \ominus F) < 2\varepsilon \min\{\|f_1\|_{L^\infty(\mathbb{R})}, \|f_2\|_{L^\infty(\mathbb{R})}\}, \tag{32}$$

*for any set $F$ having $(f_1, f_2)$ as orthogonal projections.*

The pair $(f_1, f_2)$ determines a plane set uniquely if and only if $f_{12} \equiv f_{212}$ almost everywhere. Therefore we call that the pair $(f_1, f_2)$ is *close to* a uniquely determined one when the inequality (31) holds for a small $\varepsilon > 0$.

Let us show the sketch of the proof of Theorem 3.1. Let

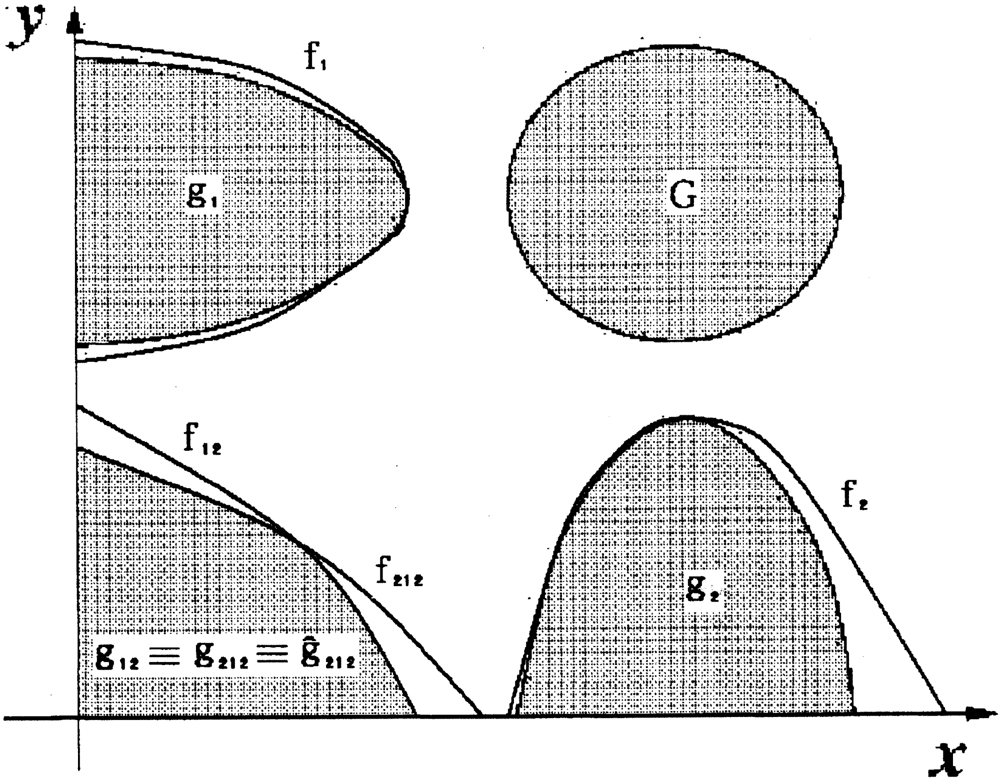
$$\begin{aligned} \hat{g}_{212}(x) &:= \min\{f_{12}(x), f_{212}(x)\}, \\ G_{yxy} &:= \{(x, y) \mid 0 < y < g_{212}(x)\}. \end{aligned} \tag{33}$$

The essential idea for the proof is to construct the pair of functions $(g_1, g_2)$ such that

$$\|f_i - g_i\|_{L^1(\mathbb{R})} < \varepsilon, \quad \text{for } i = 1, 2 \tag{34}$$

and that

$$g_{12} \equiv g_{212} \equiv \hat{g}_{212}. \tag{35}$$




By definition, the pair $(g_1, g_2)$ determines the set $G$ uniquely. The set $G$ is proved to satisfy (32). For more detail, confer [7].

The following theorem follows from Theorem 3.1.

**Theorem 3.2.** *Assume that two sets $F_1$ and $F_2$ have the same projections $(f_1, f_2)$ which are non-unique and satisfy* (31). *Then there holds*

$$\lambda_2(F_1 \ominus F_2) \leq 4\varepsilon \min\{\|f_1\|_{L^\infty(\mathbb{R})}, \|f_2\|_{L^\infty(\mathbb{R})}\}. \tag{36}$$

By virtue of Theorems 3.1 and 3.2, we obtain an answer to Problem 1.2.

**Theorem 3.3.** (an answer to Problem 1.2)
*In order to approximately reconstruct non-unique sets whose projections satisfy* (31), *we have only to approximate $G$ defined in the proof of Theorem 3.1. Since $G$ is uniquely determined, we can apply the method by Huang-Takiguchi to construct approximate sets for uniquely determined sets from their projections, as we have mentioned in the second section.*

## 4. Conclusions and open problems

Let us conclude conclusions.

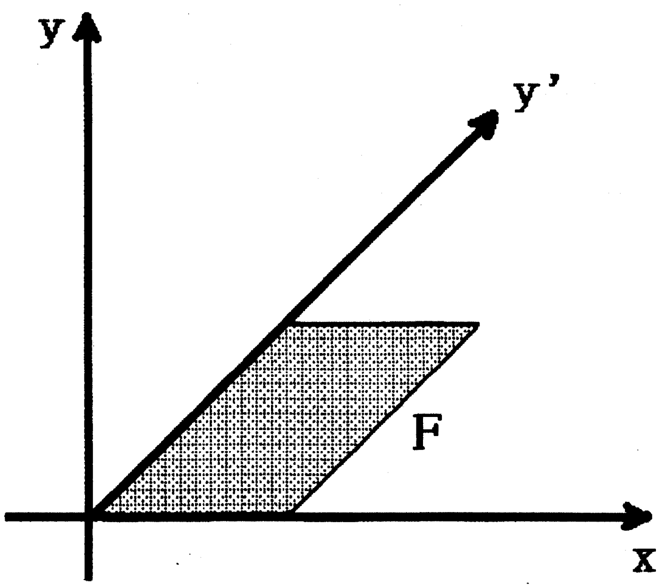
**Conclusion 4.1.**

- **(i):** *We have proved that we can approximately reconstruct the sets whose orthogonal projections are close to those of the uniquely determined ones.*
- **(ii):** *In practical applications, we may assume the existence of solutions. Therefore even if we obtain inconsistent projections by the effect of noise, the construction of approximate solution is possible by approximating $G$ in the proof of Theorem 13.*
- **(iii):** *As a method to approximate $G$, we can apply an algorithm by Huang-Takiguchi as we mentioned above.*
- **(iv):** *The results in this article do not require a priori information on the set itself. Only projections possibly with noise are required.*

In view of these, our work made the study of Problem 1.1 more practical. However, our results are still not sufficiently satisfactory, since they cover only the limited cases.

In the rest of this article, we discuss what are to be studied for further developments. If we consider orthogonal projections, there are many sets which are neither uniquely determined, nor approximated by a uniquely determined set. Consider, for instance, the set

$$F := \{(x, y) \mid y < x < y + 1,\ 0 < y < 1\}. \tag{37}$$

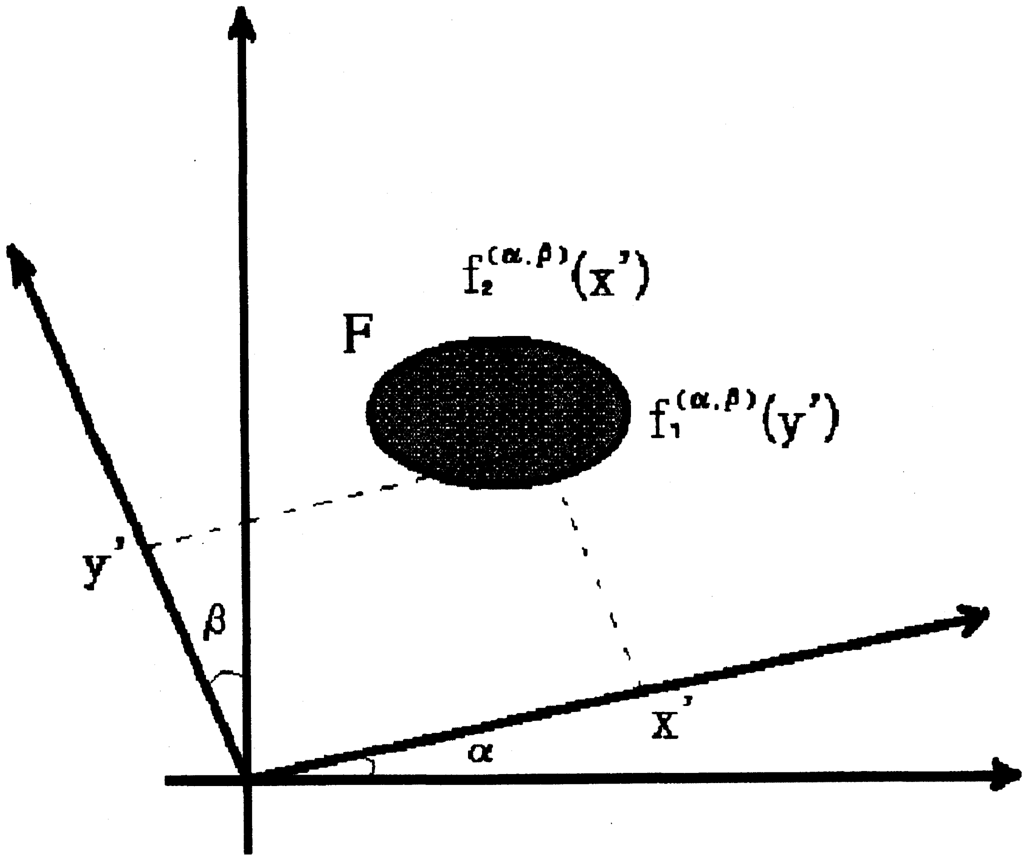



Though this set is a typical example of non-uniquely determined ones, we can reconstruct this set uniquely from its two projections if we rotate the $y$- axis by $-\pi/4$. In order to explain this example, let us prepare some definitions.

**Definition 4.1.** For $-\pi/2 < \alpha, \beta < \pi/2$ $(\alpha < \pi/2 + \beta)$, let

$$
\begin{aligned}
f_1^{(\alpha,\beta)}(y') &:= \int_{-\infty}^{\infty} f(-y' \sin\beta + t\cos\alpha, y' \sin\beta + t \sin\alpha) dt, \\
f_2^{(\alpha,\beta)}(x') &:= \int_{-\infty}^{\infty} f(x' \cos\alpha - t\cos\beta, x' \sin\alpha + t\sin\beta) dt,
\end{aligned} \tag{38}
$$

where $x' = (x', \alpha)$, $y' = (y', \pi/2 + \beta)$ in the polar coordinate.




**Definition 4.2.** For $f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)}$ defined in Definition 4.1, we define the projections of them by

$$
\begin{aligned}
f_{12}^{(\alpha,\beta)}(x) &:= \lambda_1(\{y |\ f_1^{(\alpha,\beta)}(y) \geq x\}), \\
f_{21}^{(\alpha,\beta)}(y) &:= \lambda_1(\{x |\ f_2^{(\alpha,\beta)}(x) \geq y\}), \\
f_{121}^{(\alpha,\beta)}(y) &:= \lambda_1(\{x | f_{12}^{(\alpha,\beta)}(x) \geq y\}), \\
f_{212}^{(\alpha,\beta)}(x) &:= \lambda_1(\{y | f_{21}^{(\alpha,\beta)}(y) \geq x\}).
\end{aligned} \tag{39}
$$

**Definition 4.3.**

$$
\begin{aligned}
\int_{-\infty}^{\infty} f^{(\alpha,\beta)}(x') \widehat{dx'} &:= \int_{-\infty}^{\infty} f^{(\alpha,\beta)}(t) \cos(\beta - \alpha) dt, \\
\int_{-\infty}^{\infty} f^{(\alpha,\beta)}(y') \widehat{dy'} &:= \int_{-\infty}^{\infty} f^{(\alpha,\beta)}(t) \cos(\alpha - \beta) dt.
\end{aligned} \tag{40}
$$

For these rotated axes, we obtain the following modification of the theorem by G.G. Lorentz.

**Theorem 4.1.** *Let $f_1^{(\alpha,\beta)}(y')$ and $f_2^{(\alpha,\beta)}(x')$ be non-negative, integrable functions such that*

$$\int_{-\infty}^{\infty} f_1^{(\alpha,\beta)}(y')\widehat{dy'} = \int_{-\infty}^{\infty} f_2^{(\alpha,\beta)}(x')\widehat{dx'}. \tag{41}$$

(i) *The unique case.*
*There exists a unique set in $\mathbb{R}^2$ which has $(f_1, f_2)$ as projections if and only if*

$$\int_0^c f_{12}^{(\alpha,\beta)}(x')\widehat{dx'} = \int_0^c f_{212}^{(\alpha,\beta)}(x')\widehat{dx'}, \text{ for any } c > 0. \tag{42}$$

(ii) *The non-unique case.*
*There exist plural sets having $(f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})$ as projections if and only if*

$$\int_0^c f_{12}^{(\alpha,\beta)}(x')\widehat{dx'} \geq \int_0^c f_{212}^{(\alpha,\beta)}(x')\widehat{dx'}, \text{ for any } c > 0, \tag{43}$$

*and there is a $c > 0$ for which the strict inequality holds.*
(iii) *The inconsistent case.*
*There exists no set having $(f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})$ as projections if and only if*

$$\int_0^c f_{12}^{(\alpha,\beta)}(x')\widehat{dx'} < \int_0^c f_{212}^{(\alpha,\beta)}(x')\widehat{dx'}, \text{ for some } c > 0. \tag{44}$$

Let us go back to the example (37). The pair of projections of $F$ which makes $F$ uniquely determined is of course not orthogonal. The first problem we pose is the following.

**Problem 4.1.** *For any non-unique set $F$, do there exist directions $\alpha$ and $\beta$ such that rotating the $x$-axis by $\alpha$ and rotating the $y$-axis by $\beta$ make $F$ uniquely reconstructed from the projections to the rotated axes?*

The author expects that this problem is affirmatively solved. Even if the answer to Problem 4.1 is not true, there are sets, like the one defined by (37), which are made to be uniquely determined by rotating the axes. For practical applications, we have to find such angles $\alpha$ and $\beta$ only from the projections $(f_1, f_2)$, not from $F$ itself, which is our next problem to pose.

**Problem 4.2.** *Assume that for a non-unique set $F$, do there exist such angles $\alpha$ and $\beta$ that rotating the $x$-axis by $\alpha$ and rotating the $y$-axis by $\beta$ make $F$ uniquely reconstructed from the projections to the rotated axes? If they do exist, find such angles $\alpha$ and $\beta$ from the orthogonal projections $(f_1, f_2)$ and some additional projections.*

The orthogonal projections $(f_1, f_2)$ are trivially not sufficient to determine the angles $\alpha$ and $\beta$. Hence some additional information will be necessary. We note that we do not have to obtain the exact values of $\alpha$ and $\beta$ for practical applications. Their approximate values are sufficient.

If the Problems 4.1 and 4.2 are solved, construction of approximate solutions is possible since we can modify an algorithm by Huang-Takiguchi.

**Theorem 4.2.** *An algorithm by Huang-Takiguchi mentioned in the second section works in the framework of rotated axes.*

In this theorem, we approximate sets by parallelograms, not by rectangles. The essential idea is the same as in the case of orthogonal projections.

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