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NEWTON POLYHEDRA AND THE BOUNDARY BEHAVIOR OF THE BERGMAN KERNEL

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1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^n$ and $A^2(\Omega)$ the Bergman space of Ω, that is, the Hilbert space of the L^2-holomorphic functions on Ω. The *Bergman kernel* $B(z)$ of Ω (on the diagonal) is defined by

$$B(z) = \sum_{\alpha} |\varphi_\alpha(z)|^2,$$

where $\{\varphi_\alpha\}_\alpha$ is a complete orthonormal basis of $A^2(\Omega)$. Throughout this article, we assume that the boundary $\partial\Omega$ of Ω is always C^∞-smooth. For a boundary point p, the number

$$g(p) = \sup\left\{ s > 0 \ ; \ \varliminf_{\substack{z\to p \\ z\in\Lambda}} B(z)\cdot|z-p|^s = \infty \right\}$$

is called the *growth exponent* of the Bergman kernel at p, where Λ is a nontangential cone with apex at p.

As is well known, the singularities of the Bergman kernel contain a lot of important geometrical information of the respective domains. Let us consider a fundamental question:

What kinds of geometrical characteristics of domains determine the boundary behavior of the Bergman kernel ?

There are many interesting results giving partial answers to this question. For the moment, we restrict our attention to studies about the situation for the growth of the Bergman kernel at the boundary. In the case of strongly pseudoconvex domains, the *dimension* appears in the growth exponent of the Bergman kernel in [24], [7], [8]. In the general pseudoconvex case, it is known in [30],[12] that the boundary behavior of the Bergman kernel can be estimated by using the *rank of the Levi form.* More precisely, Diederich and Herbort [10] showed that *Catlin's multitype* completely determines the growth exponent in the case of semiregular domains (which are also called h-extendible domains). Boas, Straube and Yu [2] refined their result and obtained a detailed result about the boundary limit in this case (see also [11]).

Although this multitype is an important invariant for the study of the Bergman kernel, some specific domains of finite type in $\mathbb{C}^3$ in [22],[9] show that it is not sufficient for the analysis of its singularities. Indeed Herbort [22] found a domain whose Bergman kernel has logarithmic growth and Diederich and Herbort [9] gave some class of domains with parameters to show that the growth exponent is not always determined by the multitype.

Now let us look at further essential geometrical characteristics of domains to determine the singularities of the Bergman kernel for a more general class of pseudoconvex domains containing the above examples. For this purpose, we introduce some concept of the theory of singularities into the analysis of the Bergman kernel. By doing so, we succeed to compute its asymptotic expansion. From our result, it becomes clear, that the principal term of the asymptotic expansion of the Bergman kernel is determined completely by the geometry of the *Newton polyhedron* associated with the defining functions of the domains and the *theory of toric varieties* plays important roles in the computation of its asymptotic expansion.

2. Main results

2.1. **Newton polyhedra.**

Let us introduce some concepts of the *theory of singularities* into the analysis of the Bergman kernel (see [34],[1],[31] for precise definitions).

Let $\mathbb{Z}_+$ and $\mathbb{R}_+$ be the sets of non-negative integers and real numbers, respectively. First let us recall the definition of the Newton polyhedra of functions in the real space. Let f be a real valued C^∞-smooth function in a neighborhood in $\mathbb{R}^n$ of the origin with $f(0) = 0$. Let

$$\sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha x^\alpha = \sum_{\alpha\in\mathbb{Z}_+^n} c_{\alpha_1,\ldots,\alpha_n} x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

be the Taylor expansion of f at the origin. Then the *support* of f is the set:

$$S_f = \{\alpha\in\mathbb{Z}_+^n; c_\alpha \neq 0\},$$

and the *Newton polyhedron* of f is the integral polyhedron:

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup\{\alpha+\mathbb{R}_+^n; \alpha\in S_f\} \text{ in } \mathbb{R}_+^n.$$

The *Newton diagram* $\Gamma(f)$ of f is the union of the compact faces of the Newton polyhedron $\Gamma_+(f)$. The *Newton principal part* of f is

$$f_0(x) = \sum_{\alpha\in\Gamma(f)} c_\alpha x^\alpha.$$

Now we suppose that there exists a point at which the line $\{(d,\dots,d); d > 0\}$ intersects the Newton diagram $\Gamma(f)$ and we denote this point by $Q_0 = (d_f,\dots,d_f)$. Then we call the value of d_f as the *distance* of $\Gamma(f)$. Let $\hat{m}_f$ be the number of the $(n-1)$-dimensional faces on $\Gamma(f)$ containing Q_0. Then define $m_f = \min\{\hat{m}_f, n\}$, which we call the *multiplicity* of $\Gamma(f)$.

We generalize these concepts to the case of the functions in the complex space. Let F be a real valued C^∞-smooth function in a neighborhood in $\mathbb{C}^n$ of the origin with $F(0) = 0$. Let

$$\sum_{\alpha,\beta\in\mathbb{Z}_+^n} C_{\alpha\beta} z^\alpha \bar{z}^\beta = \sum_{\alpha,\beta\in\mathbb{Z}_+^n} C_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n} z_1^{\alpha_1}\cdots z_n^{\alpha_n}\bar{z}_1^{\beta_1}\cdots\bar{z}_n^{\beta_n}$$

be the Taylor series of F at the origin. Then the *support* of F is the set:

$$S_F = \{\alpha+\beta \in \mathbb{Z}_+^n; C_{\alpha,\beta} \neq 0\},$$

and the *Newton polyhedron* of F is the integral polyhedron:

$$\tilde{\Gamma}_+(F) = \text{the convex hull of the set } \bigcup\{\alpha+\beta+\mathbb{R}_+^n; \alpha+\beta\in S_F\} \text{ in } \mathbb{R}_+^n.$$

The *Newton diagram* $\tilde{\Gamma}(F)$ of F is the union of the compact faces of the Newton polyhedron $\tilde{\Gamma}_+(F)$. The *Newton principal part* of F is

$$F_0(z) = \sum_{\alpha+\beta\in\tilde{\Gamma}(F)} C_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Now we suppose that there exists a point at which the line $\{(d,\dots,d); d > 0\}$ intersects the Newton diagram $\tilde{\Gamma}(F)$ and we denote this point by $Q_0 = (d_F,\dots,d_F)$. Then we call the value of d_F as the *distance* of $\tilde{\Gamma}(F)$. Let $\hat{m}_F$ be the number of the $(n-1)$-dimensional faces on $\tilde{\Gamma}(F)$ containing Q_0. Then define $m_F = \min\{\hat{m}_F, n\}$, which we call the *multiplicity* of $\tilde{\Gamma}(F)$.

2.2. **Main results.** Our results are concerned with the structure of singularities of the Bergman kernel for some class of pseudoconvex domains of finite type from the viewpoint of the theory of singularities.

Let F be a C^∞-smooth plurisubharmonic function on $\mathbb{C}^n$ satisfying that $F(0) = \nabla F(0) = 0$. We consider the domain:

$$\Omega_F = \{(z_0, z) = (z_0, z_1, \dots, z_n) \in \mathbb{C}\times\mathbb{C}^n; \Im(z_0) > F(z_1,\dots,z_n)\}.$$

We give the following assumptions on Ω_F.

(1) $0 \in \partial\Omega_F$ is a point of finite type (in the sense of D'Angelo [6]).
(2) $F(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n) = F(z_1,\dots,z_n)$ for any $\theta_j \in \mathbb{R}$.

(3) There are some small positive numbers c and ϵ such that $F(z) \geq c|z|^\epsilon$ for sufficiently large $|z| := (\sum_{j=1}^n |z_j|^2)^{1/2}$.

The last assumption implies that the dimension of the Bergman space $A^2(\Omega_F)$ is infinity.

Now let us mention our main results about the Bergman kernel $B(z_0, z)$ of Ω_F. First if we restrict the Bergman kernel on the vertical set to z-plane through the origin, then its singularity can be expressed as follows.

Theorem 2.1. *The Bergman kernel of the domain Ω_F has the form:*

$$B(z_0, 0) = \int_0^\infty e^{-\rho\tau} K(\tau)\tau d\tau, \tag{2.1}$$

where ρ is the imaginary part of $2z_0$ and $K(\tau)^{-1}$ has an asymptotic expansion of τ:

$$\frac{1}{K(\tau)} \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j-1} a_{j,k}\tau^{-p_j}(\log\tau)^k \quad \text{as } \tau \to \infty, \tag{2.2}$$

where the coefficients $a_{j,k}$ are real numbers. Here there exists a method of calculation of the powers p_j and m_j on the basis of the theory of toric varieties. Actually, p_j belong to finitely many arithmetic progressions constructed from positive rational numbers with $p_0 < p_1 < p_2 < \cdots$ and m_j belong to the set $\{1, \ldots, n\}$. Moreover the principal term of the asymptotic expansion (2.2) takes the form:

$$a(F_0)\tau^{-2/d_F}(\log\tau)^{m_F-1},$$

where d_F is the distance of $\tilde{\Gamma}(F)$ and m_F is the multiplicity of $\tilde{\Gamma}(F)$ as in Section 2.1 and $a(F_0)$ is a positive number depending only on the Newton principal part of F.

Remark 2.2. Since the condition of finite type implies the Newton diagram of F intersects all the coordinate axes, there exists the point Q_0 in Section 2.1 and the values of d_F and m_F can be defined.

Remark 2.3. Since the powers p_j in Theorem 2.1 belong to finitely many arithmetic progressions constructed from rational numbers, there exists a natural number m such that all the p_j belong to the set $\{k/m; k \in \mathbb{N}\}$. Actually there exists a method to give the exact value of m.

Remark 2.4. In order to correspond the well-known strongly pseudoconvex case, let us recall the result of Boutet de Monvel and Sjöstrand [3]. They computed the asymptotic expansion of the Bergman kernel for bounded strongly pseudoconvex domains $\Omega \subset \mathbb{C}^{n+1}$ by using Fourier integral operators with complex phase. Now we rewrite their result in our style. The Bergman kernel $B(z)$ has the form near the boundary:

$$B(z) = \int_0^\infty e^{-\rho\tau} K(z;\tau)\tau d\tau \qquad (z \in \mathbb{C}^{n+1}),$$

where ρ is a defining function of Ω and $K(z;\tau)$ has an asymptotic expansion of τ:

$$K(z;\tau) \sim \tau^n \sum_{j=0}^\infty a_j(z)\tau^{-j} \quad \text{as } \tau \to \infty,$$

where $a_j \in C^\infty(\bar{\Omega})$ and a_0 is positive at the boundary.

Next, in order to see the asymptotic expansion of the Bergman kernel directly, we introduce some polar coordinates. For a small $R > 0$, a nontangential cone Λ is defined by $\Lambda = \{(z_0, z); |z| < R\rho\}$ with $\rho = 2\Im(z_0)$ and set $U(R) = \{w \in \mathbb{C}^n; |w| < R\}$. We define the mapping h from $U(R) \times (0, \rho_0]$ to the cone $\Lambda \subset \mathbb{C}^{n+1}$ by $h(w,\rho) = (\rho, \rho w_1, \ldots, \rho w_n) = (\rho, \rho w) \in \Lambda$, where ρ_0 is a sufficiently small positive number such that the image of h is contained in Ω_F.

The following theorem shows that the singularity of the Bergman kernel can be expressed by a sum of combinations of $\rho^{1/m}$ and $\log(1/\rho)$ as follows:

Theorem 2.5. *The Bergman kernel of Ω_F can be written near the origin on a nontangential cone Λ as:*

$$B(h(w,\rho)) = \frac{\Phi(w,\rho)}{\rho^{2+2/d_F}(\log(1/\rho))^{m_F-1}}. \tag{2.3}$$

Here Φ admits the following asymptotic expansion:

$$\Phi(w,\rho) \sim \sum_{j=0}^\infty \sum_{k=a_j}^\infty C_{j,k}(w)\rho^{j/m}(\log(1/\rho))^{-k} \quad \text{as } \rho \to 0 \tag{2.4}$$

for $w \in U(R)$ where a_j are integers with $a_0 = 0$ and the coefficients $C_{j,k}(w)$ are polynomials of $|w_1|^2, \ldots, |w_n|^2$, $C_{0,0}(w)$ is a positive constant depending only on the Newton principal part of F and m is a natural number as in Remark 2.3.

Remark 2.6. From arguments in the proof of Theorem 2.5, more detailed structure of the asymptotic expansion (2.4) can be seen as follows. $\Phi(w,\rho)$ can be expressed as

$$\Phi(w,\rho) = \Phi^{(1)}(w,\rho) + \Phi^{(2)}(w,\rho)\log\rho,$$

where $\Phi^{(1)}$ and $\Phi^{(2)}$ admit the following asymptotic expansions:

$$\Phi^{(1)}(w,\rho) \sim \sum_{j=0}^{\infty} \sum_{k=(m_F-n)j}^{\infty} C_{j,k}^{(1)}(w)\rho^{j/m}(\log(1/\rho))^{-k} \quad \text{as } \rho \to 0,$$

$$\Phi^{(2)}(w,\rho) \sim \sum_{j=m(2+2/d_F)}^{\infty} \sum_{k=(m_F-n)j}^{\infty} C_{j,k}^{(2)}(w)\rho^{j/m}(\log(1/\rho))^{-k} \quad \text{as } \rho \to 0,$$

where the coefficients $C_{j,k}^{(1)}(w), C_{j,k}^{(2)}(w)$ are polynomials of $|w_1|^2,\ldots,|w_n|^2$ and $C_{j,k}^{(2)}(w) = 0$ if $j \neq m(2+2/d_F+l)$ $(l \in \mathbb{N})$.

Let us consider the particular case that the Newton diagram of F has only one face. This means that the principal part of F is quasihomogeneous and, moreover, the origin on $\partial\Omega_F$ is of semiregular.

Theorem 2.7. *If the Newton diagram of F has only one face and the multitype of the origin is $(1, 2m_1, \ldots, 2m_n)$, then the Bergman kernel of Ω_F can be written near the origin on a nontangential cone Λ as:*

$$B(h(w,\rho)) = \frac{\tilde{\Phi}(w,\rho)}{\rho^{2+\sum_{j=1}^{n} 1/m_j}} + \tilde{\tilde{\Phi}}(w,\rho)\log\rho.$$

Here $\tilde{\Phi}$ and $\tilde{\tilde{\Phi}}$ admit asymptotic expansions on Λ:

$$\tilde{\Phi}(w,\rho) \sim \sum_{j=0}^{\infty} \tilde{C}_j(w)\rho^{j/m}, \quad \tilde{\tilde{\Phi}}(w,\rho) \sim \sum_{j=0}^{\infty} \tilde{\tilde{C}}_j(w)\rho^{j} \quad \text{as } \rho \to 0,$$

for all $w \in U(R)$ where m is the least common multiple of $m_1,\ldots,m_n$ and the coefficients $\tilde{C}_j(w), \tilde{\tilde{C}}_j(w)$ are polynomials of $|w_1|^2,\ldots,|w_n|^2$ and $\tilde{C}_0(w)$ is a positive constant depending only on the Newton principal part of F.

Remark 2.8. Analogous results to the above theorems can be obtained in the case of the Szegö kernel.

3. Proofs of main theorems

In the argument below, the lemmas concerning asymptotic expansion of some integral are very important. But we omit their proofs (see [26]).

3.1. **Some integral formula.** For $a = (a_1,\ldots,a_n) \in \mathbb{R}_+^n$, let $|a| = a_1 + \cdots + a_n$. Let F be a C^∞-smooth plurisubharmonic function on $\mathbb{C}^n$. The weighted Hilbert space $H_\tau(\mathbb{C}^n)$ $(\tau > 0)$ consists of all entire functions $\psi : \mathbb{C}^n \to \mathbb{C}$ such that

$$\int_{\mathbb{C}^n} |\psi(z)|^2 e^{-2\tau F(z)} dV(z) < \infty,$$

where dV denotes the Lebesgue measure. If F satisfies the assumption (3) in Section 2.2, then $H_\tau(\mathbb{C}^n)$ contains z^α for all $\alpha \in \mathbb{Z}_+^n$. The reproducing kernel (on the diagonal) of $H_\tau(\mathbb{C}^n)$ is denoted by $K(z;\tau)$. We remark that the function $\tau \mapsto K(z;\tau)$ is continuous for fixed z from the result in [13]. Haslinger [20],[21] obtained an interesting relation between $K(z;\tau)$ and the Bergman kernel $B(z_0, z)$ of the domain $\Omega_F = \{(z_0, z) \in \mathbb{C}^{n+1}; \Im(z_0) > F(z)\}$ as follows:

$$B(z_0, z) = \frac{1}{2\pi}\int_0^\infty e^{-\rho\tau} K(z;\tau)\tau d\tau, \tag{3.1}$$

where ρ is the imaginary part of $2z_0$.

3.2. **Proof of Theorem 2.1.** Now we add a strong assumption (2) to the condition of $F(z)$: $F(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) = F(z_1, \ldots, z_n)$ for any $\theta_j \in \mathbb{R}$. Then we can take a complete orthonormal system for $H_\tau(\mathbb{C}^n)$ as

$$\left\{\frac{z^\alpha}{c_\alpha(\tau)}\ ;\ \alpha \in \mathbb{Z}_+^n\right\}, \quad \text{with } c_\alpha(\tau)^2 = \int_{\mathbb{C}^n} |z|^{2\alpha} e^{-2\tau F(z)} dV(z)$$

($|z|^{2\alpha} := |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n}$). Thus $K(z;\tau)$ takes the form:

$$K(z;\tau) = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{|z|^{2\alpha}}{c_\alpha(\tau)^2}.$$

From the above representation, the behavior of $K(z;\tau)$ as $\tau \to \infty$ is determined by properties of $c_\alpha(\tau)^2$.

The following is the main lemma for our theorems, which is concerned with the behavior of $c_\alpha(\tau)^2$ at infinity. Our proof of the lemma needs the theory of toric varieties.

Lemma 3.1. *If F satisfies the conditions (1)–(3) in Section 2.2, then $c_\alpha(\tau)^2$ has an asymptotic expansion for $\alpha \in \mathbb{Z}_+^n$:*

$$c_\alpha(\tau)^2 \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j - 1} a_{j,k}^{(\alpha)} \tau^{-p_j} (\log\tau)^k \qquad \text{as } \tau \to \infty, \tag{3.2}$$

where the coefficients $a_{j,k}^{(\alpha)}$ are real numbers. Here there exists a method of calculation of the powers p_j and m_j on the basis of the theory of toric varieties. Actually p_j belong to finitely many arithmetic progressions constructed from positive rational numbers with $p_0 < p_1 < p_2 < \cdots$ and m_j belong to the set $\{1, \ldots, n\}$.

Moreover the principal term of the above asymptotic expansion takes the form:

$$a_\alpha(F_0)\tau^{-\beta_\alpha}(\log\tau)^{m_\alpha - 1},$$

where $a_\alpha(F_0)$ is a positive number depending only on $\alpha \in \mathbb{Z}_+^n$ and the Newton principal part F_0 of F and the values of β_α and m_α can be determined as follows: Let $Q = (q_1, \ldots, q_n)$ be the point of the intersection of the Newton diagram $\tilde{\Gamma}(F)$ with the line joining the origin and the point $(2\alpha_1 + 2, \ldots, 2\alpha_n + 2)$. Then we have $\beta_\alpha = 2(|\alpha| + n)/|q|$ ($|q| := q_1 + \cdots + q_n$) and $m_\alpha = \min\{\hat{m}_\alpha, n\}$, where $\hat{m}_\alpha$ is the number of the $(n-1)$-dimensional faces on $\tilde{\Gamma}(F)$ containing the point Q. In particular, we have $\beta_0 = 2/d_F$ and $m_0 = m_F$, where d_F and m_F are as in Section 2.1.

Remark 3.2. From the same reason as in Remark 2.2, the values of β_α and m_α can be defined.

Now if we restrict the Bergman kernel on the set $\{(z_0, z); z = 0\} \cap \Omega_F$, then

$$B(z_0, 0) = \frac{1}{2\pi} \int_0^\infty e^{-\rho\tau} K(0; \tau) \tau d\tau.$$

Since $K(0;\tau) = c_0(\tau)^{-2}$, we can obtain Theorem 2.1 by considering the special case $\alpha = 0$ in the above lemma.

3.3. **Proof of Theorem 2.5.** Before computing asymptotic expansion, let us consider the boundary limit of the Bergman kernel in the sense in [24].

For $w \in U(R)$, $\tau > 0$, $\rho \in (0, \rho_0)$, we have

$$K(\rho w; \tau) = K(\rho w_1, \ldots, \rho w_n; \tau) = \sum_{\alpha \in \mathbb{Z}_+^n} \frac{|w|^{2\alpha}}{c_\alpha(\tau)^2} \rho^{2|\alpha|}.$$

Substituting the above sum into (3.1) and changing the integral and the sum formally, we can obtain a formal sum as follows:

$$B(h(w,\rho)) = \int_0^\infty e^{-\rho\tau} K(\rho w; \tau) \tau d\tau = \sum_{\alpha \in \mathbb{Z}_+^n} B_\alpha(\rho) |w|^{2\alpha}, \tag{3.3}$$

where

$$B_\alpha(\rho) = \rho^{2|\alpha|} \int_0^\infty e^{-\rho\tau} \frac{1}{c_\alpha(\tau)^2} \tau d\tau. \tag{3.4}$$

The sum in (3.3) is denoted by $\hat{B}(w, \rho)$. It is easy to see that the sum $\hat{B}(w,\rho)$ uniformly converges on the set $U(R) \times [\epsilon, \rho_0]$ for any $\epsilon \in (0, \rho_0]$.

From Lemma 3.1, we have

$$\frac{1}{c_\alpha(\tau)^2} = \frac{\tau^{\beta_\alpha}}{(\log \tau)^{m_\alpha - 1}} \{a_\alpha(F_0) + \epsilon(\tau)\}, \tag{3.5}$$

where $\epsilon(\tau) \to 0$ as $\tau \to \infty$. Substituting (3.5) into (3.4), then we have

$$
\begin{aligned}
&\rho^{-2|\alpha|+\beta_\alpha+2}(\log(1/\rho))^{m_\alpha-1} \cdot B_\alpha(\rho) \\
&= \rho^{\beta_\alpha+2}(\log(1/\rho))^{m_\alpha-1}\int_0^\infty e^{-\rho\tau}\frac{\tau^{1+\beta_\alpha}}{(\log\tau)^{m_\alpha-1}}\{a_\alpha(F_0)+\epsilon(\tau)\}d\tau \\
&= \int_0^\infty e^{-s}\left(\frac{\log(1/\rho)}{\log(s/\rho)}\right)^{m_\alpha-1} s^{1+\beta_\alpha}\{a_\alpha(F_0)+\epsilon(s/\rho)\}ds \\
&\longrightarrow a_\alpha(F_0)\int_0^\infty e^{-s}s^{1+\beta_\alpha}ds = \Gamma(\beta_\alpha+2)a_\alpha(F_0) =: C_\alpha(F_0) > 0 \quad \text{as } \rho \to 0.
\end{aligned} \tag{3.6}
$$

Since the value of β_α is given as in Lemma 3.1, we have

$$2|\alpha| - \beta_\alpha - 2 = 2|\alpha| - 2(|\alpha|+n)/|q| - 2 = 2|\alpha|(1-1/|q|) - 2(n/|q|+1).$$

Here the above value is denoted by $\sigma(\alpha, |q|)$. Note that $|q|$ depends on α. Since the Newton diagram $\Gamma(f)$ intersects all the coordinates axes, the value of $|\alpha|$ has the minimum and the maximum for $\alpha \in \Gamma(F)$, which are denoted by q_* and q_{**}, respectively. Moreover we have $|q| \geq 2$ from the conditions of pseudoconvexity and of finite type.

If $\alpha \neq 0$, then $\tilde{B}_\alpha(\rho) = B_\alpha(\rho)/B_0(\rho)$ tends to 0 as $\rho \to 0$. For sufficiently small $\rho > 0$, we have

$$\sum_{\alpha\in\mathbf{Z}_+^n} \tilde{B}_\alpha(\rho)|w|^{2\alpha} \leq \sum_{\alpha\in\mathbf{Z}_+^n} \tilde{B}_\alpha(\rho_0)|w|^{2\alpha} \quad \text{for } w \in U(R).$$

Thus Lebesgue's convergence theorem implies that

$$\lim_{\rho\to 0}\sum_{\alpha\in\mathbf{Z}_+^n} \tilde{B}_\alpha(\rho)|w|^{2\alpha} = \sum_{\alpha\in\mathbf{Z}_+^n}\left(\lim_{\rho\to 0}\tilde{B}_\alpha(\rho)\right)|w|^{2\alpha} = 1. \tag{3.7}$$

From (3.6),(3.7), we have

$$
\begin{aligned}
&\lim_{\rho\to 0}\rho^{2+2/d_F}(\log(1/\rho))^{m_F-1}\hat{B}(w,\rho) \\
&= \lim_{\rho\to 0}\rho^{2+2/d_F}(\log(1/\rho))^{m_F-1}B_0(\rho)\sum_{\alpha\in\mathbf{Z}_+^n}\tilde{B}_\alpha(\rho)|w|^{2\alpha} = C_0(F_0)\cdot 1.
\end{aligned}
$$

Now let us compute the asymptotic expansion of the Bergman kernel in the theorem. For sufficiently large integer N, we define

$$R_N(w,\rho) = \sum_{|\alpha|\geq N} B_\alpha(\rho)|w|^{2\alpha}.$$

Then we can write $\hat{B}(w,\rho)$ as follows:

$$\hat{B}(w,\rho) = \sum_{|\alpha|<N} B_\alpha(\rho)|w|^{2\alpha} + R_N(w,\rho). \tag{3.8}$$

From (3.6), if $|\alpha| \geq N+1$, then $\lim_{\rho\to 0}\rho^{\sigma(\alpha,q_*)}B_\alpha(\rho) = 0$. In a similar fashion to (3.7), we have

$$\lim_{\rho\to 0}\rho^{\sigma(\alpha,q_*)}\sum_{|\alpha|\geq N+1} B_\alpha(\rho)|w|^{2\alpha} = \sum_{|\alpha|\geq N+1}(\lim_{\rho\to 0}\rho^{\sigma(\alpha,q_*)}B_\alpha(\rho))|w|^{2\alpha} = 0$$

For each α with $|\alpha| = N$, there exists a positive constant C_α such that

$$|\rho^{\sigma(\alpha,q_*)}B_\alpha(\rho)| \leq C_\alpha$$

for $\rho \in [0,\rho_0]$. Thus there exist positive constants $\tilde{C}_N, C_N$ such that

$$\begin{aligned} \rho^{\sigma(\alpha,q_*)}R_N(w,\rho) &= \sum_{|\alpha|\geq N}(\rho^{\sigma(\alpha,q_*)}B_\alpha(\rho))|w|^{2\alpha} \\ &\leq \sum_{|\alpha|=N} C_\alpha|w|^{2\alpha} + \tilde{C}_N \leq C_N R^{2N} \end{aligned} \tag{3.9}$$

for $\rho \in [0,\rho_0]$. From this estimate, the remainder R_N becomes asymptotically smaller as $N\to\infty$ with respect to the variable ρ. Therefore the equation (3.8) can be regarded as an asymptotic expansion as $\rho \to 0$.

Finally we can compute the asymptotic expansion in the theorem by putting (3.8), (3.9) and the following lemma together.

Lemma 3.3. *$B_\alpha(\rho)$ takes the form:*

$$B_\alpha(\rho) = \frac{\rho^{2|\alpha|-\beta_\alpha-2}}{(\log(1/\rho))^{m_\alpha-1}}[B_\alpha^{(1)}(\rho) + B_\alpha^{(2)}(\rho)\log(1/\rho)] + B_\alpha^{(3)}(\rho),$$

where $B_\alpha^{(3)} \in C^\infty([0,\epsilon))$ and $B_\alpha^{(1)}$ and $B_\alpha^{(2)}$ admit the following asymptotic expansions:

$$B_\alpha^{(1)}(\rho) \sim \sum_{j=0}^{\infty}\sum_{k=(m_\alpha-n)j}^{\infty} B_{j,k}^{(\alpha)}\rho^{j/m}(\log(1/\rho))^{-k} \quad \textit{as } \rho\to 0,$$

$$B_\alpha^{(2)}(\rho) \sim \sum_{j=m(\beta_\alpha+2)}^{\infty}\sum_{k=(m_\alpha-n)j}^{\infty} \tilde{B}_{j,k}^{(\alpha)}\rho^{j/m}(\log(1/\rho))^{-k} \quad \textit{as } \rho\to 0,$$

where $B_{j,k}^{(\alpha)}$ and $\tilde{B}_{j,k}^{(\alpha)}$ are real numbers and, in particular, $B_{0,0}^{(\alpha)}$ is a positive number and $\tilde{B}_{j,k}^{(\alpha)} = 0$ if $j \neq m(\beta_\alpha + 2 + l)$ $(l \in \mathbb{Z}_+)$.

Proof. By using the following lemma, the above asymptotic expansion can be obtained through standard asymptotic analysis (cf. [16]).

Lemma 3.4. *For $\alpha \in \mathbb{Z}_+^n$, there exist real numbers $b_{j,k}^{(\alpha)}$ with a positive number $b_{0,0}^{(\alpha)} = a_\alpha(F_0)^{-1}$ such that*

$$\frac{1}{c_\alpha(\tau)^2} \sim \frac{\tau^{\beta_\alpha}}{(\log \tau)^{m_\alpha - 1}} \sum_{j=0}^{\infty} \sum_{k=(m_\alpha - n)j}^{\infty} b_{j,k}^{(\alpha)} \tau^{-j/m} (\log \tau)^{-k} \quad \textit{as } \tau \to \infty.$$

If $m_\alpha = 1$, then $b_{j,k}^{(\alpha)} = 0$ for $k > 0$.

Proof. A computation implies the above expansion from (3.2) in Lemma 3.1. □

3.4. **Proof of Theorem 2.9.** This theorem can be proved from the following lemma in the same fashion as in the previous section.

Lemma 3.5. *If F satisfies the conditions (1)–(3) in Section 2.2 and the Newton diagram of F has only one face, then $c_\alpha(\tau)^2$ has the asymptotic expansion:*

$$c_\alpha(\tau)^2 \sim \tau^{-\sum_{j=1}^{n} (\alpha_j + 1)/m_j} \sum_{j=0}^{\infty} a_j^{(\alpha)} \tau^{-j/m} \quad \textit{as } \tau \to \infty, \tag{3.10}$$

where the coefficients $a_j^{(\alpha)}$ are real numbers with $a_0^{(\alpha)} > 0$ and $m_1, \ldots, m_n, m$ are as in Theorem 2.7.

3.5. **Asymptotic expansion of the weighted Bergman kernel.** Let us consider the behavior of the reproducing kernel $K(z;\tau)$ of the weighted Bergman space $H_\tau(\mathbb{C}^n)$ when the parameter τ tends to infinity. From arguments in the proof of main theorems, we can obtain the following result. Analogous results have been obtained in [36],[5],[14],[15] in the strongly pseudoconvex case.

Theorem 3.6. *Suppose that F satisfies the conditions (1)–(3) in Section 2.2. Then there is a small neighborhood U of the origin such that the weighted Bergman kernel $K(z;\tau)$ has an asymptotic expansion:*

$$K(z;\tau) \sim \frac{\tau^{2/d_F}}{(\log \tau)^{m_F - 1}} \sum_{j=0}^{\infty} \sum_{k=(m_F - n)j}^{\infty} b_{j,k}(z) \tau^{-j/m} (\log \tau)^{-k} \quad \textit{as } \tau \to \infty,$$

for all $z \in U$ where the coefficients $b_{j,k}(z)$ are polynomials of $|z_1|^2, \ldots, |z_n|^2$, $b_{0,0}$ is a positive constant depending only on the principal part of F and m is as in Theorem 2.5. Moreover, if the Newton diagram of F has only one face, then

$$K(z;\tau) \sim \tau^{\sum_{j=1}^{n} 1/m_j} \sum_{j=0}^{\infty} b_j(z) \tau^{-j/m} \quad \textit{as } \tau \to \infty,$$

for all $z \in U$ where $m, m_1, \dots, m_n$ are natural numbers as in Theorem 2.7, the coefficients $b_j(z)$ are polynomials of $|z_1|^2, \dots, |z_n|^2$ and b_0 is a positive constant depending only on the principal part of F.

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