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Multiple zeta values and connection formulas of Gauss's hypergeometric functions

Takashi AOKI and Yasuo OHNO

Kinki University, Department of Mathematics

A new family of relations between sums of multiple zeta values and Riemann zeta values are established.

1 A quick review of multiple zeta values

In this section, we give an overview of multiple zeta values (cf. [2], [6]). Multiple zeta values are natural generalization of Riemann zeta values, that is, the values of the Riemann zeta function

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}$$

at integers $k>1$. Euler already knew Riemann zeta values for even positive integers:

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\ \ldots,\ \zeta(2k)=-\frac{(2\pi i)^{2k}B_{2k}}{2(2k)!},\ \ldots.$$

Here B_{2k} are Bernoulli numbers:

$$\frac{xe^x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}.$$

However, we don't know much about Riemann zeta values for odd integers $\zeta(2k+1)$.

Now the multiple zeta values, which are natural generalization of Riemann zeta values for multiple indices $\boldsymbol{k}=(k_1,k_2,\ldots,k_n)$ ($k_i\in\mathbf{Z}$, $k_i>0$), are defined as follows. If $k_1\geq 2$, $\boldsymbol{k}=(k_1,k_2,\ldots,k_n)$ is said to be admissible. For each admissible multiple index $\boldsymbol{k}$, we define two kinds of multiple zeta values $\zeta(\boldsymbol{k})$ and $\zeta^*(\boldsymbol{k})$ respectively by

$$\zeta(\boldsymbol{k})=\zeta(k_1,k_2,\ldots,k_n)=\sum_{m_1>m_2>\cdots>m_n>0}\frac{1}{{m_1}^{k_1}{m_2}^{k_2}\cdots{m_n}^{k_n}}$$

and

$$\zeta^*(\boldsymbol{k})=\zeta^*(k_1,k_2,\ldots,k_n)=\sum_{m_1\geq m_2\geq\cdots\geq m_n\geq 1}\frac{1}{{m_1}^{k_1}{m_2}^{k_2}\cdots{m_n}^{k_n}}.$$

Multiple zeta values normally mean $\zeta(\boldsymbol{k})$ in literatures. But the main subject of this article is concerned with $\zeta^*(\boldsymbol{k})$. (Note that Euler [3] was interested in $\zeta^*(\boldsymbol{k})$.) To distinguish them from ordinary multiple zeta values $\zeta(\boldsymbol{k})$, we call them multiple zeta-star values. We note that $\{\zeta(\boldsymbol{k})\}$ and $\{\zeta^*(\boldsymbol{k})\}$ are not independent over $\mathbb{Q}$. For example, we have

$$\zeta^*(k_1,k_2)=\zeta(k_1,k_2)+\zeta(k_1+k_2),\quad \zeta(k_1,k_2)=\zeta^*(k_1,k_2)-\zeta^*(k_1+k_2),$$
$$\zeta^*(k_1,k_2,k_3)=\zeta(k_1,k_2,k_3)+\zeta(k_1+k_2,k_3)+\zeta(k_1,k_2+k_3)+\zeta(k_1+k_2+k_3),$$
$$\zeta(k_1,k_2,k_3)=\zeta^*(k_1,k_2,k_3)-\zeta^*(k_1+k_2,k_3)-\zeta^*(k_1,k_2+k_3)+\zeta^*(k_1+k_2+k_3),$$

and so on. Some multiple zeta values are evaluated in terms of powers of π. For example, we have

$$\zeta(\underbrace{2,2,\dots,2}_{n})=\frac{\pi^{2n}}{(2n+1)!},\quad \zeta(\underbrace{3,1,3,1,\dots,3,1}_{2n})=\frac{2\pi^{4n}}{(4n+2)!}.$$

The former is obtained by comparing the coefficients of x^{2n} of Taylor expansion of the left-hand side with those of the expansion of the right hand side in x of the infinite product

$$\frac{\sin \pi x}{\pi x}=\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\left(1-\frac{x^2}{3^2}\right)\cdots.$$

The latter is proved by using the Gauss formula that evaluates $F(\alpha,\beta,\gamma;1)$ (cf. [2]). Here $F(\alpha,\beta,\gamma;z)$ denotes the Gauss hypergeometric function. Meanwhile, some multiple zeta values are evaluated by Riemann zeta values. For example, Euler already knew the following relations:

$$\zeta(2,1)=\zeta(3),\ \zeta(2,1,1)=\zeta(4).$$

Thus the set $\{\zeta(\boldsymbol{k})\}$ (or $\{\zeta^*(\boldsymbol{k})\}$) for all admissible indices $\boldsymbol{k}$ is not independent over $\mathbb{Q}$. Hence it is natural to consider the structure of the $\mathbb{Q}$-vector space (or $\mathbb{Q}$-algebra) spanned by $\{\zeta(\boldsymbol{k})\}$. For any multiple index $\boldsymbol{k}=(k_1,k_2,\dots,k_n)$, we set $\mathrm{wt}(\boldsymbol{k})=k_1+k_2+\cdots+k_n$, $\mathrm{dep}(\boldsymbol{k})=n$ and $\mathrm{ht}(\boldsymbol{k})=\#\{i|\ k_i>1\}$ and we call them *weight*, *depth* and *height*, respectively, of $\boldsymbol{k}$. For every integer $k>1$, we denote by Z_k the $\mathbb{Q}$-vector space spanned by $\{\zeta(\boldsymbol{k})|\boldsymbol{k}:\text{admissible and } \mathrm{wt}(\boldsymbol{k})=k\}$. We are interested in the dimension of Z_k. We know, for example,

$$Z_2=\mathbb{Q}\zeta(2)=\mathbb{Q}\pi^2,$$
$$Z_3=\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(2,1)=\mathbb{Q}\zeta(3),$$
$$Z_4=\mathbb{Q}\zeta(4)+\mathbb{Q}\zeta(3,1)+\mathbb{Q}\zeta(2,2)+\mathbb{Q}\zeta(2,1,1)=\mathbb{Q}\pi^4,$$

and all these cases have the same dimension 1. For the case where $k\ge 5$, D. Zagier proposed the following

Conjecture $\dim_{\mathbb{Q}} Z_k=d_k$. Here d_k is defined by the recursion formula

$$\begin{cases} d_0=1, d_1=0, d_2=1,\\ d_k=d_{k-2}+d_{k-3}\quad (k\ge 3).\end{cases}$$

Concerning this conjecture, the following result is known:

Theorem 1 (Goncharov, Terasoma [12]) For all $k \geq 0$, we have $\dim_{\mathbb{Q}} Z_k \leq d_k$.

This theorem is proved by using highly transcendental tools and the proof does not give enough information about concrete linear relations which should hold among multiple zeta values. Several family of linear relations for multiple zeta values had been obtained before Theorem 1 was established. The number of admissible multiple indices with weight k is 2^{k-2}. Thus there should be at least $2^{k-2} - d_k$ linear relations which hold among $\zeta(\boldsymbol{k})$'s with $\mathrm{wt}(\boldsymbol{k}) = k$. The sequence $2^{k-2} - d_k$ grows quite rapidly:

k	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
2^{k-2}	–	–	1	2	4	8	16	32	64	128	256	512	$\cdots$
d_k	1	0	1	1	1	2	2	3	4	5	7	9	$\cdots$

To state some of known results concerning linear relations for multiple zeta values, we introduce the notion of dual index. Let $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ be an admissible multi index. We rewrite $\boldsymbol{k}$ in the form

$$\boldsymbol{k} = (a_1 + 1, \underbrace{1, \ldots, 1}_{b_1 - 1}, a_2 + 1, \underbrace{1, \ldots, 1}_{b_2 - 1}, \ldots, a_s + 1, \underbrace{1, \ldots, 1}_{b_s - 1}).$$

Here $s = \mathrm{ht}(\boldsymbol{k}) \geq 1$, $a_i \geq 1$, $b_i \geq 1$ $(i = 1, \ldots s)$. Now we set

$$\boldsymbol{k}' = (b_s + 1, \underbrace{1, \ldots, 1}_{a_s - 1}, b_{s-1} + 1, \underbrace{1, \ldots, 1}_{a_{s-1} - 1}, \ldots, b_1 + 1, \underbrace{1, \ldots, 1}_{a_1 - 1})$$

and we call $\boldsymbol{k}'$ the dual index of $\boldsymbol{k}$. It is easy to see that $\boldsymbol{k}'$ is also admissible and $(\boldsymbol{k}')' = \boldsymbol{k}$.

1 (Duality) (Drinfeld, Kontsevich, Zagier) For any admissible multi index $\boldsymbol{k}$, we have

$$\zeta(\boldsymbol{k}) = \zeta(\boldsymbol{k}').$$

2 (Hoffman's relation) For any admissible multi index $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$, we have

$$\sum_{i=1}^{n} \zeta(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_n)$$
$$= \sum_{\substack{1 \leq l \leq n \\ k_l \geq 2}} \sum_{j=0}^{k_l - 2} \zeta(k_1, \ldots, k_{l-1}, k_l - j, j + 1, k_{l+1}, \ldots, k_n).$$

3 (Sum formula) (Granville, Zagier) For any integer $k > 1$ and $n \geq 1$ $(n < k)$ we have

$$\sum_{\substack{\mathrm{wt}(\boldsymbol{k}) = k, \mathrm{dep}(\boldsymbol{k}) = n \\ \boldsymbol{k}:\text{admissible}}} \zeta(\boldsymbol{k}) = \zeta(k).$$

This can be rewritten in terms of ζ^*:

$$\sum_{\substack{\mathrm{wt}(\boldsymbol{k})=k,\mathrm{dep}(\boldsymbol{k})=n\\ \boldsymbol{k}:\mathrm{admissible}}} \zeta^*(\boldsymbol{k}) = \binom{k-1}{n-1}\zeta(k). \tag{1}$$

It is known that there is a large family of relations which includes all of above relations:

4 (Ohno [9]) For any admissible multi index $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ and for any integer $l \geq 0$, we have

$$\sum_{\varepsilon_1+\cdots+\varepsilon_n=l} \zeta(k_1+\varepsilon_1, \ldots, k_n+\varepsilon_n) = \sum_{\varepsilon'_1+\cdots+\varepsilon'_{n'}=l} \zeta(k'_1+\varepsilon'_1, \ldots, k'_{n'}+\varepsilon'_{n'}), \tag{2}$$

where $\boldsymbol{k}' = (k'_1, k'_2, \ldots, k'_{n'})$ is the dual index of $\boldsymbol{k}$.

For example, if $\mathrm{wt}(\boldsymbol{k}) = 11$, there should be at least $2^{11-2} - d_{11} = 512 - 9 = 503$ relations for $\{\zeta(\boldsymbol{k})|\mathrm{wt}(\boldsymbol{k}) = 11\}$. We know that (2) gives 411 relations. Some other families of relations, such as [8], [10] and [5], are known. However we do not know the complete set of explicit relations that hold for all multiple zeta values.

2 Relations for multiple zeta-star values and the Gauss formula

Our main result is

Theorem *Let s and k be integers such that $s \geq 1$ and $k\,(\geq 2s)$. Let $I_0(k,s)$ denote the set of all admissible mutltiple indices of height s and weight k. Then we have*

$$\sum_{\boldsymbol{k}\in I_0(k,s)} \zeta^*(\boldsymbol{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\zeta(k). \tag{3}$$

Outline of the proof of Theorem Let $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ be a multi index and t a parameter. If we set

$$L^*_{\boldsymbol{k}}(t) = \sum_{m_1\geq m_2\geq\cdots\geq m_n\geq 1} \frac{t^{m_1}}{m_1{}^{k_1} m_2{}^{k_2}\cdots m_n{}^{k_n}} \qquad (|t|<1).$$

then it is clear that $L^*_{\boldsymbol{k}}(1) = \zeta^*(\boldsymbol{k})$ holds. Hence if we define X_0 by

$$X_0(k,s;t) = \sum_{\boldsymbol{k}\in I_0(k,s)} L^*_{\boldsymbol{k}}(t),$$

then $X_0(k,s;1)$ is the left-hand side of (3). For generic parameters x and z, we introduce a generating function

$$\Phi_0(t) = \sum_{k,s\geq 0} X_0(k,s;t)x^{k-2s}z^{2s-2}.$$

After some calculation, we see that $\Phi_0(t)$ is a unique power series solution of the following differential equation:

$$t^2(1-t)\frac{d^2\Phi_0}{dt^2} + t\Big((1-t)(1-x)-x\Big)\frac{d\Phi_0}{dt} + (x^2-z^2)\,\Phi_0 = t. \tag{4}$$

Let us construct directly the unique solution

$$\Phi_0(t) = \Phi_0(x,z;t) = \sum_{n=1}^{\infty} a_n t^n$$

of (4). Substituting this into (4) and comparing the coefficientof each power of t, we have the recursion relation for $\{a_n\}$ and thus we get

$$a_n = \frac{\Gamma(n)\Gamma(n-x)\Gamma(1-x-z)\Gamma(1-x+z)}{\Gamma(1-x)\Gamma(1-x-z+n)\Gamma(1-x+z+n)}.$$

Here $\Gamma(z)$ is the gamma function. Therefore we have $\Phi_0(1) = \sum_{n=1}^{\infty} a_n$. To evaluate this sum, we rewrite a_n in the following form:

$$a_n = \sum_{l=1}^{n}\left(\frac{A_{n,l}^{(+)}}{x+z-l} + \frac{A_{n,l}^{(-)}}{x-z-l}\right),$$

where

$$A_{n,l}^{(\pm)} = (-1)^l\binom{n-1}{l-1}\frac{(\pm z-l+1)(\pm z-l+2)\cdots(\pm z-l+n-1)}{(\pm 2z-l+1)(\pm 2z-l+2)\cdots(\pm 2z-l+n)}.$$

Hence we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\left(\frac{A_{n,l}^{(+)}}{x+z-l} + \frac{A_{n,l}^{(-)}}{x-z-l}\right)\\ &= \sum_{l=1}^{n}\left(\sum_{n=l}^{\infty} A_{n,l}^{(+)}\frac{1}{x+z-l} + \sum_{n=l}^{\infty} A_{n,l}^{(-)}\frac{1}{x-z-l}\right).\end{aligned}$$

We can calculate the sum of $A_{n,l}^{(\pm)}$ with respect to n as follows:

$$\begin{aligned}\sum_{n=l}^{\infty} A_{n,l}^{(\pm)} &= (-1)^l\sum_{n=0}^{\infty}\frac{(l-1+n)!(\pm z-l+1)(\pm z-l+2)\cdots(\pm z+n-1)}{n!(l-1)!(\pm 2z-l+1)(\pm 2z-l+2)\cdots(\pm 2z+n)}\\ &= (-1)^l\frac{(\pm z-l+1)(\pm z-l+2)\cdots(\pm z-1)}{(\pm 2z-l+1)(\pm 2z-l+2)\cdots(\pm 2z)}F(l,\pm z,\pm 2z+1,1).\end{aligned}$$

Here $F(\alpha,\beta,\gamma;t)$ denotes the Gauss hypergeometric function. Using the Gauss formula

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

we get

$$\sum_{n=l}^{\infty}A_{n,l}^{(\pm)}=\pm\frac{(-1)^l}{z}.$$

Hence we have

$$\sum_{n=1}^{\infty}a_n=\frac{1}{z}\sum_{l=1}^{\infty}(-1)^l\left(\frac{1}{x+z-l}-\frac{1}{x-z-l}\right).$$

We expand the right-hand side in x and z and take the coefficient of $x^{k-2s}z^{2s-2}$, which is equal to

$$2\binom{k-1}{2s-1}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l^k}.$$

If we rewrite this by using the formula $\displaystyle\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l^k}=(1-2^{1-k})\zeta(k)$, then we obtain the right-hand side of (3).

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