数理解析研究所講究録 1412 巻 2005 年 199-209

Minimal filtered free resolutions for analytic D-modules (Extended abstract)

Michel Granger
Université d'Angers, France
Toshinori Oaku (大阿久 俊則)
Tokyo Woman's Christian University (東京女子大学・文理学部)

概要

We define the notion of a minimal filtered free resolution for a filtered module over the ring $\mathcal{D}^{(h)}$, a homogenization of the ring $\mathcal{D}$ of analytic differential operators. This provides us with analytic invariants attached to a (bi)filtered $\mathcal{D}$-module. We also give an effective argument using a generalization of the division theorem in $\mathcal{D}^{(h)}$ due to Assi-Castro-Granger (2001), by which we obtain an upper bound for the length of minimal filtered resolutions.

1 Minimal free resolutions of $\mathcal{D}^{(h)}$-modules

We denote by $\mathcal{D}$ the ring of germs of analytic linear differential operators (of finite order) at the origin of $\mathbb{C}^n$. Fixing the canonical coordinate system $x = (x_1, \dots, x_n)$ of $\mathbb{C}^n$, we can write an element P of $\mathcal{D}$ in a finite sum $P = \sum_{\beta\in\mathbb{N}^n} a_\beta(x)\partial^\beta$ with $a_\beta(x)$ belonging to $\mathbb{C}\{x\}$, the ring of convergent power series. Here we use the notation $\partial^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$ with $\partial_i = \partial/\partial x_i$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, where $\mathbb{N} = \{0, 1, 2, \dots\}$.

Following Assi-Castro-Granger [ACG], we introduce the homogenized ring $\mathcal{D}^{(h)}$ of $\mathcal{D}$ as follows (in [ACG] it is denoted $\mathcal{D}[t]$): Introducing a new variable h, we define the homogenization of $P \in \mathcal{D}$ above to be

$$P^{(h)} := \sum_\beta a_\beta(x)\partial^\beta h^{m-|\beta|} \quad \text{with} \quad m = \operatorname{ord} P := \max\{|\beta| = \beta_1 + \cdots + \beta_n \mid a_\beta(x) \neq 0\}.$$

We regard this operator as an element of the ring $\mathcal{D}^{(h)}$, which is the $\mathbb{C}$-algebra generated by $\mathbb{C}\{x\}$, $\partial_1, \dots, \partial_n$, and h with the commuting relations

$$ha = ah, \quad h\partial_i = \partial_i h, \quad \partial_i\partial_j = \partial_j\partial_i, \quad \partial_i a - a\partial_i = \frac{\partial a}{\partial x_i}h$$

for any $a \in \mathbb{C}\{x\}$ and $i, j \in \{1, \dots, n\}$.

Note that $\mathcal{D}^{(h)}$ is isomorphic to the Rees ring of $\mathcal{D}$ with respect to the order filtration, i.e., the filtration by the degree in ∂ of operators, and is (left and right) Noetherian. The ring $\mathcal{D}^{(h)}$ is a non-commutative graded $\mathbb{C}$-algebra with the grading

$$\mathcal{D}^{(h)} = \bigoplus_{d\geq 0}(\mathcal{D}^{(h)})_d \quad \text{with} \quad (\mathcal{D}^{(h)})_d := \bigoplus_{|\beta|+k=d} \mathbb{C}\{x\}\partial^\beta h^k.$$

An element P of $(\mathcal{D}^{(h)})_d$ is said to be homogeneous of degree $\deg P := d$. A left $\mathcal{D}^{(h)}$-module M is a graded $\mathcal{D}^{(h)}$-module if M is written as a direct sum $M = \oplus_{k\in\mathbb{Z}} M_k$ of $\mathbb{C}\{x\}$-modules such that $(\mathcal{D}^{(h)})_d M_k \subset M_{k+d}$ holds for any k, d. If M is finitely generated, there exists an integer k_0 such that $M_k = 0$ for $k \leq k_0$. Let M and N be graded left $\mathcal{D}^{(h)}$-modules. Then a map $\psi : M \to N$ is a homomorphism of graded $\mathcal{D}^{(h)}$-modules (of degree 0) if ψ is $\mathcal{D}^{(h)}$-linear and satisfies $\psi(M_i) \subset N_i$ for any $i \in \mathbb{Z}$.

The ring $\mathcal{D}^{(h)}$ has a unique maximal graded two-sided ideal

$$\mathcal{I}^{(h)} := (\mathbb{C}\{x\}x_1 + \cdots + \mathbb{C}\{x\}x_n) \oplus \bigoplus_{d\geq 1}(\mathcal{D}^{(h)})_d.$$

The quotient ring $\mathcal{D}^{(h)}/\mathcal{I}^{(h)}$ is isomorphic to $\mathbb{C}$.

Lemma 1.1 (Nakayama's lemma for $\mathcal{D}^{(h)}$) *Let M be a finitely generated graded left $\mathcal{D}^{(h)}$-module. Then $\mathcal{I}^{(h)}M = M$ implies $M = 0$.*

In view of this lemma, the classical theory of minimal free resolutions for modules over a local ring extends to graded left $\mathcal{D}^{(h)}$-modules.

By a *graded free $\mathcal{D}^{(h)}$-module $\mathcal{L}$* of rank r, we mean a graded $\mathcal{D}^{(h)}$-module

$$\mathcal{L} = (\mathcal{D}^{(h)})^r[\mathbf{n}] := \bigoplus_{d\in\mathbb{Z}} \left((\mathcal{D}^{(h)})_{d-n_1} \oplus \cdots \oplus (\mathcal{D}^{(h)})_{d-n_r}\right)$$

with an $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, which we call the *shift vector* for $\mathcal{L}$. The unit vectors $(1, 0, \ldots, 0), \cdots, (0, \ldots, 0, 1) \in \mathcal{L}$ are called the canonical generators of $\mathcal{L}$. An element $P = (P_1, \ldots, P_r)$ of $(\mathcal{D}^{(h)})^r[\mathbf{n}]$ is said to be homogeneous of degree $\deg[\mathbf{n}](P) = d$ if each P_i is homogeneous of degree $d - n_i$.

Definition 1.2 (minimal free resolution) Let M be a graded free left $\mathcal{D}^{(h)}$-module. An exact sequence

$$\cdots \longrightarrow \mathcal{L}_2 \xrightarrow{\psi_2} \mathcal{L}_1 \xrightarrow{\psi_1} \mathcal{L}_0 \xrightarrow{\psi_0} M \to 0$$

of graded left $\mathcal{D}^{(h)}$-modules, where $\mathcal{L}_i$ is a graded free $\mathcal{D}^{(h)}$-module of rank r_i, is called a *minimal free resolution* of M if the maps in the induced complex

$$\cdots \longrightarrow (\mathcal{D}^{(h)}/\mathcal{I}^{(h)}) \otimes_{\mathcal{D}^{(h)}} \mathcal{L}_2 \to (\mathcal{D}^{(h)}/\mathcal{I}^{(h)}) \otimes_{\mathcal{D}^{(h)}} \mathcal{L}_1 \to (\mathcal{D}^{(h)}/\mathcal{I}^{(h)}) \otimes_{\mathcal{D}^{(h)}} \mathcal{L}_0$$

(of $\mathbb{C}$-vector spaces) are all zero; i.e., the entries of ψ_i ($i \geq 1$) regarded as a matrix belong to $\mathcal{I}^{(h)}$. By Nakayama's lemma, this is equivalent to the condition that the images by ψ_i of the canonical generators of $\mathcal{L}_i$ is a minimal generating set of the kernel of ψ_{i-1}, or of M if $i = 0$, for each $i \geq 0$. We call the ranks $r_0, r_1, \ldots$ the Betti numbers of M.

A minimal free resolution exists and is unique up to isomorphism. This follows from Nakayama's lemma as in the case with a local ring (see e.g., Theorem 20.2 of [E]). In particular, the Betti numbers and the associated shift vectors (up to permutation of their components) are invariants of M.

Example 1.3 In two variables $(x_1, x_2) = (x, y)$, let $I^{(h)}$ be the left ideal of $\mathcal{D}^{(h)}$ generated by

$$x^3 - y^2, \quad 2x\partial_x + 3y\partial_y + 6h, \quad 3x^2\partial_y + 2y\partial_x$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and put $M := \mathcal{D}^{(h)}/I^{(h)}$. Its dehomogenization $\mathcal{D}/(I^{(h)}|_{h=1})$ (see Section 3) is the local cohomology group supported by $x^3 - y^2 = 0$. A minimal free resolution of M is given by

$$0 \to (\mathcal{D}^{(h)})^2[(1,1)] \xrightarrow{\psi_2} (\mathcal{D}^{(h)})^3[(0,1,1)] \xrightarrow{\psi_1} \mathcal{D}^{(h)} \xrightarrow{\psi_0} M \to 0$$

with ψ_0 being the canonical projection and

$$\psi_1 = \begin{pmatrix} x^3 - y^2 \\ 2x\partial_x + 3y\partial_y + 6h \\ 3x^2\partial_y + 2y\partial_x \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} -2\partial_x & x^2 & -y \\ 3\partial_y & y & -x \end{pmatrix}.$$

2 Minimal filtered free resolutions of $\mathcal{D}^{(h)}$-modules

We call $(u, v) = (u_1, \dots, u_n, v_1, \dots, v_n) \in \mathbb{Z}^{2n}$ a *weight vector* if $u_i + v_i \geq 0$ and $u_i \leq 0$ hold for $i = 1, \dots, n$. For an element P of $\mathcal{D}^{(h)}$ of the form

$$P = \sum_{k \geq 0, \alpha, \beta \in \mathbb{N}^n} a_{\alpha\beta k} x^\alpha \partial^\beta h^k \quad (a_{\alpha\beta k} \in \mathbb{C}),$$

where the sum is finite with respect to k and β, we define the (u, v)-order of P by

$$\mathrm{ord}_{(u,v)}(P) := \max\{\langle u, \alpha\rangle + \langle v, \beta\rangle \mid a_{\alpha\beta k} \neq 0 \text{ for some } k\}$$

with $\langle u, \alpha\rangle = u_1\alpha_1 + \cdots + u_n\alpha_n$. We also define the (u, v)-order of $P \in \mathcal{D}$ in the same way. (If $P = 0$, we define its (u, v)-order to be $-\infty$.) Then the (u, v)-filtrations of $\mathcal{D}$ and $\mathcal{D}^{(h)}$ are defined by

$$F_k^{(u,v)}(\mathcal{D}) := \{P \in \mathcal{D} \mid \mathrm{ord}_{(u,v)}(P) \leq k\}, \qquad F_k^{(u,v)}(\mathcal{D}^{(h)}) := \{P \in \mathcal{D}^{(h)} \mid \mathrm{ord}_{(u,v)}(P) \leq k\}$$

for $k \in \mathbb{Z}$.

For a $\mathbb{C}$-subspace I of $\mathcal{D}$ or of $\mathcal{D}^{(h)}$, put $F_k(I) := F_k^{(u,v)}(\mathcal{D}) \cap I$ or $F_k^{(u,v)}(\mathcal{D}^{(h)}) \cap I$ respectively, and define

$$\mathrm{gr}_k^{(u,v)}(I) := F_k^{(u,v)}(I)/F_{k-1}^{(u,v)}(I).$$

Then the graded rings with respect to the (u, v)-filtration are defined by

$$\mathrm{gr}^{(u,v)}(\mathcal{D}) := \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k^{(u,v)}(\mathcal{D}), \qquad \mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)}) := \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k^{(u,v)}(\mathcal{D}^{(h)}).$$

For an element P of $F_k^{(u,v)}(\mathcal{D})$ or of $F_k^{(u,v)}(\mathcal{D}^{(h)})$, we denote by $\sigma^{(u,v)}(P) = \sigma_k^{(u,v)}(P)$ the residue class of P in $\mathrm{gr}_k^{(u,v)}(\mathcal{D})$ or in $\mathrm{gr}_k^{(u,v)}(\mathcal{D}^{(h)})$, and call it the (u, v)-principal symbol of P.

The ring $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$ is Noetherian and has a structure of bigraded $\mathbb{C}$-algebra

$$\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)}) = \bigoplus_{d\geq 0} \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k^{(u,v)}((\mathcal{D}^{(h)})_d).$$

A nonzero element of $\mathrm{gr}_k^{(u,v)}((\mathcal{D}^{(h)})_d)$ is called bihomogeneous of bidegree (d, k). In general, if I is a graded left ideal of $\mathcal{D}^{(h)}$, then $\mathrm{gr}^{(u,v)}(I) := \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k^{(u,v)}(I)$ is a bigraded left ideal of $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$ with respect to the bigraded structure. In particular, $\mathrm{gr}^{(u,v)}(\mathcal{I}^{(h)})$ is the unique maximal bigraded (two-sided) ideal of $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$. We can define the notion of a minimal free resolution of a bigraded $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$-module by virtue of the following

Lemma 2.1 (Nakayama's lemma for $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$) *Let M' be a finitely generated bigraded left $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$-module. Then $\mathrm{gr}^{(u,v)}(\mathcal{I}^{(h)})M' = M'$ implies $M' = 0$.*

For a graded left $\mathcal{D}^{(h)}$-module $M = \oplus_{d\in\mathbf{Z}} M_d$, a family $\{F_k(M)\}_{k\in\mathbf{Z}}$ of $\mathbb{C}\{x\}$-submodules of M satisfying

$$F_k(M) \subset F_{k+1}(M), \quad \bigcup_{k\in\mathbf{Z}} F_k(M) = M, \quad F_l^{(u,v)}(\mathcal{D}^{(h)})F_k(M) \subset F_{k+l}(M)$$

$$F_k(M) = \bigoplus_{d\in\mathbf{Z}} F_k(M_d) \quad \text{with} \quad F_k(M_d) := F_k(M) \cap M_d \quad (k, l, d \in \mathbb{Z})$$

is called a (graded) (u, v)-*filtration* of M. The graded module of M associated with this filtration is a bigraded left $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$-module defined by

$$\mathrm{gr}(M) := \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k(M) = \bigoplus_{d\geq 0} \bigoplus_{k\in\mathbf{Z}} \mathrm{gr}_k(M_d),$$

$$\mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M), \quad \mathrm{gr}_k(M_d) := F_k(M_d)/F_{k-1}(M_d).$$

A nonzero element of $\mathrm{gr}_k(M_d)$ is said to be bihomogeneous of bidegree (d, k). For an element f of M, we put $\mathrm{ord}_{(u,v)} f := \inf\{k \in \mathbb{Z} \mid f \in F_k(M)\}$.

A (u, v)-filtration $\{F_k(M)\}_{k\in\mathbf{Z}}$ on a graded $\mathcal{D}^{(h)}$-module M is said to be *good* if there exist homogeneous elements $f_1, \ldots, f_r$ of M and integers $m_1, \ldots, m_r$ such that

$$F_k(M) = F_{k-m_1}^{(u,v)}(\mathcal{D}^{(h)})f_1 + \cdots + F_{k-m_r}^{(u,v)}(\mathcal{D}^{(h)})f_r \qquad (\forall k \in \mathbb{Z}).$$

Then $\mathrm{gr}(M)$ is finitely generated. A (good) filtration of a left $\mathcal{D}$-module and the associated graded module are defined in the same way, but without assuming any homogeneity for M and f_i.

By assigning $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, which we call a *shift vector* for the (u,v)-filtration, we define good filtrations on $\mathcal{D}^r$ and on $(\mathcal{D}^{(h)})^r$ by

$$F_k^{(u,v)}[\mathbf{m}](\mathcal{D}^r) := \{(P_1, \dots, P_r) \in D^r \mid \mathrm{ord}_{(u,v)}(P_i) + m_i \le k \ (i = 1, \dots, r)\},$$
$$F_k^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r) := \{(P_1, \dots, P_r) \in (\mathcal{D}^{(h)})^r \mid \mathrm{ord}_{(u,v)}(P_i) + m_i \le k \ (i = 1, \dots, r)\}$$

respectively. For an element $P = (P_1, \dots, P_r)$ of $(\mathcal{D}^{(h)})^r$, we put

$$\mathrm{ord}_{(u,v)}[\mathbf{m}](P) := \min\{k \in \mathbb{Z} \mid P \in F_k^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r)\}$$

and denote by $\sigma^{(u,v)}[\mathbf{m}](P) = \sigma_k^{(u,v)}[\mathbf{m}](P)$ the residue class of $P \in F_k^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r)$ in $\mathrm{gr}_k^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r) := F_k^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r)/F_{k-1}^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r)$.

The free modules $\mathcal{D}^r$ and $(\mathcal{D}^{(h)})^r$ equipped with these filtrations are called (u,v)-filtered free modules. For a (u,v)-filtered free module $\mathcal{L} = (\mathcal{D}^{(h)})^r$ with the above filtration, we put

$$F_k(\mathcal{L}) := F_k^{(u,v)}[\mathbf{m}]((\mathcal{D}^{(h)})^r) \qquad (k \in \mathbb{Z}).$$

The graded free module $\mathcal{L} = (\mathcal{D}^{(h)})^r[\mathbf{n}]$ equipped with this filtration is called a (u,v)-*filtered graded free module* and denoted by $\mathcal{L} = (\mathcal{D}^{(h)})^r[\mathbf{n}][\mathbf{m}]$ in order to explicitly refer to the shift vectors.

Definition 2.2 (minimal filtered free resolution) Let M be a graded left $\mathcal{D}^{(h)}$-module with a (u,v)-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then a (u,v)-*filtered free resolution* of M is an exact sequence

$$\cdots \longrightarrow \mathcal{L}_2 \xrightarrow{\psi_2} \mathcal{L}_1 \xrightarrow{\psi_1} \mathcal{L}_0 \xrightarrow{\psi_0} M \longrightarrow 0 \tag{2.1}$$

of graded left $\mathcal{D}^{(h)}$-modules with (u,v)-filtered graded free $\mathcal{D}^{(h)}$-modules $\mathcal{L}_i$ which induces an exact sequence

$$\cdots \longrightarrow F_k(\mathcal{L}_2) \xrightarrow{\psi_2} F_k(\mathcal{L}_1) \xrightarrow{\psi_1} F_k(\mathcal{L}_0) \xrightarrow{\psi_0} F_k(M) \longrightarrow 0$$

for any $k \in \mathbb{Z}$. Then (2.1) induces an exact sequence

$$\cdots \longrightarrow \mathrm{gr}(\mathcal{L}_2) \xrightarrow{\overline{\psi}_2} \mathrm{gr}(\mathcal{L}_1) \xrightarrow{\overline{\psi}_1} \mathrm{gr}(\mathcal{L}_0) \xrightarrow{\overline{\psi}_0} \mathrm{gr}(M) \longrightarrow 0 \tag{2.2}$$

of bigraded $\mathrm{gr}^{(u,v)}(\mathcal{D}^{(h)})$-modules. The filtered free resolution (2.1) is called a *minimal* (u,v)-*filtered free resolution* of M if (2.2) is a minimal free resolution of $\mathrm{gr}(M)$.

This last condition means that, as a matrix, all entries of $\overline{\psi}_i$ $(i \ge 1)$ belong to $\mathrm{gr}^{(u,v)}(\mathcal{I}^{(h)})$ or equivalently that the image of the set of canonical generators of $\mathrm{gr}(\mathcal{L}_i)$ is a minimal set of generators of $\mathrm{Ker}\,\overline{\psi}_{i-1}$ for $i \ge 0$, and of $\mathrm{gr}(M)$ for $i = 0$. The existence of a minimal bigraded free resolution as (2.2) can be proved exactly in the same way as in the non-filtered case of Section 1, by using Lemma 2.1.

Note that a minimal $(\mathbf{0},\mathbf{1})$-filtered free resolution with $(\mathbf{0},\mathbf{1}) = (0,\ldots,0,1,\ldots,1)$ is simply a minimal free resolution.

Theorem 2.3 (existence and uniqueness of minimal filtered resolutions) *Let M be a graded left $\mathcal{D}^{(h)}$-module with a good (u,v)-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then there exists a minimal filtered free resolution*

$$\cdots \to \mathcal{L}_3 \xrightarrow{\psi_3} \mathcal{L}_2 \xrightarrow{\psi_2} \mathcal{L}_1 \xrightarrow{\psi_1} \mathcal{L}_0 \xrightarrow{\psi_0} M \to 0$$

of M. Moreover, a minimal filtered free resolution of M is unique up to isomorphism; i.e., if

$$\cdots \to \mathcal{L}'_3 \xrightarrow{\psi'_3} \mathcal{L}'_2 \xrightarrow{\psi'_2} \mathcal{L}'_1 \xrightarrow{\psi'_1} \mathcal{L}'_0 \xrightarrow{\psi'_0} M \to 0$$

is another minimal (u,v)-filtered free resolution of M, then there exist filtered graded $\mathcal{D}^{(h)}$-isomorphisms $\theta_i : \mathcal{L}_i \to \mathcal{L}'_i$, i.e., a graded $\mathcal{D}^{(h)}$-isomorphism (of degree 0) satisfying $\theta_i(F_k(\mathcal{L}_i)) = F_k(\mathcal{L}'_i)$ for any $k \in \mathbb{Z}$, such that the diagram

$$\begin{array}{ccccccccc}
\cdots \to \mathcal{L}_3 & \xrightarrow{\psi_3} & \mathcal{L}_2 & \xrightarrow{\psi_2} & \mathcal{L}_1 & \xrightarrow{\psi_1} & \mathcal{L}_0 & \xrightarrow{\psi_0} & M \\
\downarrow{\scriptstyle \theta_3} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \| \\
\cdots \to \mathcal{L}'_3 & \xrightarrow{\psi'_3} & \mathcal{L}'_2 & \xrightarrow{\psi'_2} & \mathcal{L}'_1 & \xrightarrow{\psi'_1} & \mathcal{L}'_0 & \xrightarrow{\psi'_0} & M
\end{array}$$

is commutative. Moreover, the length of a minimal filtered free resolution is $\leq 2n+1$, i.e, $\psi_{2n+2} = 0$.

Corollary 2.4 *For a graded $\mathcal{D}^{(h)}$-module M with a good (u,v)-filtration, the rank r_i, the shifts $\mathbf{n}_i$ and $\mathbf{m}_i$ attached to the i-th free module $\mathcal{L}_i$ in a minimal (u,v)-filtered resolution of M are unique (up to permutation of the components of each shift).*

As an application, let us take a germ $f(x)$ of analytic function at $0 \in \mathbb{C}^n$ and let I be the left ideal generated by $t - f(x)$, $\partial_i + (\partial f/\partial x_i)\partial_t$ $(i = 1,\ldots,n)$ with $\partial_i = \partial/\partial x_i$ and $\partial_t = \partial/\partial t$ in the ring $\mathcal{D}^{(h)}$ for the $n+1$ variables (t,x). Consider the graded $\mathcal{D}^{(h)}$-module $M := \mathcal{D}^{(h)}/I = \mathcal{D}^{(h)}\delta(t-f)$, where $\delta(t-f)$ is the residue class of 1. Note that the dehomogenization $M|_{h=1}$ is the local cohomology group supported by $t - f = 0$.

Put $(u,v) = (-1,0,\ldots,0;1,0,\ldots,0)$, which corresponds to the V-filtration with respect to $t = 0$. Then M has a good V-filtration $F_k(M) := F_k^{(u,v)}(\mathcal{D}^{(h)})\delta(t-f)$ for $k \in \mathbb{Z}$. This filtered module M is uniquely determined up to isomorphism by the divisor defined by f. The following examples were computed by using software kan/sm1 [T].

Example 2.5 Using the variables (t,x,y), put $f = x^3-y^2$. Then a minimal $(-1,0,0;1,0,0)$-filtered free resolution of $M := \mathcal{D}^{(h)}\delta(t-f)$ is given by

$$0 \to \mathcal{D}^{(h)}[\mathbf{n}_3][\mathbf{m}_3] \xrightarrow{\psi_3} (\mathcal{D}^{(h)})^5[\mathbf{n}_2][\mathbf{m}_2] \xrightarrow{\psi_2} (\mathcal{D}^{(h)})^5[\mathbf{n}_1][\mathbf{m}_1] \xrightarrow{\psi_1} \mathcal{D}^{(h)}[\mathbf{n}_0][\mathbf{m}_0] \xrightarrow{\psi_0} M \to 0$$

with shift vectors

$$\mathbf{n}_0 = (0), \quad \mathbf{n}_1 = (1,0,1,1,1), \quad \mathbf{n}_2 = (1,1,2,1,1), \quad \mathbf{n}_3 = (2),$$
$$\mathbf{m}_0 = (0), \quad \mathbf{m}_1 = (1,0,1,0,0), \quad \mathbf{m}_2 = (0,0,1,1,1), \quad \mathbf{m}_3 = (1).$$

The homomorphisms are given by the following matrices (ψ_0 is the canonical projection):

$$\psi_1 = \begin{pmatrix} \underline{-2y\partial_t} + \partial_y \\ \underline{-x^3+y^2+t} \\ \underline{3x^2\partial_t + \partial_x} \\ \underline{3x^2\partial_y + 2y\partial_x} \\ \underline{-6t\partial_t - 2x\partial_x - 3y\partial_y - 6h} \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 0 & \underline{-2\partial_x} & \underline{2t} & \underline{y} & \underline{x^2} \\ \underline{3t} & \underline{-3\partial_y} & 0 & \underline{-x} & \underline{-y} \\ \underline{\partial_x} & 0 & \underline{-\partial_y} & \underline{\partial_t} & 0 \\ \underline{-3x^2} & 0 & \underline{-2y} & 1 & 0 \\ \underline{3y} & \underline{6\partial_t} & \underline{2x} & 0 & 1 \end{pmatrix},$$

$$\psi_3 = \Big(\underline{-6y\partial_t} + 3\partial_y, \quad \underline{-6x^2\partial_t} - 2\partial_x, \quad \underline{-6x^3+6y^2+6t}, \quad \underline{-6t\partial_t - 2x\partial_x - 3y\partial_y - 5h},$$
$$\underline{-3x^2\partial_y - 2y\partial_x}\Big).$$

The underlined parts constitute the induced homomorphisms $\overline{\psi}_i$. Since the b-function of x^3-y^2 has -1 as the only integer root, we get a minimal free resolution of Example 1.3 by restricting the above resolution to $t=0$ by using ($(\mathbf{0},1)$-homogenized version of) Algorithms 5.4 and 7.3 of [OT1].

Example 2.6 The ranks of minimal V-filtered resolutions of $\mathcal{D}^{(h)}\delta(t-f)$ for several f of three variables x,y,z are as follows:

f	r_0	r_1	r_2	r_3	r_4	r_5
xyz	1	7	12	7	1	0
$x^3+y^3+z^3$	1	8	12	7	2	0
$x^3+y^2z^2$	1	8	13	8	2	0

3 Minimal filtered free resolutions of $\mathcal{D}$-modules

The aim of this section is to study minimal filtered free resolutions of $\mathcal{D}$-modules. The main result is that for a $\mathcal{D}$-module with a good $((\mathbf{0},1),(u,v))$-bifiltration, its 'minimal' bifiltered free resolution can be defined uniquely up to isomorphism, via homogenization. Note that the (u,v)-filtrations on $\mathcal{D}$-modules are more delicate than those on $\mathcal{D}^{(h)}$-modules so that the arguments in the preceding section do not work in general.

The notion of a good bifiltration was introduced, for example, in [Sa], [LM]. The results below should be more or less 'well-known to specialists', but we did not find them explicitly stated in the literature.

3.1 Correspondence between $(0,1)$-filtered $\mathcal{D}$-modules and graded $\mathcal{D}^{(h)}$-modules

For an element $P = \sum_{\beta,k} a_{\beta k}(x)\partial^\beta h^k$ of $\mathcal{D}^{(h)}$, we define its dehomogenization to be the element $\rho(P) := P|_{h=1} = \sum_{\beta,k} a_{\beta k}(x)\partial^\beta$ of $\mathcal{D}$. This defines a surjective ring homomorphism

$$\rho : \mathcal{D}^{(h)} \ni P \longmapsto P|_{h=1} \in \mathcal{D},$$

which gives $\mathcal{D}$ a structure of two-sided $\mathcal{D}^{(h)}$-module. Let $M' = \oplus_{d\in\mathbb{Z}} M'_d$ be a graded left $\mathcal{D}^{(h)}$-module of finite type. Then we define its dehomogenization to be $\rho(M') := \mathcal{D} \otimes_{\mathcal{D}^{(h)}} M'$ as a left $\mathcal{D}$-module. Moreover, $\rho(M')$ has a good $(\mathbf{0},\mathbf{1})$-filtration (with $(\mathbf{0},\mathbf{1}) = (0,\dots,0;1,\dots,1) \in \mathbb{Z}^{2n}$) defined by

$$F_k^{(\mathbf{0},\mathbf{1})}(\rho(M')) := 1 \otimes M'_k := \{1 \otimes f \in \mathcal{D} \otimes M'_k \mid f \in M'_k\} \qquad (k \in \mathbb{Z}) \tag{3.1}$$

in view of the relations $P \otimes f = 1 \otimes (P^{(h)} f)$ and $\deg P^{(h)} = \operatorname{ord} P$ for $P \in \mathcal{D}$. For a graded homomorphism $\psi : M' \to N'$ of graded left $\mathcal{D}^{(h)}$-modules, $1 \otimes \psi : \rho(M') \to \rho(N')$ is a filtered $\mathcal{D}$-homomorphism with the filtrations defined above. If M' and N' are free modules, then $1 \otimes \psi$ is obtained by the substitution $\psi|_{h=1}$ of the matrix ψ. This defines a functor ρ from the category of graded $\mathcal{D}^{(h)}$-modules of finite type, to the category of $\mathcal{D}$-modules with good $(\mathbf{0},\mathbf{1})$-filtrations.

A converse functor is given by the construction of so-called Rees modules. With a commutative variable T, the Rees ring of $\mathcal{D}$ with respect to the $(\mathbf{0},\mathbf{1})$-filtration is defined to be the $\mathbb{C}$-algebra

$$R_{(\mathbf{0},\mathbf{1})}(\mathcal{D}) := \bigoplus_{k\geq 0} F_k^{(\mathbf{0},\mathbf{1})}(\mathcal{D}) T^k \subset \mathcal{D}[T],$$

which is isomorphic to $\mathcal{D}^{(h)}$ by the correspondence $a_\beta(x)\partial^\beta h^k \leftrightarrow a_\beta(x)\partial^\beta T^{k+|\beta|}$. Hence we identify $R_{(\mathbf{0},\mathbf{1})}(\mathcal{D})$ with $\mathcal{D}^{(h)}$. For a left $\mathcal{D}$-module with a good $(\mathbf{0},\mathbf{1})$-filtration $F_k^{(\mathbf{0},\mathbf{1})}(M)$, its Rees module is defined by

$$R_{(\mathbf{0},\mathbf{1})}(M) := \bigoplus_{k\in\mathbb{Z}} F_k^{(\mathbf{0},\mathbf{1})}(M) T^k,$$

which is a left $\mathcal{D}^{(h)}$-module of finite type with the action

$$a(x)\partial^\beta h^k (f_j T^j) = (a(x)\partial^\beta f_j) T^{j+k+|\beta|} \qquad (a(x) \in \mathbb{C}\{x\},\ f_j \in F_j^{(\mathbf{0},\mathbf{1})}(M)).$$

If $\varphi : M \to N$ is a $(\mathbf{0},\mathbf{1})$-filtered homomorphism, this naturally induces a graded homomorphism

$$\varphi^{(h)} : R_{(\mathbf{0},\mathbf{1})}(M) \longrightarrow R_{(\mathbf{0},\mathbf{1})}(N),$$

which is thought of as the homogenization of φ with respect to both filtrations.

Definition 3.1 A left $\mathcal{D}^{(h)}$-module M' is called *h-saturated* if $hf = 0$ implies $f = 0$ for any homogeneous element f of M'.

It is easy to see that the Rees module of a $(\mathbf{0},\mathbf{1})$-filtered $\mathcal{D}$-module M is h-saturated.

Proposition 3.2 *The functors ρ and $R_{(\mathbf{0},\mathbf{1})}$ give an equivalence between the category of h-saturated graded $\mathcal{D}^{(h)}$-modules of finite type, and the category of $\mathcal{D}$-modules with good $(\mathbf{0},\mathbf{1})$-filtrations. Moreover these functors are exact, with exactness meaning filtered exactness in the second category.*

From the arguments above, we have

Theorem 3.3 *Let M be a left $\mathcal{D}$-module with a good $(\mathbf{0},\mathbf{1})$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then there exists a $(\mathbf{0},\mathbf{1})$-filtered free resolution*

$$\cdots \longrightarrow \mathcal{L}_3 \xrightarrow{\varphi_3} \mathcal{L}_2 \xrightarrow{\varphi_2} \mathcal{L}_1 \xrightarrow{\varphi_1} \mathcal{L}_0 \xrightarrow{\varphi_0} M \to 0 \tag{3.2}$$

with $\mathcal{L}_i = \mathcal{D}^{r_i}[\mathbf{n}_i]$ being the free module equipped with the $(\mathbf{0},\mathbf{1})$-filtration $\{F_d^{(\mathbf{0},\mathbf{1})}[\mathbf{n}_i](\mathcal{L}_i)\}_{d\in\mathbb{Z}}$ with $\mathbf{n}_i \in \mathbb{Z}^{r_i}$ such that

$$\cdots \to R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_3) \xrightarrow{\varphi_3^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_2) \xrightarrow{\varphi_2^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_1) \xrightarrow{\varphi_1^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_0) \xrightarrow{\varphi_0^{(h)}} R_{(\mathbf{0},\mathbf{1})}(M) \to 0$$

is a minimal free resolution of $R_{(\mathbf{0},\mathbf{1})}(M)$. Moreover, such a free resolution as (3.2) is unique up to $(\mathbf{0},\mathbf{1})$-filtered isomorphism. Note that $R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_i)$ is isomorphic to the graded free module $(\mathcal{D}^{(h)})^{r_i}[\mathbf{n}_i]$.

3.2 Correspondence between bifiltered $\mathcal{D}$-modules and (u,v)-filtered graded $\mathcal{D}^{(h)}$-modules

In order to define the bifiltration, we will assume here that $(u,v) \neq (\mathbf{0},\mathbf{1})$. A $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltration of a $\mathcal{D}$-module M is a double sequence $\{F_{d,k}(M)\}_{d,k\in\mathbb{Z}}$ of $\mathbb{C}\{x\}$-submodules of M such that

$$F_{d,k}(M) \subset F_{d+1,k}(M) \cap F_{d,k+1}(M), \quad \bigcup_{d,k\in\mathbb{Z}} F_{d,k}(M) = M,$$

$$(F_{d'}^{(\mathbf{0},\mathbf{1})}(\mathcal{D}) \cap F_{k'}^{(u,v)}(\mathcal{D}))F_{d,k}(M) \subset F_{d+d',k+k'}(M) \quad \text{for any } d,d',k,k' \in \mathbb{Z}.$$

Definition 3.4 Let M be a left $\mathcal{D}$-module. Then a bifiltration $\{F_{d,k}(M)\}$ is called a *good $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltration* if there exist $f_1,\dots,f_l \in M$ and $n_i, m_i \in \mathbb{Z}$ such that

$$F_{d,k}(M) = (F_{d-n_1}^{(\mathbf{0},\mathbf{1})}(\mathcal{D}) \cap F_{k-m_1}^{(u,v)}(\mathcal{D}))f_1 + \cdots + (F_{d-n_l}^{(\mathbf{0},\mathbf{1})}(\mathcal{D}) \cap F_{k-m_l}^{(u,v)}(\mathcal{D}))f_l$$

holds for any $d,k \in \mathbb{Z}$.

Forgetting the (u,v)-filtration gives (functorially) a $(\mathbf{0},\mathbf{1})$-filtration on M, by setting $F_d^{(\mathbf{0},\mathbf{1})}(M) = \bigcup_{k\in\mathbb{Z}} F_{d,k}(M)$, and this functor sends good bifiltered modules to good filtered modules.

Now let M' be a h-saturated graded left $\mathcal{D}^{(h)}$-module with a good (u,v)-filtration $F_k^{(u,v)}(M')$. We can find $f'_j \in M'$ homogeneous of degree n_j, and integers m_j so that

$$F_k^{(u,v)}(M') = F_{k-m_1}^{(u,v)}(\mathcal{D}^{(h)})f'_1 + \cdots + F_{k-m_r}^{(u,v)}(\mathcal{D}^{(h)})f'_r$$

holds for any $k \in \mathbb{Z}$. Then $\rho(M')$ has a good bifiltration defined by

$$F_{d,k}(\rho(M')) \;:=\; 1 \otimes F_k^{(u,v)}(M'_d) \;=\; \sum_{j=1}^{r} \left(F_{d-n_j}^{(\mathbf{0},\mathbf{1})}(\mathcal{D}) \cap F_{k-m_j}^{(u,v)}(\mathcal{D})\right)(1 \otimes f'_j).$$

We remark that this is not in general the intersection of the d-th term of the $(\mathbf{0},\mathbf{1})$-filtration on $\rho(M')$, which we denoted by $1 \otimes M'_d$, and the k-th term of the (u,v)-filtration

$$F_k^{(u,v)}(\rho(M')) := 1 \otimes F_k^{(u,v)}(M') := \{1 \otimes f \in \mathcal{D} \otimes_{\mathcal{D}^{(h)}} M' \mid f \in F_k^{(u,v)}(M')\}.$$

Conversely, if M is a left $\mathcal{D}$-module with a good bifiltration $\{F_{d,k}(M)\}$, then its Rees module $R_{(\mathbf{0},\mathbf{1})}(M)$ with respect to its $(\mathbf{0},\mathbf{1})$-filtration has a good (u,v)-filtration

$$F_k^{(u,v)}(R_{(\mathbf{0},\mathbf{1})}(M)) := \bigoplus_{d\in\mathbb{Z}} F_{d,k}(M)T^d \;=\; \sum_{j=1}^{r} F_{k-m_j}^{(u,v)}(\mathcal{D}^{(h)})(f_j T^{n_j}),$$

where f_j, m_j, n_j are as in Definition 3.4.

Proposition 3.5 *The functors ρ and $R_{(\mathbf{0},\mathbf{1})}$ give an equivalence between the category of h-saturated graded $\mathcal{D}^{(h)}$-modules with good (u,v)-filtrations and the category of $\mathcal{D}$-modules with good $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltrations. Furthermore these functors are exact in the sense of filtered and bifiltered categories respectively.*

In conclusion we obtain

Theorem 3.6 *Let M be a left $\mathcal{D}$-module with a good $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltration $\{F_{d,k}(M)\}$. Then there exists a bifiltered free resolution*

$$\cdots \to \mathcal{L}_3 \xrightarrow{\varphi_3} \mathcal{L}_2 \xrightarrow{\varphi_2} \mathcal{L}_1 \xrightarrow{\varphi_1} \mathcal{L}_0 \xrightarrow{\varphi_0} M \to 0 \tag{3.3}$$

with $\mathcal{L}_i = \mathcal{D}^{r_i}[\mathbf{n}_i][\mathbf{m}_i]$ being the free module equipped with the bifiltration $\{F_k^{(\mathbf{0},\mathbf{1})}[\mathbf{n}_i](\mathcal{L}_i) \cap F_k^{(u,v)}[\mathbf{m}_i](\mathcal{L}_i)\}$ with $\mathbf{n}_i, \mathbf{m}_i \in \mathbb{Z}^{r_i}$ such that

$$\cdots \to R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_3) \xrightarrow{\varphi_3^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_2) \xrightarrow{\varphi_2^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_1) \xrightarrow{\varphi_1^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_0) \xrightarrow{\varphi_0^{(h)}} R_{(\mathbf{0},\mathbf{1})}(M) \to 0$$

is a minimal (u,v)-filtered free resolution of the (u,v)-filtered module $R_{(\mathbf{0},\mathbf{1})}(M)$. Moreover, such a free resolution as (3.3) is unique up to bifiltered isomorphism. Note that $R_{(\mathbf{0},\mathbf{1})}(\mathcal{L}_i)$ is isomorphic to the (u,v)-filtered graded free module $(\mathcal{D}^{(h)})^{r_i}[\mathbf{n}_i][\mathbf{m}_i]$.

参考文献

[ACG] Assi, A., Castro-Jiménez, F.J., Granger M., The analytic standard fan of a $\mathcal{D}$-module. J. Pure Appl. Algebra **164** (2001), 3–21.

[B] D. Bayer, The division algorithm and the Hilbert scheme, Ph. D. Thesis, Harvard University, 1982.

[E] Eisenbud, D., Commutative Algebra with a View Toward Algebraic Geometry. Springer, New York, 1995.

[GO] Granger, M., Oaku, T., Minimal filtered free resolutions and division algorithms for analytic D-modules. Prépublications du département de mathématiques, Univ. Angers, No. 170 (2003).

[LM] Laurent, Y., Monteiro Fernandes, T., Systèmes différentiels fuchsiens le long d'une sous-variété. Publ. RIMS, Kyoto Univ. **24** (1988), 397–431.

[OT1] Oaku, T., Takayama, N., Algorithms for D-modules — restriction, tensor product, localization, and local cohomology groups. J. Pure Appl. Algebra **156** (2001), 267-308.

[OT2] Oaku, T., Takayama, N., Minimal free resolutions of homogenized D-modules. J. Symbolic Computation **32** (2001), 575–595.

[Sa] Sabbah, C., Proximité évanescente, I. La structure polaire d'un $\mathcal{D}$-module. Compositio Math. **62** (1987), 283–328.

[Sch] Schapira, P., Microdifferential Systems in the Complex Domain. Springer, Berlin, 1985.

[T] Takayama, N., Kan/sm1, `http://www.math.kobe-u.ac.jp/KAN/`, 1991–2003.