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HÖRMANDER FORM AND UNIQUENESS FOR THE CAUCHY PROBLEM

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Hörmander proved a uniqueness theorem in the characteristic Cauchy problem for single differential equations with C^∞ coefficients under the condition of principal normality of the equation and a certain convexity condition of the initial surface (see [H, Section 8.9]). Robbiano and Zuily [RZ] and Tataru [T] generalized Hörmander's theorem in the case of C^∞ and partly analytic coefficients with relaxing the principal normality condition for analytic variables. These works concern only single differential equations with simple characteristics. We will here discuss how to formulate and prove a similar result for "systems" of differential equations, and in fact we give a definition of the notion of what we call the Hörmander form and briefly explain how to give a new geometric proof to Hörmander's theorem for systems with analytic coefficients.

Let M be an open subset of $\mathbf{R}^n$, and $x_0 \in M$. Let φ be a real-valued analytic function such that $\varphi(x_0) = 0$ and $d\varphi(x_0) \neq 0$. Let $\mathcal{M}$ be a system of differential equations with analytic coefficients (i.e., a coherent $\mathcal{D}_X$-module, X being a complex neighborhood of M), and $\mathrm{Ch}(\mathcal{M})$ denotes its characteristic variety (without counting multiplicity).

For functions f and g on T^*X, we denote

$$\frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \zeta} = \sum_{\nu=1}^{n} \frac{\partial f}{\partial \zeta_\nu}\frac{\partial g}{\partial z_\nu}, \quad \partial^2\varphi \frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial \zeta} = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 \varphi}{\partial z_\mu \partial z_\nu}\frac{\partial f}{\partial \zeta_\mu}\frac{\partial g}{\partial \zeta_\nu}$$

by fixing a system of coordinates (z, ζ) of T^*X.

Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on M. Let $p = id\varphi(x_0)$.

Theorem 1. *Assume $p \in \mathrm{Ch}(\mathcal{M})$, and let $f_1, \dots, f_r$ be generators of defining equations of $\mathrm{Ch}(\mathcal{M})$ at p. Suppose that the hermitian quadratic form, which we will call the Hörmander form,*

$$\sum_{1 \le j,k \le r} \left\{ \frac{1}{2\sqrt{-1}} \left(\frac{\partial f_j}{\partial \zeta} \frac{\partial f_k^c}{\partial z} + \frac{\partial f_j}{\partial z} \frac{\partial f_k^c}{\partial \zeta} \right)(p) + \partial^2 \varphi \frac{\partial f_j}{\partial \zeta} \frac{\partial f_k^c}{\partial \zeta}(p) \right\} \xi^j \overline{\xi^k}$$

in $\xi^1, \dots, \xi^r$, where f^c denotes the complex conjugate of f, has a positive eigenvalue. Then we have

$$\Gamma_{\{\varphi \ge 0\}} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_{x_0} = 0,$$

that is, if u is a hyperfunction solution germ to $\mathcal{M}$ at x_0 and $u = 0$ on $\varphi < 0$, then $u = 0$.

If $\mathcal{M}$ is a single differential equation with simple characteristics, this theorem is a special case of the result of [RZ] (the case with no purely C^∞ variables).

A simple, geometric proof to the above theorem is given without the Carleman estimate and instead by the structure theorem, described in terms of generalized Levi form, of microfunction solution sheaves of micro-differential systems (see [SKK, Chap. II]). By adding a new complex variable t (which is regarded as complex conjugate to φ), we show that the Hörmander form for the system $\mathcal{M}$ is obtained from the generalized Levi form of the product differential system $\mathcal{D}_{\mathbb{C}}\delta(t) \hat{\otimes} \mathcal{M}$. In view of the proof, it seems natural to consider the Hörmander form as an imaginary counterpart of the generalized Levi form for the system of micro-differential equations, and this enables us to restate the theorem in a natural manner (with its proof left as it is) so that it interpolates between Hörmander's theorem (for the analytic case) and the following much simpler result (which is an immediate conclusion from the structure theorem of micro-differential systems of [SKK]).

Theorem 2. *Let $p = id\varphi(x_0)$. Suppose $p \in \mathrm{Ch}(\mathcal{M})$, and assume that the generalized Levi form $L_{\mathcal{M}}(p)$ of $\mathcal{M}$ at p is not zero:*

$$L_{\mathcal{M}}(p) \ne 0.$$

Then we have

$$\Gamma_{\{\varphi \ge 0\}} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_{x_0} = 0.$$

For the proof, we do not use the theory of FBI transform (which is used in [RZ] as a main tool) but in an essential way the sheaf $\mathcal{C}_{A|X}$ of microfunctions for boundary value problems. For more details, see [U].

Remark. Recently Tamura [Ta] extended the result of [RZ] to the system of differential equations with smooth and partly analytic coefficients. He used the Poisson bracket

$$\{A, B\} = \sum_{\nu=1}^{n} \left(\frac{\partial A}{\partial \zeta_\nu} \frac{\partial B}{\partial z_n} - \frac{\partial B}{\partial z_\nu} \frac{\partial A}{\partial \zeta_n} \right)$$

for square matrices A and B of functions on T^*X instead of the Hörmander form and followed the same argument as in [RZ].

References

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