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Some Doubly Infinite and Mixed Infinite Sums derived from The N- Fractional Calculus of A Logarithmic Function (with Some Examinations)

Katsuyuki Nishimoto , Susana S. de Romero
and Ana I. Prieto

Abstract

In this article theorems for some doubly infinite and mixed infinite sums derived from the N-fractional calculus of a logarithmic function are reported.

Moreover some numerical examinations for the theorems are reported too.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}d\zeta \qquad (\nu \notin Z^-), \qquad (1)$$

$$(f)_{-m} = \lim_{\nu \to -m}(f)_\nu \qquad (m \in Z^+), \qquad (2)$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $z \in C$, $\nu \in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu} = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_{C} \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z), \quad [\, Refer\ to\ (1)\,] \qquad (3)$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \qquad (m \in Z^{+}), \qquad (4)$$

and define the binary operation $\circ$ *as*

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \qquad (\alpha, \beta \in R), \qquad (5)$$

then the set

$$\left\{ N^{\nu} \right\} = \left\{ N^{\nu} \mid \nu \in R \right\} \qquad (6)$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^{\nu})^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^{ν} *, for the function* f *such that* $f \in F = \left\{ f;\ 0 \neq |f_{\nu}| < \infty, \nu \in R \right\}$, *where* $f = f(z)$ *and* $z \in C$.
(*vis.* $-\infty < \nu < \infty$).
(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index* ν *" for the set of* F . (*F.O.G.* ; Fractional calculus operator group)

Theorem C. *Let*

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} \quad (\nu \in R). \qquad (7)$$

Then the set S *is a commutative ring for the function* $f \in F$, *when the identity*

$$N^{\alpha} + N^{\beta} = N^{\gamma} \quad (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \qquad (8)$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} (z-c)^{\beta - \alpha} \qquad \left(\left| \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

(ii) $$(\log(z-c))_{\alpha} = -e^{-i\pi\alpha} \Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \qquad (|\Gamma(\alpha)| < \infty),$$

where $z - c \neq 0$ in (i), and $z - c \neq 0, 1$ in (ii) and (iii). (Γ; Gamma function),

(iv) $$(u \cdot v)_{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \left(\begin{array}{l} u = u(z), \\ v = v(z) \end{array} \right).$$

§ 1. Doubly Infinite Sum and Mixed One

In the following $\alpha, \beta \in R$.

Theorem 1. *Let*

$$M(\alpha, \beta ; k, m) := \frac{\Gamma(\alpha + k)\Gamma(k+m)\Gamma(\beta+1)\Gamma(\beta-\alpha-m)}{k! \cdot m!\, \Gamma(\alpha)\Gamma(k)\Gamma(\beta+1-m)\Gamma(-\alpha)} . \tag{1}$$

(i) *When* $\beta \notin Z_0^+$, *we have the following doubly infinite sums ;*

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} M(\alpha, \beta ; k, m) \left(\frac{z-c}{z}\right)^m \left(\frac{c}{z}\right)^k \subseteq \frac{\Gamma(\beta-\alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c}\right)^{\alpha-\beta} , \tag{2}$$

where

$$z-c \neq 0, 1, \quad z \neq 0, 1, \quad |(z-c)/z| < 1, \quad |c/z| < 1,$$

and

$$|\Gamma(\alpha)|, \quad \left|\frac{\Gamma(\beta-\alpha-m)}{\Gamma(-\alpha)}\right| < \infty .$$

The identity (notation $=$ *) holds for* $(\alpha - \beta) \in Z$.

(i i) *When* $s \in Z^+$, *we have the following mixed infinite sums ;*

$$\sum_{k=0}^{\infty} \sum_{m=0}^{s} M(\alpha, s ; k, m) \left(\frac{z-c}{z}\right)^m \left(\frac{c}{z}\right)^k \subseteq \frac{\Gamma(s-\alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c}\right)^{\alpha-s} , \tag{3}$$

where

$$z-c \neq 0, 1, \quad z \neq 0, 1, \quad |c/z| < 1, \quad |(z-c)/z| < \infty,$$

and

$$|\Gamma(\alpha)| < \infty .$$

The identity (notation $=$ *) holds for* $(\alpha - s) \in Z$.

Proof of (i). We have

$$\log \frac{z-c}{z} = \log\left(1 - \frac{c}{z}\right) = -\sum_{k=1}^{\infty} \frac{c^k}{k} z^{-k} \quad \left(\left|\frac{-c}{z}\right| < 1\right). \tag{4}$$

Operate N-fractional calculus operator N^{α} to the both sides of (4), we obtain

$$z^{-\alpha} - (z-c)^{-\alpha} = -\sum_{k=1}^{\infty} \frac{c^k}{k!} \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} z^{-k-\alpha} \quad (|\Gamma(\alpha)| < \infty), \tag{5}$$

since

$$N^{\alpha}\left(\log\frac{z-c}{z}\right) = \left(\log(z-c) - \log z\right)_{\alpha} \quad (z-c \neq 0,1.\ z \neq 0,1.) \tag{6}$$

$$= e^{-i\pi\alpha}\Gamma(\alpha)\,\{z^{-\alpha} - (z-c)^{-\alpha}\} \quad (|\,\Gamma(\alpha)\,| < \infty\,) \tag{7}$$

and

$$N^{\alpha}\left(z^{-k}\right) = \left(z^{-k}\right)_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(k+\alpha)}{\Gamma(k)}\,z^{-k-\alpha} \quad \left(\left|\frac{\Gamma(k+\alpha)}{\Gamma(k)}\right| < \infty\right) \tag{8}$$

by Lemmas (i i) and (i).

Therefore, we have

$$(z-c)^{\alpha} - z^{\alpha} = -\sum_{k=1}^{\infty}\frac{c^k}{k!}\cdot\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}(z-c)^{\alpha}z^{-k}, \tag{9}$$

hence

$$z^{\alpha} = \sum_{k=0}^{\infty}\frac{c^k}{k!}\cdot\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}(z-c)^{\alpha}z^{-k}, \tag{10}$$

from (5).

Next operate N- fractional calculus operator N^{β} to the both sides of (10), we obtain

$$\left(z^{\alpha}\right)_{\beta} = \sum_{k=0}^{\infty}\frac{c^k}{k!}\cdot\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\left((z-c)^{\alpha}z^{-k}\right)_{\beta}, \ . \tag{11}$$

$$\left(z^{\alpha}\right)_{\beta} = e^{-i\pi\beta}\frac{\Gamma(\beta-\alpha)}{\Gamma(-\alpha)}\,z^{\alpha-\beta} \quad \left(\left|\frac{\Gamma(\beta-\alpha)}{\Gamma(-\alpha)}\right| < \infty\right), \tag{12}$$

$$\left((z-c)^{\alpha}z^{-k}\right)_{\beta} = \sum_{m=0}^{\infty}\frac{\Gamma(\beta+1)}{m!\,\Gamma(\beta+1-m)}\left((z-c)^{\alpha}\right)_{\beta-m}(z^{-k})_{m}, \ . \tag{13}$$

Now we have

$$\left((z-c)^{\alpha}\right)_{\beta-m} = e^{-i\pi(\beta-m)}\frac{\Gamma(\beta-\alpha-m)}{\Gamma(-\alpha)}\,(z-c)^{\alpha-\beta+m} \quad \left(\left|\frac{\Gamma(\beta-\alpha-m)}{\Gamma(-\alpha)}\right| < \infty\right), \tag{14}$$

and

$$\left(z^{-k}\right)_{m} = e^{-i\pi m}\frac{\Gamma(k+m)}{\Gamma(k)}\,z^{-k-m}, \tag{15}$$

by Lemma (i v) and (i), respectively.

Therefore, we obtain

$$\frac{\Gamma(\beta-\alpha)}{\Gamma(-\alpha)} z^{\alpha-\beta}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(k+m)\Gamma(\beta+1)\Gamma(\beta-\alpha-m)}{m!\Gamma(k)\Gamma(\beta+1-m)\Gamma(-\alpha)} \left(\frac{z-c}{z}\right)^m \left(\frac{c}{z}\right)^k (z-c)^{\alpha-\beta} \quad (16)$$

from (10) ~ (15).

Therefore, we have

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} M(\alpha,\beta\,;k,m) \left(\frac{z-c}{z}\right)^m \left(\frac{c}{z}\right)^k = \frac{\Gamma(\beta-\alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c}\right)^{\alpha-\beta}, \quad (17)$$

from (16).

However the LHS (left hand side) of (17) is always one valued function, on the contrary the RHS (right hand side) of (17) is many valued function for $(\alpha-\beta)\notin \boldsymbol{Z}$ and one valued one for $(\alpha-\beta)\in \boldsymbol{Z}$.

Hence we must calculate as

$$\left(\frac{z}{z-c}\right)^{\alpha-\beta} = \left(e^{i2n\pi}\frac{z}{z-c}\right)^{\alpha-\beta} \quad \left(\begin{array}{c} n\in \boldsymbol{Z} \\ (\alpha-\beta)\notin \boldsymbol{Z} \end{array}\right) \quad (18)$$

because we are now being in the field of complex analysis.

Moreover, when $(\alpha-\beta)\in \boldsymbol{Z}$ both of the LHS and RHS of (17) are one valued functions respectively. In this case we have (17) strictly.

Therefore, we obtain (2) from (17), considering (18) finally.

Proof of (i i). Set $\beta = s\in \boldsymbol{Z}^+$ in (2), we have then (3) clearly, under the comditions.

§ 2. Some Numerical Examinations for Theorem 1

[I] Examination of Theorem 1. (2)

Set

$$c = 1, \quad z = 10, \quad \alpha = 1/4 \quad \text{and} \quad \beta = 1/2$$

in Theorem 1. (2) , we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} M(1/4, 1/2\,;k, m)\left(\frac{1}{10}\right)^{k}\left(\frac{9}{10}\right)^{m}$$

$$\subseteq -\frac{1}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)}\left(e^{i2n\pi}\frac{9}{10}\right)^{1/4} \qquad (n \in \boldsymbol{Z}) \tag{1}$$

$$= \begin{cases} -0.72044\cdots & (\text{for } n = 0) \quad (2) \\ -i\,0.72044\cdots & (\text{for } n = 1) \quad (3) \\ \;\;0.72044\cdots & (\text{for } n = 2) \quad (4) \\ \;\;i\,0.72044\cdots & (\text{for } n = 3) \quad (5) \end{cases}$$

When $\alpha, \beta, c, z \in \boldsymbol{R}$, the left hand side (LHS) of (3) is real, then we must choose (2) and (4) from the set $\{(2), (3), (4), (5)\}$.

Now we have

$$M(1/4, 1/2\,;\,0, 0)\left(\frac{1}{10}\right)^{0}\left(\frac{9}{10}\right)^{0} \qquad \left(\begin{array}{c}\text{first term of}\\ \text{the LHS of (1)}\end{array}\right)$$

$$= \frac{\Gamma(1/4)}{\Gamma(-1/4)} = -\frac{1}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)} < 0\,. \tag{6}$$

Then choosing (2) from the set { (2), (4) }, since the sign of the double infinite sum of LHS of (1) is decided by the sign of its first term (with $k = m = 0$), when

$$\left|M_{k,m}\left(\frac{1}{10}\right)^{k}\left(\frac{9}{10}\right)^{m}\right| > \left|M_{k+1,\,m+1}\left(\frac{1}{10}\right)^{k+1}\left(\frac{9}{10}\right)^{m+1}\right|, \quad M_{k,m} = M(\alpha, \beta; k, m),$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} M(1/4, 1/2\,;\,k, m)\left(\frac{1}{10}\right)^{k}\left(\frac{9}{10}\right)^{m} = -\,0.72044\cdots \tag{7}$$

from (2), considering (6).

Indeed we have

$$\text{LHS of (7)} = \sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)}{k!\,\Gamma(\frac{1}{4})}\left(\frac{1}{10}\right)^k\sum_{m=0}^{\infty}\frac{\Gamma(k+m)\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}-m)}{m!\,\Gamma(k)\Gamma(\frac{1}{2}+1-m)\Gamma(-\frac{1}{4})}\left(\frac{9}{10}\right)^m \qquad (8)$$

$$= \frac{\Gamma(\frac{1}{4})}{\Gamma(-\frac{1}{4})}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)}{\Gamma(\frac{1}{4})}\left(\frac{1}{10}\right)^k + \frac{\Gamma(-\frac{3}{4})}{2!\,\Gamma(-\frac{1}{4})}\left(\frac{9}{10}\right)\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)\cdot k}{k!\,\Gamma(\frac{1}{4})}\left(\frac{1}{10}\right)^k$$

$$-\frac{1}{2!\,2^2}\cdot\frac{\Gamma(-\frac{7}{4})}{\Gamma(-\frac{1}{4})}\left(\frac{9}{10}\right)^2\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)\cdot k(k+1)}{k!\,\Gamma(\frac{1}{4})}\left(\frac{1}{10}\right)^k$$

$$+\frac{3}{3!\,2^3}\cdot\frac{\Gamma(-\frac{11}{4})}{\Gamma(-\frac{1}{4})}\left(\frac{9}{10}\right)^3\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)\cdot k(k+1)(k+2)}{k!\,\Gamma(\frac{1}{4})}\left(\frac{1}{10}\right)^k$$

$$-\frac{3\cdot 5}{4!\,2^3}\cdot\frac{\Gamma(-\frac{15}{4})}{\Gamma(-\frac{1}{4})}\left(\frac{9}{10}\right)^4\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{4}+k)\cdot k(k+1)(k+2)(k+3)}{k!\,\Gamma(\frac{1}{4})}\left(\frac{1}{10}\right)^k$$

$$+\cdots\cdots\cdots\cdots \qquad (9)$$

$$= \frac{1}{(-4)}\cdot\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left\{1+\frac{1}{40}+\frac{5}{2!\,40^2}+\frac{5\cdot 9}{3!\,40^3}+\frac{5\cdot 9\cdot 13}{4!\,40^4}+\frac{5\cdot 9\cdot 13\cdot 17}{5!\,40^5}+\cdots\right\}$$

$$+\frac{1}{2\cdot 3}\cdot\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left(\frac{9}{10}\right)\left\{0+\frac{1}{40}+\frac{5}{40^2}+\frac{5\cdot 9}{2!\,40^3}+\frac{5\cdot 9\cdot 13}{3!\,40^4}+\frac{5\cdot 9\cdot 13\cdot 17}{4!\,40^5}+\cdots\right\}$$

$$+\frac{1}{2!\,3\cdot 7}\cdot\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left(\frac{9}{10}\right)^2\left\{0+\frac{2}{40}+\frac{3\cdot 5}{40^2}+\frac{4(5\cdot 9)}{2!\,40^3}+\frac{5(5\cdot 9\cdot 13)}{3!\,40^4}+\frac{6(5\cdot 9\cdot 13\cdot 17)}{4!\,40^5}+\cdots\right\}$$

$$+\frac{1}{3\cdot 7\cdot 11}\cdot\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left(\frac{9}{10}\right)^3\left\{0+\frac{2\cdot 3}{40}+\frac{3\cdot 4\cdot(5)}{40^2}+\frac{4\cdot 5(5\cdot 9)}{2!\,40^3}+\frac{5\cdot 6(5\cdot 9\cdot 13)}{3!\,40^4}+\frac{6\cdot 7(5\cdot 9\cdot 13\cdot 17)}{4!\,40^5}+\cdots\right\}$$

$$+\frac{5}{2!\,3\cdot 7\cdot 11\cdot 15}\cdot\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\left(\frac{9}{10}\right)^4\left\{0+\frac{2\cdot 3\cdot 4}{40}+\frac{3\cdot 4\cdot 5(5)}{40^2}+\frac{4\cdot 5\cdot 6(5\cdot 9)}{2!\,40^3}+\frac{5\cdot 6\cdot 7(5\cdot 9\cdot 13)}{3!\,40^4}+\frac{6\cdot 7\cdot 8(5\cdot 9\cdot 13\cdot 17)}{4!\,40^5}+\cdots\right\}$$

$$+ \cdots\cdots\cdots \tag{10}$$

$$= -\ (0.75941\cdots) + (0.01265\cdots) + (0.00205\cdots)$$

$$+ (0.00182\cdots) + (0.00117\cdots) + \cdots\cdots \tag{11}$$

$$= -\ 0.74172\cdots \quad . \tag{12}$$

[II] Examination of Theorem 1. (3) for $(\alpha - s) \notin \boldsymbol{Z}$

Set

$$c = 1,\quad z = 3,\quad \alpha = 1/2 \quad \text{and} \quad s = 1$$

in Theorem 1. (3), we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{1} M(1/2, 1\,; k, m) \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^m$$

$$\subseteq -\frac{1}{2}\left(e^{i\,2n\pi}\frac{2}{3}\right)^{1/2} \qquad (n \in \boldsymbol{Z}) \tag{13}$$

$$= \begin{cases} -0.408240\cdots & (\text{for } n = 0) \quad (14) \\ \ \ 0.408240\cdots & (\text{for } n = 1) \quad (15). \end{cases}$$

Now we have

$$M(1/2, 1\,; 0, 0)\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^0 \qquad \left(\begin{matrix}\text{first term of} \\ \text{the LHS of (13)}\end{matrix}\right)$$

$$= \frac{\Gamma(1/2)}{\Gamma(-1/2)} = -\frac{1}{2} < 0 \ . \tag{16}$$

Then choosing (14) from the set { (14), (15) }, since the sign of the infinite mixed sum of LHS of (13) is decided by the sign of its first term (with $k = m = 0$), when

$$\left| M_{k,m}\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^m \right| > \left| M_{k+1,\,m+1}\left(\frac{1}{3}\right)^{k+1} \left(\frac{2}{3}\right)^{m+1} \right| , \quad M_{k,m} = M(1/2, 1\,; k, m),$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{1} M(1/2, 1\,;\, k, m) \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^m = -\ 0.408240\cdots \tag{17}$$

from (14), considering (16).

Indeed we have

$$\text{LHS of (17)} = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)}{k!\,\Gamma(\frac{1}{2})}\left(\frac{1}{3}\right)^k \left(-\frac{1}{2}+k\cdot\frac{2}{3}\right) \tag{18}$$

$$= -\frac{1}{2} + \frac{1}{2\cdot 3}\cdot\frac{1}{6} + \frac{1}{2!}\cdot\frac{1}{2^2\cdot 3}\cdot\frac{5}{6} + \frac{1}{3!}\cdot\frac{1}{2^3\cdot 3^2}\cdot\frac{9}{6}$$

$$+\frac{1}{4!}\cdot\frac{5\cdot 7}{2^4\cdot 3^3}\cdot\frac{13}{6} + \frac{1}{5!}\cdot\frac{5\cdot 7\cdot 9}{2^5\cdot 3^4}\cdot\frac{17}{6} + \cdots\cdots \tag{19}$$

$$= -0.5 + (0.02777\cdots) + (0.034722\cdots) + (0.017361\cdots)$$

$$+(0.007314\cdots) + (0.002869\cdots) + (0.0001083\cdots) + \cdots\cdots \tag{20}$$

$$= -\,0.40887\cdots \quad . \tag{21}$$

[I I I] Examination of Theorem 1. (3) for $(\alpha - s) \in \boldsymbol{Z}$

Set

$$c = 1, \quad z = 3, \quad \alpha = 2 \quad \text{and} \quad s = 1$$

in Theorem 1. (3) , we obtain

$$\sum_{k=0}^{\infty}\sum_{m=0}^{1} M(2,1\,;k,m)\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^m = \frac{\Gamma(-1)}{\Gamma(-2)}\left(\frac{3}{2}\right) = -3\ , \tag{22}$$

without the ad hoc shown in [I] and [I I] in which the RHS of (1) and (13) are many valued ones.

That is, in this case both sides of § 1. (3) are one valued functions respectively, then we have the notation = in **§ 1.** (3) .

Indeed we have

$$\text{LHS of (22)} = \sum_{k=0}^{\infty}\frac{\Gamma(2+k)}{k!\,\Gamma(k)}\left(\frac{1}{3}\right)^k \sum_{m=0}^{1}\frac{\Gamma(k+m)\Gamma(-1-m)}{m!\,\Gamma(2-m)\Gamma(-2)}\left(\frac{2}{3}\right)^m \tag{23}$$

$$= -2\sum_{k=0}^{\infty}(k+1)\left(\frac{1}{3}\right)^k + \frac{2}{3}\sum_{k=0}^{\infty}k(k+1)\left(\frac{1}{3}\right)^k \tag{24}$$

$$= -2\left(1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\frac{5}{3^4}+\cdots\right)$$

$$+\frac{2}{3}\left(0+\frac{2}{3}+\frac{2\cdot 3}{3^2}+\frac{3\cdot 4}{3^3}+\frac{4\cdot 5}{3^4}+\frac{5\cdot 6}{3^5}+\cdots\right) \quad (25)$$

$$= -2(2.2478\cdots)+\frac{2}{3}(2.2313\cdots) \quad (26)$$

$$= -3.00807\cdots \quad . \quad (27)$$

§ 3. Commentary

1. Notice that the LHS of § 1. (2) is always one valued function, on the contrary its RHS is many valued function for $(\alpha-\beta)\notin Z$ and one valued one for $(\alpha-\beta)\in Z$.

And notice that when both of the LHS and the RHS of § 1. (2) are one valued functions respectively, namely in the case of $(\alpha-\beta)\in Z$, we have the identity (notation =) in § 1. (2) always.

2. " Fractional calculus " is essentially a problem in the field of complex analysis. We should not forget that we are now being in the field of fractional calculus, that is, in that of complex analysis.

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Katsuyuki Nishimoto
Institute of Applied Mathematics
Descartes Press Co.
2 - 13 - 10 Kaguike, Koriyama
963 - 8833 Japan

Susana S. de Romero
and Ana I. Prieto
C.I.M.A., Universidad del Zulia
Apartado 10482, Maracaibo
Venezuela