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Mixed Type Duality in Mathematical Programming Involving generalized Convex set Functions*

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Abstract

A mixed type dual problem for minimax fractional programming concerning set functions is constructed from the variety of incomplete Lagramgian dual. In this note, we establish the duality theorems for mixed type dual of a given programming problem under nonsmooth generalized convex set functions.

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- The whole paper will be appeared elsewhere.

1. Introduction

In order to know when a feasible solution of a programming problem could be an optimal. Many authors effort to find the sufficient optimality conditions. It is often to establish the converse of the necessary optimality condition by some extra assumptions. After the sufficient optimality conditions (usually various type) are established, one could employ the sufficient optimality theorems to constitute the dual models relative to primal problem, and then prove the weak, strong, and strict converse duality theorem between the primal and the dual problems.

At times, these duality forms are difficult to understand the motivation for writing the dual exactly in the model given by more general dual constitution, but only for the requirement in mathematical analysis. Reason follows from these various type duality, a question is rised that whether we can constitute a mixed type dual to integrate these duality (cf. [1-2])

In this paper we will constitute a mixed type dual for a minimax fractional programming of set functions (see [6-11], [15] and [19-20] etc.)

At first we consider the following fractional programming problem with set functions:

$$(FP) \qquad \min_{\Omega} \max_{1\leq i\leq p} \; F_i(\Omega)/G_i(\Omega)$$

$$s.t. \quad \Omega \in S \;\; and \;\; H_j(\Omega) \leq 0, \;\; j \in M = \{1,2,3\ldots,m\}$$

where S is a convex subfamily of measurable subsets in an atomless finite measure space (X,Γ,μ); F_i, $-G_i$, $1 \leq i \leq p$ and H_j, $1 \leq j \leq m$ are convex set functions defined on S. Without loss of generality, we may assume that all $G_i > 0$ and all $F_i \geqslant 0$ on S in (FP). Then the mixed type dual model can be contructed as the following form:

$$(MD) \qquad \max \;\; (y^T F(U) + z^T_{M_0} H(U)) / y^T G(U)$$

$$\begin{aligned} s.t. \quad & U \in S, \\ & 0 \in y^T G(U)[\partial(y^T F)(U) + \partial(z^T H)(U)] \\ & \qquad - \partial(y^T G)(U)[y^T F(U) + z^T_{M_0} H(U)] + N_S(U), \\ & z^T_{M_\alpha} H(U) \geqslant 0, \qquad\qquad \alpha = 1,2,\ldots k, \\ & y \in I \equiv \left\{ \alpha \in R^m_+ \mid \alpha = (\alpha_1 \ldots \alpha_m), \;\; \sum_{i=1}^{m} \alpha_i = 1 \right\} \end{aligned}$$

where

$M_\alpha \subset M$, $\alpha = 0,1,2\ldots,k$ with $M_\alpha \cap M_\beta = \varnothing$ if $\alpha \neq \beta$ and $\cup_{\alpha=0}^{k} M_\alpha = M$;

$z^T_{M\alpha} H(U) = \sum_{j\in M_\alpha} z_j H_j(U)$ and $\partial(z^T_{M_\alpha} H)(U) = \sum_{j\in M_\alpha} z_j \partial H_j(U)$

(cf. Morean Rockafellar Theorem (Lai and Lin [8]));

$N_S(U) = \{f \in L_1(X,\Gamma,\mu) \mid \chi_\Omega - \chi_U, f > \le 0,\ \forall \Omega \in S\}$;

(a normal cone at U with respect to S)

$$F(U) = (F_1(U),\dots,F_p(U))^T,\ G(U) = (G_1(U),\dots,G_p(U))^T,$$
$$H(U) = (H_1(U),\dots,H_m(U))^T.$$

This mixed type dual (MD) includes the Wolfe type dual and Mond-Weir type dual as the special cases. Actually,

(1) As $M_0 = M,\ M_\alpha = \varnothing,\ \alpha = 1,\dots,k,$ then (MD) is reduced to the Wolfe type dual:

$$(WD) \qquad \max\ (y^TF(U) + z^TH(U))/y^TG(U)$$
$$\text{s.t.}\quad 0 \in y^TG(U)[\partial(y^TF)(U) + \partial(z^TH)(U)] - \partial(y^TG)(U)[y^tF(U) + z^T_{M_0}H(U)] + N_S(U),$$
$$y \in I \equiv \left\{\alpha \in R^p_+ \mid \sum_{i=1}^m \alpha_i = 1\right\}, \qquad z \in R^m_+ .$$

(2) As $M_0 = \varnothing,\ M_1 = M,\ M_\alpha = \varnothing,\ \alpha = 2,\dots,k,$ then (MD) is reduced to the Mond-Weir type dual:

$$(MWD) \qquad \max\ y^TF(U)/y^TG(U)$$
$$\text{s.t.}\quad 0 \in y^TG(U)[\partial(y^TF)(U) + \partial(z^TH)(U)] - \partial(y^TG)(U)y^tF(U) + N_S(U),$$
$$z^TH(U) \ge 0,\ y \in I.$$

The formation of (MD) is motivated from the incomplete Lagrangian dual as in next Section 2. The main task on the mixed type dual problem (MD) is to eastablish the weak, strong, and strict converse duality theorems in Section 5. In sections 3 and 4, we will mention brevity for the basic behavior of set functions and generalized convex set functions in the frame work for our requirement.

2. Incomplete Lagrangian Dual

In usual constrained programming problem, it may be sonsidered as follows:

$$(P) \qquad \min\ f(x),\ f : \mathbb{R}^n \to \mathbb{R}$$
$$\text{s.t.}\quad x \in \mathbb{R}^n \text{and } h(x) \le 0,\ h : \mathbb{R}^n \to \mathbb{R}^m.$$

It is well known that the Lagrangian dual is given as

$$(LD) \qquad \max\ [f(u) + \lambda^Th(u)]$$

$$s.t. \quad f'(u) + \lambda^T h'(u) = 0, \qquad \lambda^T h(u) = 0 \, .$$

provided that objective and constrained functions in (P) are differentiable.

One sees that all constraints of (P) is contained in the objective of (LD). Now if we consider part of the constrains of (P) in the objective of (LD) and the remained constraints still left in the constraints, it then forms a new Lagrangian dual, namely an *incomplete Lagrangian dual problem* which we state as the following problem:

$$(ILD) \qquad \max \ [f(u) + \lambda_J^T h_J(u)]$$
$$s.t. \ \ h_K(u) \leq 0, \ \ K = M \setminus J, \ \ M = J \cup K$$
$$\lambda_M^T h(u) = 0, \ \lambda_M = \lambda \in \mathbb{R}_+^n$$

where M is regarded as the m member of the constraints (see Bector et al. [2]).

From (ILD), one can see that the variety of J as well as K in M will formate a various duality, and it will reduce a mixed type dual involving some know duality forms, like the Wolfe type and the Mond-Weir type dual that are special scases of the mixed type dual. we will consider in this paper for more general mixed dual occured in minimax fractional programming problem with set functions for generalized convexity (cf. Lai and Liu [11]), and the nondifferentiable set functions will satisfy the constraints in (MD) by subdifferentiable situations.

3. Convexity and Subdifferentiability for Set Functions

In this section all symbols and definitions concerning set functions can refere to [6-11], [15-16] and [19-20]. For convenience, we recall some of that for our requirement. Throughout we consider an atomless finite measure space (X, Γ, μ) with $L^1(X, \mu)$ separable. That is, $\mu(X) < \infty$ and for any $A \in \Gamma, \mu(A) > 0$, we always hvae a nonempty subset $B \subset A$ such that $\mu(B) > 0$; and for any $\Omega \in \Gamma, \mu(\Omega) = \int_X \chi_\Omega d\mu < \infty$. It follows that for any $\Omega \in \Gamma$, there corresponds a characteristic function $\chi_\Omega \in L^\infty = (L^1)^* \subset L^1$. Furthmore, by the separability of L^1, all discusion in our requirement, we need only use a sequence $\{\Omega_n\}$ in Γ. The *convexity* of a subfamily $S \subset \Gamma$ can be defined as follows :

for any $\Omega, \ \Lambda \in S$ and $\lambda \in [0,1]$, there are associate sequences $\{\Omega_n\} \subset \Omega \setminus \Lambda$ and $\{\Lambda_n\} \subset \Lambda \setminus \Omega$ such that

$$\circledast \begin{cases} \chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Omega \setminus \Lambda} \text{ and } \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda \setminus \Omega} \\ \Rightarrow \chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Omega + (1-\lambda)\chi_\Lambda \end{cases}$$

Since measure space is not linear and has no topology in general, the convexity, continuity, and differentiability for set functions on any subfamily S of measurable subsets in X are need to redefined which are different from usual.

A set function $F : S \to \mathbb{R}$ is *convex* if for any $(\Omega, \Lambda, \lambda) \in S \times S \times [0,1]$, there associated a sequence $V_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)$ with property ⊛ such that

$$\lim_{n\to\infty} \sup \ F(U_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda).$$

- F is *continuous* at Ω if there is a sequence $\{\Omega_n\} \subset \Gamma$ such that

$$\lim_{n\to\infty} F(\Omega_n) = F(\Omega) \qquad \textit{whenever} \quad \chi_{\Omega_n} \xrightarrow{w^*} \chi_{\Omega}.$$

- F is *subdifferentiable* at $\Omega_0 \in S$, if there is a $f \in L^1(X, \mu)$ such that

$$F(\Omega) - F(\Omega_0) \geq \langle \chi_\Omega - \chi_{\Omega_0}, f \rangle \qquad \forall \Omega \in S.$$

This $f \in L^1$ is called a *subgradient* of F at Ω_0.

The set of all subgradients f of F at Ω_0, denoted by

$$\partial F(\Omega_0) = \{f \in L^1 | F(\Omega) - F(\Omega_0 \geq \langle \chi_\Omega - \chi_{\Omega_0}, f \rangle \qquad \textit{for any } \Omega \in S\}$$

is called the *subdifferential* of F at Ω_0.

It is known that $\partial F(\Omega_0)$ is a singleton set if and only if F is differentiable.

Based on the above preparation, we are proceed to next section for optimality condition in the minimax fractional programming problem (FP) for set functions.(cf. Lai and Liu [11])

4. Optimality Conditions, Generalized Convexity

Concerning the necessary optimalily conditions, it can be described as follows:

If Ω^* is an optimal solution of (FP) having constraint qnalification and F_i, $-G_i$ $(i = 1,2,\ldots,p)$, H_j $(j = 1,2,\ldots,m)$, are proper convex set functions, then there exist $y \in \mathbb{R}^p_+$ and $z \in \mathbb{R}^m_+$ such that (Kuhn-Tucker type)conditions hold:

$$\begin{aligned} 0 \in\ & y^T G(\Omega^*)[\partial(y^T F)(\Omega^*) + \partial(z^T H)(\Omega^*)] \\ & - \partial(y^T G)(\Omega^*)\, y^T F(\Omega^*) + N_S(\Omega^*), \\ & z^T H(\Omega^*) = 0, \end{aligned} \tag{1}$$

where $N_S(\Omega^*)$ is the normal cone in L^1 at Ω^*.

Furthermore, if the optimal value of (FP) gives

$$\lambda^* = \max_{1\leq i\leq p} F_i(\Omega^*)/G_i(\Omega^*) = y^T F(\Omega^*)/y^T G(\Omega^*) \qquad \textit{for } y \in I$$

then (1) becomes

$$0 \in [\partial(y^T F)(\Omega^*) - \lambda^* \partial(y^T G)(\Omega^*)] + \partial(z^T H)(\Omega^*) + N_S(\Omega^*), \tag{2}$$

and

$$y^T[F(\Omega^*) - \lambda^* G(\Omega^*)] = 0 \ , \tag{3}$$

$$z^T H(\Omega^*) = 0 \ . \tag{4}$$

In addition if $0 \notin \partial(z^T H)(\Omega^*) + N_S(\Omega^*)$, then the relation (2) ~(4) hold and $y \in I$. In this case such a point $\Omega^* \in S$ is called regular. Conversely, one may ask whether a feasible solution of (FP) satisfying the Kuhn-Tucker type condition (2) with the relations (3) and (4), would be an optimal for (FP) ? We will see that there some extra assumptions, like convexity/ generalized convexity are reqired.

In [11] Lai and Liu defined $(\mathfrak{F},\rho,\theta)$-convexity. For convenience, we recall that a set function $F : \Gamma \to \mathbb{R}$ is $(\mathfrak{F},\rho,\theta)$-*convex* at Ω_o if

$$F(\Omega) - F(\Omega_0) \geq \mathfrak{F}(\Omega,\Omega_0;f) + \rho\, \theta(\Omega,\Omega_0) \tag{5}$$

for $f \in \partial F(\Omega_0)(\subset L^1)$, $\Omega \in \Gamma$ and $\rho \in \mathbb{R}$. Here $\mathfrak{F} : \Gamma \times \Gamma \times L^1(X,M) \to \mathbb{R}$ is sublinear with respect to (w. r. t. for short) the 3rd argument and $\theta : \Gamma \times \Gamma \to \mathbb{R}_+$ be such that $\theta(\Omega_1,\Omega_2) = 0$ only if $\Omega_1 = \Omega_2$. F is called $(\mathfrak{F},\rho,\theta)$ *-quasiconvex* (*prestrict quasiconvex*) if (by the inequality (5))

$$F(\Omega) - F(\Omega_0) \leq 0 (< 0) \Rightarrow \mathfrak{F}(\Omega,\Omega_0;f) + \rho\theta(\Omega,\Omega_0) \leq 0 \tag{6}$$

F is called $(\mathfrak{F},\rho,\theta)$ *-pseudoconvex* (*-strict pseudoconvex*) if

$$\mathfrak{F}(\Omega,\Omega_0;f) + \rho\theta(\Omega,\Omega_0) \geq 0 \Rightarrow F(\Omega) - F(\Omega_0) \geq 0 (> 0) \tag{7}$$

It is known that there several sufficient optimality conditions for (FP) are established(cf. [11]) under several generalized convexity. We state these conditions related to duality theorems in the mixed problem (MD) as following.

For a feasible solution (U,y,z) in (MD), denote a functional on S by

$$D(\bullet) \ = y^T G(U)[y^T F(\bullet) \ + z^T_{M_0} H(\bullet)] \ - y^T G(\bullet)[y^T F(U) + z^T_{M_0} H(U)]. \tag{8}$$

Then the following dulaity theorems are established.

Theorem 1 (weak duality)

Let Ω and (U,y,z) be the feasibe solutions of (FP) and (MD) respectively. Further, assume that $\mathfrak{F}(\Omega, U; -\eta) \geq 0$ for each $\eta \in N_S$ and suppose that any one of the following conditions holds:

(a) $y^T H$ is $(\mathfrak{F},\rho_1,\theta)$-convex, $-y^T G$ is $(\mathfrak{F},\rho_2,\theta)$-convex, $z^T_{M\alpha} H$ is $(\mathfrak{F},\rho_{3\alpha},\theta)$-convex for each $\alpha = 0,1,2,\ldots,k$ and

$$y^T G(U) P_1 + [y^T F(U) + z^T_{M\alpha} H(U)]\rho_2 + y^T G(U) \sum_{\alpha=0}^{k} \rho_{3\alpha} \geq 0,$$

(b) $D(\bullet)$ is $(\mathfrak{F},\rho,\theta)$ –pseudoconvex, $z_{M\alpha}^T H$ is $(\mathfrak{F},\rho,\theta)$ –quasiconvex for each $\alpha = 1,2,\ldots,k,$ and $\rho_1 + y^T G(U)\sum_{\alpha=1}^{k} \rho_{2\alpha} \geq 0,$

(c) D is $(\mathfrak{F},\rho,\theta)$ –quasiconvex, $z_{M\alpha}^T H$ is strictly $(\mathfrak{F},\rho,\theta)$ –pseudoconvex, for each $\alpha = 1,2,\ldots,k,$ and $\rho_1 + y^T G(U)\sum_{\alpha=1}^{k} \rho_{2\alpha} \geq 0,$

(d) D is prestrictly $(\mathfrak{F},\rho,\theta)$ –quasiconvex, $z_{M\alpha}^T H$ is $(\mathfrak{F},\rho,\theta)$ –quasiconvex, for each $\alpha = 1,2,\ldots,k,$ and $\rho_1 + y^T G(U)\sum_{\alpha=1}^{k} \rho_{2\alpha} > 0,$

(e) $D + y^T G(U)\sum_{\alpha=1}^{k} z_{M_\alpha}^T H$ is $(\mathfrak{F},\rho,\theta)$ –pseudoconvex and $\rho \geq 0,$

(f) $D + y^T G(U)\sum_{\alpha=1}^{k} z_{M_\alpha}^T H$ is prestrictly $(\mathfrak{F},\rho,\theta)$ –quasiconvex and $\rho > 0,$ then

$$\max_{1\leq i\leq p} \frac{F_i(\Omega)}{G_i(\Omega)} \geq (y^T F(U) + z_{M_0}^T H(U))/y^T G(U).$$

We prove this theorem for brevity under hyperthesis (a) only, and omit the others.

Proof. On the case of hyperthesis (a).
The objective of problem (FP) is actually

$$\min_{\Omega} \varphi(\Omega)$$

$$\text{with} \qquad \varphi(\Omega) = \max_{1\leq i\leq p} \frac{F_i(\Omega)}{G_i(\Omega)} = \max_{y\in I} \frac{y^T F(\Omega)}{y^T G(\Omega)}.$$

Suppose on the contrary that

$$\varphi(\Omega) < (y^T F(U) + z_{M_\alpha}^T H(U))/y^T G(U).$$

Then

$$\frac{y^T F(\Omega)}{y^T G(\Omega)} < (y^T F(U) + z_{M\alpha}^T H(U))/y^T G(U)$$

or

$$y^T G(U) y^T F(\Omega) - y^T G(\Omega)[y^T F(U) + z_{M_\alpha}^T H(U)] < 0.$$

Since $z_{M_\alpha}^T H(U) \leq 0, \quad y^T G(U) > 0, \quad \cup_{\alpha=0}^{k} M_\alpha = M$ and the constraint inequality in (MD), one can reduce that

$$y^T G(U)[y^T F(\Omega) + z_{M\alpha}^T H(U)] - y^T G(\Omega)[y^T F(U) + z_{M\alpha}^T H(U)] < y^T G(U) z_{M\alpha}^T H(\Omega) \leq 0.$$

It follows that

$$D(\Omega) < 0 = D(U)$$

As the Kuhn-Tucker type condtion held in (MD), one can find that there exist $f \in \partial(y^T F)(U),\ h_\alpha \in \partial(z_{M\alpha}^T H)(U), \alpha = 1,2,\ldots,k,\ g \in \partial(-y^T G(U))$ and $\eta \in N_S(U)$ such that

$$y^TG(U)(f+\sum_{\alpha=0}^{k}h_\alpha)+[y^TF(U)+z_{M\alpha}^TH(U)]\,g+\eta\;=\;0.$$

By the sublinearity of $\mathfrak{F}$ on the 3rd argument, we have

$$\mathfrak{F}(\Omega,U;y^TG(U)(f+\sum_{\alpha=0}^{k}h_\alpha)+[y^TF(U)+z_{M\alpha}^TH(U)]\,g+\eta)\;=\;0$$

or

$$\mathfrak{F}(\Omega,U;y^TG(U)(f+\sum_{\alpha=0}^{k}h_\alpha)+[y^TF(U)+z_{M\alpha}^TH(U)]\,g)\;=\;\mathfrak{F}(\Omega,U;-\eta)\geq 0.$$

From hyperthesis (a), the $(\mathfrak{F},\rho_j,\theta)$-convexity for j=1,2,3, it follows that

$$0>D(\Omega)\geq(y^TG(U)\rho_1+[y^TF(U)+z_{M_\alpha}^T(U)]\rho_2+y^TG(U)\sum_{\alpha=0}^{k}\rho_{3\alpha})\,\theta(\Omega,U).$$

This contradicts the fact

$$y^TG(U)\rho_1+[y^TF(U)+z_{M\alpha}^T(U)]\rho_2+y^TG(U)\sum_{\alpha=0}^{k}\rho_{3\alpha}\geq 0$$

since $\theta(\Omega,U)>0$. ■

Note that if $M_0=M,\ M_\alpha=\varnothing\ \ \forall\alpha,$ then (b)=(e), (c)=(d)=(f) in Theorem 1. and so (MD)=(WD).

Corollary 1.1 (wolfe type weak duality) [11,theorem 4.1]

Let Ω and (U,y,z) be the feasible solutions of (FP) and (WD) respectively. Suppose that any one of (a), (b) and (c) holds. Then

$$\varphi(\Omega)\geq(y^TF(U)+z^TH(U)/y^TG(U). \qquad ■$$

Note that if $M_0=\phi,\ M_\alpha=M_1=M,$ then (MD)=(MWD) . Denote by

$$D(\bullet)\;=y^TG(U)y^TF(\bullet)\;-y^TG(\bullet)F(U).$$

Corollary 1.2 (Mond-Weir type weak duality)[11. Theorem 5.1]

. Let Ω and (U,y,z) be feasible solutions of (FP) and (MWD) respectively. Suppose that if any one of the condtions (a) ~(f) holds, then

$$\varphi(\Omega)\geq y^TF(U)/y^TG(U) \qquad ■$$

For convenience, we choose the functional

$$\mathfrak{F}(\Omega,\Omega_0;f) = \langle x_\Omega - x_{\Omega_0}, f\rangle \quad for\ \Omega,\Omega_0 \in S\ and\ f \in L^1.$$

Taking $\rho > 0$, then $(\mathfrak{F},\rho,\theta)$ –convexity is called $(\mathfrak{F}^*,\rho,\theta)$ –convexity (c.f. [9]). It is known that if a real valued function F is $(\mathfrak{F}^*,\rho,\theta)$ –convex at Ω_0, then F is convex at Ω_0. (cf. Lai and Liu [9, Theorem 3.2]). According to the above preparation, we have the following strong duality between (FP) and (MD).

Theorem 2 (Strong duality)

Let F_i, $-G_i$, $i = 1,2,\ldots,p$ and H_j, $j = 1,2,\ldots,m$ are $(\mathfrak{F}^*,\rho,\theta)$ –convex on S. Suppose that $\Omega^* \in S$ is regular (FP)-optimal solution, then there exist $y^* \in I$ and $z^* \in \mathbb{R}^m_+$ such that (Ω^*,y^*,z^*) is (MD)-feasible. Furthermore if the conditions of Theorem 1 are fulfilled for (MD)-feasible, then (Ω^*,y^*,z^*) is (MD)-optimal, and min(FP) =max(MD).

Remark 1

The Wolfe type strong duality [11, Theorem 4.2], and Mond-Weir type strong duality [11, Theorem 5.2] are special cases of Theorem 2.

Theorem 3 (Strict converse duality)

Let Ω^1 and (Ω^*,y^*,z^*) be optimal solutions of (P) and (MD), respectively. Suppose that the assumptions of Theorem 2 are fulfilled and $\mathfrak{F}(\Omega^1,\Omega^*;-h) \geq 0$ for each $h \in N_S(\Omega^*)$. Let y^* insteal of y in $D(\bullet)$. Further if any one of the following conditions holds :

(a) D is strictly $(\mathfrak{F},\rho,\theta)$ –pseudoconvex. $z^T_{M\alpha}H$ is $(\mathfrak{F},\rho_{2\alpha},\theta)$ –quasiconvex for each $\alpha \in \{1,2,\ldots,k\}$, and $\rho_1 + y^{*T}G(\Omega^*)\sum_{\alpha=1}^{k}\rho_{2\alpha} \geq 0$.

(b) $D + y^{*T}G(\Omega^*)\sum_{\alpha=1}^{k} z^{*T}_{M_\alpha}H$ is strictly $(\mathfrak{F},\rho,\theta)$ –pseudoconvex, and $\rho \geq 0$.

Then $\Omega^1 = \Omega^*$, and min(FP) =max(MD).

Remark 2

The Wolfe type strict converse duality [11, Theorem 4.3] and the Mond-Weir type strict converse duality [11, Theorem 5.3] are special cases of Theorem 3.

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